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41 FETI-DP Methods for Elliptic Problems with Discontinuous Coefficients in Three Dimensions

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Introduction

Farhat, Lesoinne, Le Tallec, Pierson, and Rixen [FLLT$^+$01] have recently introduced a dual–primal FETI (FETI-DP) algorithm suitable for second order elliptic problems in the plane and for plate problems. A convergence analysis in the case of benign coefficients is given by Mandel and Tezaur [MT01]. Numerical experiments show a poor performance for this algorithm in three dimensions; cf. [FLLT$^+$01]. Recent experiments with alternative algorithms are reported in [FLP00, Pie00]. We give a brief description of our own recent work in the third section; see [KWD01] for many more details.

The remainder of this paper is organized as follows. In the next section , we introduce our elliptic problems and the basic geometry of the decomposition. In the third section, we present results on new dual–primal FETI methods for problems with discontinuous coefficient in three dimensions; see [KWD01].

Elliptic model problem, finite elements, and geometry

Let $\Omega \subset \mathbf{R}^3$, be a bounded, polyhedral region, let $\partial\Omega_D \subset \partial\Omega$ be a closed set of positive measure, and let $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$ be its complement. We impose homogeneous Dirichlet and general Neumann boundary conditions, respectively, on these two subsets and introduce the Sobolev space $H_0^1(\Omega, \partial\Omega_D) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$.

We decompose Ω into non-overlapping subdomains $\Omega_i, i = 1, \ldots, N$, also known as substructures, and each of which is the union of shape-regular elements with the finite element nodes on the boundaries of neighboring subdomains matching across the interface $\Gamma := \overline{\left(\bigcup_{i=1}^N \partial\Omega_i\right)} \setminus \partial\Omega$. The interface Γ is decomposed into subdomain faces, regarded as open sets, which are shared by two subregions, edges which are shared by more than two subregions and the vertices which form the endpoints of edges. We denote faces of Ω_i by $\mathcal{F}^{ij}$, edges by $\mathcal{E}^{ik}$, and vertices by $\mathcal{V}^{i\ell}$.

For simplicity, we will only consider a piecewise linear, conforming finite element approximation of the following scalar, second order model problem:

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Find $u \in H_0^1(\Omega, \partial\Omega_D)$, such that

$$a(u,v) = f(v) \quad \forall v \in H_0^1(\Omega, \partial\Omega_D), \tag{1}$$

where

$$a(u,v) = \sum_{i=1}^{N} \rho_i \int_{\Omega_i} \nabla u \cdot \nabla v dx, \quad f(v) = \sum_{i=1}^{N} \Big(\int_{\Omega_i} f v dx \; + \; \int_{\partial\Omega_i \cap \partial\Omega_N} g_N v ds \Big). \tag{2}$$

where g_N is the Neumann boundary data defined on $\partial\Omega_N$; it provides a contribution to the load vector of the finite element problem. We assume that ρ_i is a positive constant on each subregion Ω_i.

In our theoretical analysis, we assume that each subregion Ω_i is the union of a number of shape regular tetrahedral coarse elements and that the number of such tetrahedra is uniformly bounded for each subdomain. Thus, the subregions are not very thin and we can also easily show that the diameters of any pair of neighboring subdomains are comparable.

We also make a number of technical assumptions on the intersection of the boundary of the substructures and $\partial\Omega_D$; see [KWD01]. The sets of nodes in Ω_i, on $\partial\Omega_i$, and on Γ are denoted by $\Omega_{i,h}, \partial\Omega_{i,h}$, and Γ_h, respectively.

We denote the standard finite element space of continuous, piecewise linear functions on Ω_i by $W^h(\Omega_i)$. For simplicity, we assume that the triangulation of each subdomain is quasi uniform. The diameter of Ω_i is H_i, or generically, H. We denote the corresponding finite element trace spaces by $W_i := W^h(\partial\Omega_i), i = 1, \ldots, N$, and by $W := \prod_{i=1}^N W_i$ the associated product space. We will often consider elements of W which are discontinuous across the interface.

The finite element approximation of the elliptic problem is continuous across Γ and we denote the corresponding subspace of W by $\widehat{W}$. We note that while the stiffness matrix K and its Schur complement S, obtained from K by elimination of the interior subdomain variables, which both correspond to the product space W generally are singular those of $\widehat{W}$ are not.

For the dual–primal FETI methods, we will use additional, intermediate subspaces $\widetilde{W}$ of W for which a relatively small number of continuity constraints are enforced across the interface throughout the iteration. One of the benefits of working in $\widetilde{W}$, rather than in W, is that certain related Schur complements $\widetilde{S}$ and S_Δ are positive definite.

As in previous work on Neumann–Neumann and FETI algorithms, a crucial role is played by *the weighted counting functions* $\mu_i \in \widehat{W}$, which are associated with the individual subdomain boundaries $\partial\Omega_i$; cf., e.g., [DSW96, DW95]. In present context they will be used in the definition of certain diagonal scaling matrices. These functions are defined, for $\gamma \in [1/2, \infty)$, and for $x \in \Gamma_h \cup \partial\Omega_h$, by a sum of contributions from Ω_i, and its relevant next neighbors

$$\mu_i(x) = \begin{cases} \sum_{j \in \mathcal{N}_x} \rho_j^\gamma(x) & x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}, \\ \rho_i^\gamma(x) & x \in \partial\Omega_{i,h} \cap (\partial\Omega_h \setminus \Gamma_h), \\ 0 & x \in (\Gamma_h \cup \partial\Omega_h) \setminus \partial\Omega_{i,h}. \end{cases} \tag{3}$$

Here, $\mathcal{N}_x$ is the set of indices of the subregions which have x on its boundary. We note that any node of Γ_h belongs either to two faces, more than two edges, or to the vertices of several substructures.

The pseudo inverses $\mu_i^\dagger$ are defined, for $x \in \Gamma_h \cup \partial\Omega_h$, by

$$\mu_i^\dagger(x) = \begin{cases} \mu_i^{-1}(x) & \text{if } \mu_i(x) \neq 0, \\ 0 & \text{if } \mu_i(x) = 0. \end{cases}$$

New Dual–Primal FETI methods

In previous studies of dual–primal FETI methods for problems in two dimensions, see Farhat, Lesoinne, Le Tallec, Pierson, and Rixen [FLLT$^+$01] and Mandel and Tezaur [MT01], the constraints on the degrees of freedom associated with the vertices of the substructures are enforced, i.e., the corresponding degrees of freedom have been added to the primal set of variables, while all the constraints associated with the edge nodes are enforced only at the convergence of the iterative method. In each step of the iteration a fully assembled linear subsystem is solved. In a simple two–dimensional case, this subsystem corresponds to all the interior and cross point variables; these variables can be eliminated at a modest expense since we can first eliminate all the interior variables, in parallel across the subdomains, resulting in a Schur complement for the cross point variables which can be shown to be sparse. It has a dimension which equals the number of subdomain vertices which do not belong to $\partial\Omega_D$.

In their recent paper, Mandel and Tezaur [MT01] established a condition number bound of the form $C(1 + \log(H/h))^2$ for the resulting FETI method equipped with a Dirichlet preconditioner which is very similar to those used for the older FETI methods and which is built from local solvers on the subregions with zero Dirichlet conditions at the vertices of the subregions. They also established a corresponding result for a fourth-order elliptic problem in the plane.

The same algorithm is also defined for three dimensions but it does not perform well. This is undoubtedly related to the poor performance of many vertex-based iterative substructuring methods; see [DSW94, Section 6.1] and [KWD01]. Recently, Farhat, Lesoinne, and Pierson added edge and face constraints to this basic algorithm, see [FLP00], and improved the performance.

In the present study, as well as in others of FETI–DP methods, it is convenient to work in subspaces $\widetilde{W} \subset W$ for which sufficiently many constraints are enforced so that the resulting leading diagonal block matrix of the saddle point problem, though no longer block diagonal, is strictly positive definite. We will explain how this can be accomplished and also introduce two subspaces, $\widehat{W}_\Pi \subset \widehat{W}$ and $\widetilde{W}_\Delta$, corresponding to a primal and a dual part of the space $\widetilde{W}$. These subspaces will play an important role in the description and analysis of our iterative method. The direct sum of these spaces equals $\widetilde{W}$, i.e.,

$$\widetilde{W} = \widehat{W}_\Pi \oplus \widetilde{W}_\Delta. \tag{4}$$

The second subspace, $\widetilde{W}_\Delta$, is the direct sum of local subspaces $\widetilde{W}_{\Delta,i}$ of $\widetilde{W}$ where each subdomain Ω_i contributes a subspace $\widetilde{W}_{\Delta,i}$; only its $i-th$ component in the sense of the product space $\widetilde{W}$ is nontrivial.

In the description of our algorithms, we will need certain standard finite element cutoff functions $\theta_{\mathcal{E}^{ik}}$, $\theta_{\mathcal{F}^{ij}}$, and $\theta_{\mathcal{V}^{i\ell}}$. The first two are the discrete harmonic functions which equal 1 on $\mathcal{E}_h^{ik}$ and $\mathcal{F}_h^{ij}$, respectively, and which vanish elsewhere on Γ_h; $\theta_{\mathcal{V}^{i\ell}}$ denotes the piecewise discrete harmonic extension of the standard nodal basis function associated with the vertex $\mathcal{V}^{i\ell}$. These cutoff functions are also used in the analysis of the methods; see [KWD01].

We are now ready to describe our algorithms in terms of pairs of subspaces.

Algorithm A: The primal subspace, $\widehat{W}_\Pi$, is spanned by the nodal finite element basis functions $\theta_{\mathcal{V}^{i\ell}}$. The local subspace $\widetilde{W}_{\Delta,i}$ is defined in terms of the subspace of W_i of elements which vanish at the subdomain vertices, i.e., by

$$\widetilde{W}_{\Delta,i} := \{u \in W_i : u(\mathcal{V}^{i\ell}) = 0 \ \forall \mathcal{V}^{i\ell} \in \partial\Omega_i\}.$$

Hence, $\widetilde{W}$ is the subspace of W of functions that are continuous at the subdomain vertices.

Algorithm B: The primal subspace, $\widehat{W}_\Pi$, is spanned by the vertex nodal finite element basis functions $\theta_{\mathcal{V}^{i\ell}}$ and the cutoff functions $\theta_{\mathcal{E}^{ik}}$ and $\theta_{\mathcal{F}^{ij}}$ associated with all the individual edges and faces, respectively, of the interface. The local subspaces $\widetilde{W}_{\Delta,i}$ are defined as the subspaces of W_i where the values at the subdomain vertices vanish together with the averages $\overline{u}_{\mathcal{E}^{ik}}$ and $\overline{u}_{\mathcal{F}^{ij}}$, i.e., by

$$\widetilde{W}_{\Delta,i} := \{u \in W_i : u(\mathcal{V}^{i\ell}) = 0, \overline{u}_{\mathcal{E}^{ik}} = 0, \overline{u}_{\mathcal{F}^{ij}} = 0 \ \forall \mathcal{V}^{i\ell}, \mathcal{E}^{ik}, \mathcal{F}^{ij} \subset \partial\Omega_i\}.$$

Hence, $\widetilde{W}$ is the subspace of W of functions that are continuous at the subdomain vertices and have the same averages $\overline{u}_{\mathcal{E}^{ik}}$ and $\overline{u}_{\mathcal{F}^{ij}}$ independently of which component of $u \in \widetilde{W}$ is used in the the evaluation of these averages. Here the averages $\bar{u}_{\mathcal{E}^{ik}}$ and $\bar{u}_{\mathcal{F}^{ij}}$, which by assumption take on unique values $\forall u_h \in \widetilde{W}$, are defined by,

$$\bar{u}_{\mathcal{E}^{ik}} = \frac{\int_{\mathcal{E}^{ik}} u ds}{\int_{\mathcal{E}^{ik}} 1 ds} \quad \text{and} \quad \bar{u}_{\mathcal{F}^{ij}} = \frac{\int_{\mathcal{F}^{ij}} u dx}{\int_{\mathcal{F}^{ij}} 1 dx}. \tag{5}$$

Algorithm C: The primal subspace, $\widehat{W}_\Pi$, is spanned by the vertex nodal finite element basis functions $\theta_{\mathcal{V}^{i\ell}}$ and the cutoff functions $\theta_{\mathcal{E}^{ik}}$ defined on all the edges of Γ. The local subspaces $\widetilde{W}_{\Delta,i}$ are defined as the subspaces of W_i where the values at the subdomain vertices vanish together with the averages $\overline{u}_{\mathcal{E}^{ik}}$, i.e., by

$$\widetilde{W}_{\Delta,i} := \{u \in W_i : u(\mathcal{V}^{i\ell}) = 0, \overline{u}_{\mathcal{E}^{ik}} = 0, \ \forall \mathcal{V}^{i\ell}, \mathcal{E}^{ik} \subset \partial\Omega_i\}.$$

Hence, $\widetilde{W}$ is the subspace of W of functions that are continuous at the subdomain vertices and have common averages $\overline{u}_{\mathcal{E}^{ik}}$ for the individual edges. The number of degrees of freedom of the corresponding primal subspace $\widehat{W}_\Pi$ is therefore equal to the sum of the number of vertices and the number of edges; this $\widehat{W}_\Pi$ will be of lower dimension than the primal space of Algorithm B.

The number of constraints enforced in all the iterations of Algorithms B and C is substantially larger than when only the vertex constraints are satisfied as in Algorithm A, but we are still able to work with a uniformly bounded number of such constraints for each substructure. In order to put this in perspective, we consider Algorithms B and C in the very regular case of cubic substructures. There are then seven global variables for each interior substructure in the case of Algorithm B since there are eight vertices, each shared by eight cubes, twelve edges, each shared by four, and six faces each shared by a pair of substructures. The count for Algorithm C is four. We note that the counts would be different, relative to the number of substructures, in the case of tetrahedral subregions.

It is useful to distinguish between the continuity constraints at the vertices and the other constraints. The latter are sometimes called optional constraints since they are not needed to guarantee solvability of the subproblems if there are enough vertex constraints. The vertex constraints are enforced in the subassembly process, for the primal problem, outlined above. The optional constraints could be similarly incorporated after a change of variables. Another possibility, which we advocate, is to introduce an additional set of Lagrange multipliers which are computed exactly in each iteration to enforce the required optional constraints of the primal subspace; see Farhat, Lesoinne, and Pierson [FLP00], where this approach is used. For a more detailed description of this approach, we refer to section 4.2, especially formulae (24)-(28), of that paper.

We are able to show as strong a result for Algorithm C as for Algorithm B. It is therefore natural to attempt to drop additional constraints, i.e., further decrease the primal subspace $\widetilde{W}_\Pi$ while attempting to preserve the fast convergence of the FETI-DP method. This leads to the introduction of our final algorithm.

Algorithm D: The primal subspace $\widehat{W}_\Pi$, is defined in terms of constraints associated with a subset of the edges and vertices of the interface. We first describe the requirements on a minimal set of primal constraints which we have found necessary to give a complete proof of a good bound for Algorithm D. For each face, we should have at least one designated, primal edge. Additionally, for all pairs of substructures Ω_i, Ω_j, which have an edge in common, we must have an acceptable *edge path* between the two subdomains. An acceptable edge path is a path from Ω_i to Ω_j, possibly via several other subdomains, Ω_k, which have the edge $\mathcal{E}^{ij}$ in common and such that their coefficients satisfy $TOL * \rho_k \geq \min(\rho_i, \rho_j)$ for some chosen tolerance TOL. The path can only pass from one subdomain to another through an edge designated as primal. Finally, we consider all pairs of substructures which have a vertex $\mathcal{V}^{i\ell}$ but not a face or an edge in common. Then, we assume that either $\mathcal{V}^{i\ell}$ is a primal vertex or that we have an acceptable edge path of the same nature as above, except that we can be more lenient and only insist on $TOL * \rho_k \geq (h_k/H_k) \min(\rho_i, \rho_j)$. A possible algorithm of selecting the set of primal constraints is given in [KWD01].

We can now formulate our FETI–DP algorithms. The primal part of the algorithm is based on the exact elimination of all unknowns of the primal subspace as well as the interior variables. The remaining system is written in terms of a Schur complement $\widetilde{S}$. Thus, for all the algorithms, we arrive at this reduced problem after eliminating the primal variables associated with the interior nodes, the vertex nodes designated as primal, as well as the Lagrange multipliers related to the optional constraints. This Schur complement $\widetilde{S}$ can also be defined in terms of a minimum property; cf. [KWD01]. Analogously, we get from the load vectors associated with each subdomain a reduced right hand side $\tilde{f}_\Delta$. We can now reformulate the original finite element problem, reduced to the degrees of freedom of the second subspace $\widetilde{W}_\Delta$, as a minimization problem with constraints given by the requirement of continuity across Γ_h:

Find $u_\Delta \in \widetilde{W}_\Delta$, such that

$$\left. \begin{array}{c} J(u_\Delta) := \frac{1}{2}\langle \widetilde{S} u_\Delta, u_\Delta \rangle - \langle \tilde{f}_\Delta, u_d \rangle \to \min \\ B_\Delta u_\Delta = 0 \end{array} \right\} . \tag{6}$$

The matrix B_Δ is constructed from $\{0, 1, -1\}$ such that the values of the solution u_Δ, associated with more than one subdomain, coincide when $B_\Delta u_\Delta = 0$. These constraints are very simple and just express that the nodal values coincide across the interface; in comparison with the one-level FETI method, see, e.g., [KW01], we can drop some of the constraints, in particular those associated with the vertex nodes of the primal space. However, we will otherwise use all possible constraints and thus work with a fully redundant set of Lagrange multipliers as in [KW01, section 5].

By introducing a set of Lagrange multipliers $\lambda \in V := range\,(B_\Delta)$, to enforce the constraints $B_\Delta u_\Delta = 0$, we obtain a saddle point formulation of (6), which is similar to that of the one-level FETI method; see, e.g., Klawonn and Widlund [KW01]. We use that $\widetilde{S}$ is invertible and eliminate the subvector u_Δ, and obtain the following system for the dual variable:

$$F\lambda = d, \tag{7}$$

where

$$F := B_\Delta \widetilde{S}^{-1} B_\Delta^t$$

and the right hand side

$$d := B_\Delta \widetilde{S}^{-1} \tilde{f}_\Delta.$$

To define the FETI–DP Dirichlet preconditioner, we need to introduce an additional set of Schur complement matrices,

$$S_\Delta^{(i)} := K_{\Delta\Delta}^{(i)} - K_{\Delta I}^{(i)} (K_{II}^{(i)})^{-1} K_{I\Delta}^{(i)}, \quad i = 1, \ldots, N,$$

Here, $K_{\Delta\Delta}^{(i)}$ is the principal minor of the stiffness matrix after the change of variables and it is related to the variables of $\widetilde{W}_\Delta$. The associated block–diagonal matrix is denoted by

$$S_\Delta := diag_{i=1}^N (S_\Delta^{(i)}).$$

We can compute the action of S_Δ on a vector from the second subspace $\widetilde{W}_\Delta$ by solving local problems with solutions that are constrained to vanish or to have zero average at the designated, primal variables, as required by the algorithm in question; these constraints can be enforced by using Lagrange multipliers or a partial change of basis.

We also introduce diagonal scaling matrices $D_\Delta^{(i)}$ that operate on the Lagrange multiplier spaces. Each element on the main diagonal corresponds to a Lagrange multiplier which enforces continuity between the nodal values of some $w_i \in \widetilde{W}_i$ and $w_j \in \widetilde{W}_j$ at some point $x \in \Gamma_h$. This diagonal element is defined as $\rho_j^\gamma(x) \mu_j^\dagger(x)$. Finally, we define a scaled jump operator by

$$B_{D,\Delta} := [D_\Delta^{(1)} B_\Delta^{(1)}, \ldots, D_\Delta^{(N)} B_\Delta^{(N)}].$$

As in Klawonn and Widlund [KW01, section 5], we solve the dual system (7) using the preconditioned conjugate gradient algorithm with the preconditioner

$$M^{-1} := B_{D,\Delta} S_\Delta B_{D,\Delta}^t. \tag{8}$$

The dual–primal FETI method is now the standard preconditioned conjugate gradient algorithm for solving the preconditioned system

$$M^{-1} F\lambda = M^{-1} d.$$

This definition of M clearly depends on the choice of the subspaces $\widehat{W}_\Pi$ and $\widetilde{W}_\Delta$ for the different algorithms.

A proof of the following theorem can be found in Klawonn, Widlund, and Dryja [KWD01].

Theorem 1 *The condition numbers of the preconditioned FETI–DP Algorithms B and C satisfy*

$$\kappa(M^{-1}F) \leq C\,(1 + \log(H/h))^2$$

and the condition number of Algorithm D satisfies

$$\kappa(M^{-1}F) \leq C\,\max(1, TOL)\,(1 + \log(H/h))^2.$$

Here, C is independent of h, H, γ, and the values of the ρ_i.

Remark 1 *A weaker condition number estimate, with an additional factor H/h, can be given for Algorithm A; see [KWD01].*

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