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52 Iterative substructuring methods for incompressible and nonisothermal flows using the $k-\epsilon$ turbulence model

T. Knopp[1], G. Lube[2], H. Müller[3]

Introduction

We consider the parallel solution of the incompressible Navier-Stokes equations coupled with the energy equation. For turbulent flows, the k/ϵ model is used. The iterative process requires the fast solution of advection-diffusion-reaction and Oseen type problems. These linearized problems are discretized using stabilized FEM. We apply an iterative substructuring method which couples the subdomain problems via Robin-type interface conditions. Then we apply the approach to the simulation of indoor air flow problems.
The mathematical model under consideration is the incompressible, nonisothermal (Reynolds averaged) Navier-Stokes problem in a bounded polyhedral domain $\Omega \subset \mathbf{R}^d$. For turbulent flows we apply the $k-\epsilon$ model, cf. [CS99, MP94, Mue99]. Turbulent effects are modelled as additional turbulent viscosity $\nu_t = C_\mu \frac{k^2}{\epsilon}$ and thermal diffusivity $a_t = \frac{\nu_t}{Pr_t}$, using the turbulent kinetic energy k and turbulent dissipation ϵ. Bouyancy effects are taken into account using the Boussinesq approximation.
The velocity $\vec{u}$, the (reduced) pressure p, and the temperature θ, and in the turbulent case, the quantities k and ϵ are solutions of the coupled nonlinear system

$$\left\{\begin{array}{rcl} \partial_t\vec{u} - \vec{\nabla}\cdot(2\nu_e S(\vec{u})) + (\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p &=& -\beta\theta\vec{g} \\ \vec{\nabla}\cdot\vec{u} &=& 0 \\ \partial_t\theta + (\vec{u}\cdot\vec{\nabla})\theta - \vec{\nabla}\cdot(a_e\vec{\nabla}\theta) &=& \dot{q}^V/c_p \\ \partial_t k + (\vec{u}\cdot\vec{\nabla})k - \vec{\nabla}\cdot(\nu_k\vec{\nabla}k) &=& P_k + G - \epsilon \\ \partial_t\epsilon + (\vec{u}\cdot\vec{\nabla})\epsilon - \vec{\nabla}\cdot(\nu_\epsilon\vec{\nabla}\epsilon) + C_2\epsilon^2 k^{-1} &=& C_1\epsilon k^{-1}(P_k + G) \end{array}\right. \quad (1)$$

with constants $C_1, C_2, C_\mu, C_t, Pr_t, Pr_k, Pr_\epsilon$, effective viscosities $\nu_e = \nu + \nu_t$, $a_e = a + a_t, \nu_k = \nu + \frac{\nu_t}{Pr_k}, \nu_\epsilon = \nu + \frac{\nu_t}{Pr_\epsilon}$, production and bouyancy terms

$$P_k := 2\nu_t|S(\vec{u})|^2, \quad G := C_t\beta\frac{\nu_t}{Pr_t}\vec{g}\cdot\vec{\nabla}\theta \quad \text{with } S(\vec{u}) := \frac{1}{2}(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T).$$

In laminar flows we set $k \equiv 0$ and skip the $k-\epsilon$ equations in (1). The boundary is divided into inlet, outlet and wall zones Γ_-, Γ_+ and Γ_0 depending on the sign of $\vec{u}\cdot\vec{n}$. Using $\tau = 2\nu_e S(\vec{u})$, we set in *laminar* flows

$$(\tau - pI)\vec{n} = \tau_n\vec{n} \text{ on } \Gamma_N \subset \Gamma_- \cup \Gamma_+, \qquad \vec{u} = \vec{u}_b \text{ on } \Gamma_D \subset \Gamma_- \cup \Gamma_+, \quad (2)$$

[1]University of Göttingen, Math. Departm., D-37073 Göttingen, knopp@math.uni-goettingen.de
[2]University of Göttingen, Math. Departm., D-37073 Göttingen, lube@math.uni-goettingen.de
[3]Aerotec Engrg. GmbH Hamburg

with $\Gamma_D \cap \Gamma_N = \emptyset$ and $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \overline{\Gamma}_- \cup \overline{\Gamma}_+$. On Γ_0, we prescribe either the tangential stresses and the normal velocity or a no-slip condition

$$(i)\ \ (I - \vec{n} \otimes \vec{n})\tau\vec{n} = \vec{\tau}_t, \quad \vec{u} \cdot \vec{n} = 0, \qquad \text{or } \ (ii)\ \ \vec{u} = \vec{0} \quad \text{on } \Gamma_0. \tag{3}$$

In this paper, we consider only case (i) with $\Gamma_D = \emptyset$. For θ we set

$$\theta = \theta_{in} \quad \text{on } \Gamma_-\ ,\ a_e \vec{\nabla}\theta \cdot \vec{n} = 0 \text{ on } \Gamma_+\ ,\ a_e \vec{\nabla}\theta \cdot \vec{n} = \dot{q}_0/c_p \ \text{ on } \Gamma_0. \tag{4}$$

For *turbulent* flows, we apply the concept of wall functions in a neighbourhood Ω_δ of Γ_0 containing at least the so-called viscous sub-layer. Firstly, as usual in wall law theory, the r.h.s. $\vec{\tau}_t$ in (3) and $\dot{q}_0$ in (4) are modified. We set $\vec{\tau}_t = U_*^2 \vec{u}/||\vec{u}||$ and seek $(U_*, \dot{q}_0)$ as solutions of coupled nonlinear equations. Secondly, for the $k - \epsilon$ equations the computational domain is $\Omega \backslash \Omega_\delta$. Dirichlet data are prescribed on Γ_- and on the artificial boundary $\Gamma_\delta = \partial\Omega_\delta \cap \Omega$. A no-flow condition is specified on Γ_+. A computational algorithm has to control that Γ_δ, being discretized with mesh points with minimal distance to Γ_0, belongs to the so-called log-layer. For details see [KLGR00, Mue99].

Discretization, decoupling, and linearisation

Semidiscretization in time of system (1):
We are mainly interested in the long-term behaviour of the model. So we apply the backward Euler scheme on a partition $\{t_m\}_{m=0}^M$ of $[0,T]$ with $t_0 = 0,\ t_M = T$. We use the abbreviation $F^m = F(t_m) \equiv F(t_m, \cdot)$ for a function F. The time derivative $\partial_t F(t_m)$ is approximated by $\partial_t^m F = (F^m - F^{m-1})/\triangle_m$ with time-step $\triangle_m = t_m - t_{m-1}$. We arrive at the semidiscrete system

$$\begin{aligned}
\partial_t^m \vec{u} - \vec{\nabla} \cdot (2\nu_e^m S(\vec{u}^m)) + (\vec{u}^m \cdot \vec{\nabla})\vec{u}^m + \vec{\nabla} p^m &= -\beta\theta^m \vec{g} \\
\vec{\nabla} \cdot \vec{u}^m &= 0 \\
\partial_t^m \theta + (\vec{u}^m \cdot \vec{\nabla})\theta^m - \vec{\nabla} \cdot (a_e^m \vec{\nabla}\theta^m) &= (\dot{q})^{V^m}/c_p \\
\partial_t^m k + (\vec{u}^m \cdot \vec{\nabla})k^m - \vec{\nabla} \cdot (\nu_k^m \vec{\nabla} k^m) &= P_k^m + G^m - \epsilon^m \\
\partial_t^m \epsilon + (\vec{u}^m \cdot \vec{\nabla})\epsilon^m - \vec{\nabla} \cdot (\nu_\epsilon^m \vec{\nabla}\epsilon^m) + C_2 \frac{(\epsilon^m)^2}{k^m} &= C_1 \frac{\epsilon^m}{k^m}(P_k^m + G^m).
\end{aligned} \tag{5}$$

Decoupling and linearization:
We use a block Gauss-Seidel method for the iterative decoupling of (5). A second upper index denotes the iteration step. Furthermore we replace $\partial_t^m F$ by $\tilde{\partial}_t^m F := (F^{m,i} - F^{m-1})/\triangle_m$. Given $\vec{u}^{m,0}$, $p^{m,0}$, $\theta^{m,0}$, $k^{m,0}$, $\epsilon^{m,0}$ as the solutions of the previous time step, the algorithm reads:

(1) Initialization: Set $it_{dlc} \longleftarrow 1$.

(2) Set $i \longleftarrow it_{dlc}$ and update turbulent viscosity $\nu_t^m \longleftarrow \nu_t^m(k^{m,i-1}, \epsilon^{m,i-1})$. Update $U^*, \dot{q}_0$ according to (3),(4) using $\vec{u}^{m,i-1}$ and $\theta^{m,i-1}$.

(3) Update ν_e^m and solve the linearized Navier-Stokes-equation

$$\begin{aligned}
\tilde{\partial}_t^m \vec{u} + (\vec{u}^{m,i-1} \cdot \vec{\nabla})\vec{u}^{m,i} - \vec{\nabla} \cdot (2\nu_e^m S(\vec{u}^{m,i})) + \vec{\nabla} p^{m,i} &= -\beta\theta^{m,i-1}\vec{g} \\
\vec{\nabla} \cdot \vec{u}^{m,i} &= 0
\end{aligned}$$

(4) Update a_e^m and solve the θ-equation.

$$\tilde{\partial}_t^m \theta + (\vec{u}^{m,i} \cdot \vec{\nabla})\theta^{m,i} - \vec{\nabla} \cdot (a_e^m \vec{\nabla}\theta^{m,i}) \;=\; (\dot{q}^V)^m / c_p$$

(5) Update ν_k^m, P_k^m, G^m using $\vec{u}^{m,i}$, $\theta^{m,i}$ and solve the k-equation.

$$\tilde{\partial}_t^m k + (\vec{u}^{m,i} \cdot \vec{\nabla})k^{m,i} - \vec{\nabla} \cdot (\nu_k^m \vec{\nabla} k^{m,i}) = P_k^m + G^m - \epsilon^{m,i-1}$$

(6) Update P_k^m, G^m, ν_ϵ^m using $\vec{u}^{m,i}$, $\theta^{m,i}$, $k^{m,i}$ and solve the ϵ-equation.

$$\tilde{\partial}_t^m \epsilon + (\vec{u}^{m,i} \cdot \vec{\nabla})\epsilon^{m,i} - \vec{\nabla} \cdot (\nu_\epsilon^m \vec{\nabla}\epsilon^{m,i}) + C_2 \frac{\epsilon^{m,i-1}}{k^{m,i}} \epsilon^{m,i} = C_1 \frac{\epsilon^{m,i-1}}{k^{m,i}} (P_k^m + G^m)$$

(7) Stopping-criterion for linearization cycle : If $it_{dlc} < max_{dlc}$ and if stopping criteria for $\{\vec{u}^{m,i}\}_i$, $\{\theta^{m,i}\}_i$, $\{k^{m,i}\}_i$, $\{\epsilon^{m,i}\}_i$ are not yet fulfilled, then set $it_{dlc} \longleftarrow it_{dlc} + 1$ and goto (2). Otherwise goto next time step.

Linearized kernels:
The iterative scheme requires the solution of two basic model problems. First, the linearized equations for θ, k and ϵ are *advection-diffusion problems* with non-constant viscosity of the general form :

$$\left\{ \begin{array}{rcll} Lu \equiv -\vec{\nabla} \cdot (\nu \vec{\nabla} u) + (\vec{b} \cdot \vec{\nabla})u + cu &=& f & \text{in } \tilde{\Omega} \\ u &=& g & \text{on } \tilde{\Gamma}_D \\ \nu \vec{\nabla} u \cdot \vec{n} &=& h & \text{on } \tilde{\Gamma}_N . \end{array} \right. \tag{6}$$

For θ we set $\tilde{\Omega} = \Omega$, $\tilde{\Gamma}_D = \Gamma_-$, $\tilde{\Gamma}_N = \Gamma_0 \cup \Gamma_+$, $h|_{\Gamma_0} = \dot{q}_0 / c_p$, $h|_{\Gamma_+} = 0$. For k and ϵ set $\tilde{\Omega} = \Omega \backslash \Omega_\delta$, $\tilde{\Gamma}_D = (\Gamma_- \cap \partial\tilde{\Omega}) \cup \Gamma_\delta$ with appropriate g and $\tilde{\Gamma}_N = \Gamma_+$ with $h = 0$. The other data are given in the following table.

equation	u	ν	$\vec{b}$	cu	f
for θ	$\theta^{m,i}$	a_e^m	$\vec{u}^{m,i}$	$\theta^{m,i}/\triangle_m$	$\dot{q}^V/c_p + \theta^{m-1}/\triangle_m$
for k	$k^{m,i}$	ν_k^m	$\vec{u}^{m,i}$	$k^{m,i}/\triangle_m$	$(P_k^m + G^m) - \epsilon^{m,i-1}$ $+k^{m-1}/\triangle_m$
for ϵ	$\epsilon^{m,i}$	ν_ϵ^m	$\vec{u}^{m,i}$	$C_2 \frac{\epsilon^{m,i-1}}{k^{m,i}} \epsilon^{m,i}$ $+\epsilon^{m,i}/\triangle_m$	$C_1 \frac{\epsilon^{m,i-1}}{k^{m,i}} (P_k^m + G^m)$ $+\epsilon^{m-1}/\triangle_m$

Later on, we simply write Ω and omit the indices of viscosities and production terms.

The linearized Navier-Stokes-equation is an *Oseen*-type problem with a positive reaction term and non-constant viscosity:

$$\begin{array}{rcll} L_O(\vec{a}, \vec{u}, p) \equiv -\vec{\nabla} \cdot (2\nu S(\vec{u})) + (\vec{a} \cdot \vec{\nabla})\vec{u} + c\vec{u} + \vec{\nabla} p &=& \vec{f} & \text{in } \Omega \\ \vec{\nabla} \cdot \vec{u} &=& 0 & \text{in } \Omega \\ (\tau - pI)\vec{n} &=& \tau_n \vec{n} & \text{on } \Gamma_- \cup \Gamma_+ \\ (I - \vec{n} \otimes \vec{n})\tau\vec{n} = \vec{\tau}_t, \quad \vec{u} \cdot \vec{n} &=& 0 & \text{on } \Gamma_0 . \end{array} \tag{7}$$

Comparison with step (3) of the algorithm yields $\vec{u} = \vec{u}^{m,i}$, $\nu = \nu_e$, $\vec{a} = \vec{u}^{m,i-1}$, $c = \triangle_m^{-1}$, $p = p^{m,i}$, $\vec{f} = -\beta\theta^{m,i-1}\vec{g} + \triangle_m^{-1}\vec{u}^{m-1}$.

Stabilized finite element discretization of (6)-(7):
Assume an admissible triangulation $\mathcal{T}_h$ of the Lipschitz domain Ω and define finite element subspaces $X_h^l \equiv \{v \in C(\bar{\Omega}) \mid v|_K \in \Pi_l(K)\ \forall K \in \mathcal{T}_h\}$, $l \in \mathbf{N}$.
For the *advection-diffusion-reaction problem* (6), for simplicity with $g = 0$ on Γ_D, we apply the Galerkin-FEM with SUPG-stabilization:

$$\text{Find } u \in V_h = \{v \in X_h^l \mid v|_{\Gamma_D} = 0\} \text{ s.t.: } b^s(u,v) = l^s(v) \quad \forall v \in V_h\ , \tag{8}$$

$$\begin{aligned} b^s(u,v) &= \int_\Omega \Big(\nu\vec{\nabla}u\cdot\vec{\nabla}v + (\vec{b}\cdot\vec{\nabla})u\ v + cuv\Big)\,dx + \sum_{T\in\mathcal{T}_h}\int_T \delta_T Lu(\vec{b}\cdot\vec{\nabla})v\ dx \\ l^s(v) &= \int_\Omega fv\,dx + \int_{\Gamma_N} hv\,ds + \sum_{T\in\mathcal{T}_h}\int_T \delta_T f\ (\vec{b}\cdot\vec{\nabla})v\ dx \end{aligned}$$

with appropriate parameter set $\{\delta_T\}_T$, see [KLGR00]. The SUPG solutions may suffer from local crosswind oscillations in layers, hence negative values of k or ϵ can occur. As a remedy, we add in a consistent way crosswind diffusion thus leading to the (nonlinear) shock-capturing method, for details see [CS99].

For the *Oseen-problem* (7), we define the discrete spaces $\mathbf{V}_h \times Q_h = (X_h^r)^d \times X_h^s$ with $r, s \in \mathbf{N}$. The Galerkin FEM requires the (bi)linear forms

$$\mathcal{A}(U,V) = a(\vec{u},\vec{v}) + b(\vec{v},p) - b(\vec{u},q)\ , \qquad \mathcal{L}(V) = L(\vec{v}).$$

with $U = (\vec{u},p)$, $V = (\vec{v},q)$ and $b(\vec{v},p) = -\int_\Omega p(\vec{\nabla}\cdot\vec{v})\,dx$. Furthermore set

$$\begin{aligned} a(\vec{u},\vec{v}) &= \int_\Omega 2\nu S(\vec{u}) : \vec{\nabla}\vec{v} + ((\vec{a}\cdot\vec{\nabla})\vec{u} + c\vec{u})\cdot\vec{v}\,dx + \int_{\Gamma_0}(pI - \vec{n}\otimes\vec{n}\tau)\vec{n}\cdot\vec{v}\,ds \\ L(\vec{v}) &= \int_\Omega \vec{f}\cdot\vec{v}\,dx + \int_{\Gamma_-\cup\Gamma_+}\tau_n\vec{n}\cdot\vec{v}\,ds + \int_{\Gamma_0}\vec{\tau}_t\cdot\vec{v}\,ds. \end{aligned}$$

When using equal order ansatz functions $r = s$, the discrete Babuska-Brezzi condition is not satisfied. This problem is circumvented using a pressure (PSPG) stabilization. In addition, divergence and a SUPG stabilization is used to deal with dominating first order terms. More precisely, we set

$$\begin{aligned} \mathcal{A}^s(U,V) &= \mathcal{A}(U,V) + \sum_{T\in\mathcal{T}_h}\int_T \Big[L_O(\vec{a},\vec{u},p)\ \Big(\delta_{1u}^T\vec{a}\cdot\vec{\nabla})\vec{v}\ + \delta_{1p}^T\ \vec{\nabla}q\Big)\,dx \\ &\qquad + \int_T \delta_{2u}^T\ (\vec{\nabla}\cdot\vec{u})(\vec{\nabla}\cdot\vec{v})\ dx\Big] \\ \mathcal{L}^s(V) &= \mathcal{L}(V) + \sum_{T\in\mathcal{T}_h}\int_T \vec{f}\Big(\delta_{1u}^T(\vec{a}\cdot\vec{\nabla})\vec{v} + \delta_{1p}^T\vec{\nabla}q\Big)\ dx. \end{aligned}$$

Finally, the stabilized problem to the Oseen equation (7) reads

$$\text{Find } U = (\vec{u},p) \in \mathbf{V}_h \times Q_h, \text{ s.t. } \mathcal{A}^s(U,V) = \mathcal{L}^s(V)\ \forall V \in \mathbf{V}_h \times Q_h. \tag{9}$$

For the choice of the stabilization parameters δ_{1u}^T, δ_{2u}^T and δ_{1p}^T see [KLGR00].

Domain decomposition of the linearized problems

Here we apply a nonoverlapping domain decomposition method with Robin interface conditions to the basic linearized problems (6), (7). Consider a nonoverlapping partition of Ω into convex, polyhedral subdomains being aligned with the finite element mesh, i.e.

$$\bar{\Omega} = \cup_{k=1}^{N} \bar{\Omega}_k, \quad \Omega_k \cap \Omega_j = \emptyset \quad \forall k \neq j\ , \quad \forall K \in \mathcal{T}_h\ \exists k\ :\ K \subset \Omega_k.$$

Furthermore, set $\Gamma_k := \partial\Omega_k \backslash \partial\Omega,\ \Gamma_{jk} := \partial\Omega_j \cap \partial\Omega_k,\ j \neq k$, where Γ_{kj} is identified with Γ_{jk}. Assume, for simplicity, that the partition is stripwise.

For the (continuous) *advection-diffusion-reaction* problem (6) the DDM reads:
for given u_k^n from iteration step n on each Ω_k, seek (in parallel) for u_k^{n+1}

$$\begin{cases} Lu_k^{n+1} = f & \text{in } \Omega_k \\ u_k^{n+1} = 0 & \text{on } \Gamma_D \cap \partial\Omega_k \\ \nu\vec{\nabla}u_k^{n+1} \cdot \vec{n}_k = h & \text{on } \Gamma_N \cap \partial\Omega_k \\ \Phi_k(u_k^{n+1}) = \theta\Phi_k(u_j^n) + (1-\theta)\Phi_k(u_k^n) & \text{on } \Gamma_{jk},\ j = 1, \ldots, N,\ j \neq k. \end{cases} \tag{10}$$

$\theta \in (0, 1]$ is a relaxation parameter. The interface function is specified as

$$\Phi_k(u) = \nu\vec{\nabla}u \cdot \vec{n}_k + (-\frac{1}{2}\vec{b} \cdot \vec{n}_k + z_k)u. \tag{11}$$

Let $V_{k,h}$, b_k^s and l_k^s denote the restrictions of V_h, b^s and l^s to Ω_k, respectively. $W_{kj,h}$ is the restriction of V_h to the interface part Γ_{kj}. Furthermore, $\langle\cdot,\cdot\rangle_{\Gamma_{kj}}$ is the inner product in $L^2(\Gamma_{kj})$ or, whenever needed, the dual product between $(W_{kj,h})^*$ and $W_{kj,h}$. The *fully discretized* DDM reads for $k = 1, \ldots, N$:

Parallel computation step : *Find $u_k^{n+1} \in V_{k,h}$ such that $\forall\ v_k \in V_{k,h}$*

$$b_k^s(u_k^{n+1}, v_k) + \langle(-\frac{1}{2}\vec{b} \cdot \vec{n}_k + z_k)u_k^{n+1}, v_k\rangle_{\Gamma_k} = l_k^s(v_k) + \sum_{j(\neq k)} \langle\Lambda_{jk}^n, v_k\rangle_{\Gamma_{kj}}.$$

Communication step : *For all $j \neq k$, update the Lagrangian multipliers*

$$\langle\Lambda_{kj}^{n+1}, \phi\rangle_{\Gamma_{kj}} = \langle\theta(z_k + z_j)u_k^{n+1} - \theta\Lambda_{jk}^n + (1-\theta)\Lambda_{kj}^n, \phi\rangle_{\Gamma_{kj}} \quad \forall\phi \in W_{kj,h}.$$

The analysis of the method, given in [LMO00], can be easily extended to the case of non-constant viscosity ν: The algorithm is well-posed if $z_k = z_j > 0$. The sequences $\{u_k^n\}_n$, $k = 1, ..., N$ converge strongly to the restrictions of the global discrete solution to Ω_k w.r.t. the stabilized energy norm induced by the symmetric part of $b_k^s(\cdot,\cdot)$.

Furthermore, an *a posteriori* estimate allows to control the convergence on subdomains via jumps of discrete DD solutions across the interface. Besides this estimate yields the following design of the interface function

$$z_k = \frac{1}{2}|\vec{b} \cdot \vec{n}_k| + R_k \tag{12}$$

with strictly positive $R_k = \mathcal{O}(\sqrt{\nu})$ depending on problem data. Formula (12) is compatible with the vanishing viscosity limit $\nu \to 0$. Moreover, it is shown in [LMO00] that (12) allows

a considerable acceleration of convergence. More precisely, the lower (and certain moderate) frequencies of the error are quickly damped. In this range, formula (12) is surprisingly sharp w.r.t. data. The convergence speed slows down when the level of the discretization error is reached. An acceleration of the method w.r.t. higher frequencies of the error is under consideration.

For the *Oseen problem* (7) we use the abbreviation $\pi_{t,k} := I - \vec{n}_k \otimes \vec{n}_k$. Then the DDM is defined as follows:
for given $(\vec{u}_k^n, p_k^n)$ *from step* n *on each* Ω_k*, seek (in parallel) for* $(\vec{u}_k^{n+1}, p_k^{n+1})$

$$\begin{cases} L_O(\vec{a}, \vec{u}_k^{n+1}, p_k^{n+1}) &= \vec{f} & \text{in } \Omega_k \\ \vec{\nabla} \cdot \vec{u}_k^{n+1} &= 0 & \text{in } \Omega_k \\ (\tau_k^{n+1} - p_k^{n+1} I)\vec{n}_k &= \tau_n \vec{n}_k & \text{on } \partial\Omega_k \cap (\Gamma_- \cup \Gamma_+) \\ \pi_{t,k}\tau_k^{n+1}\vec{n}_k &= \vec{\tau}_t, \quad -\vec{u}_k^{n+1} \cdot \vec{n}_k = 0 & \text{on } \partial\Omega_k \cap \Gamma_0 \\ \Phi_k(\vec{u}_k^{n+1}, p_k^{n+1}) &= \theta\Phi_k(\vec{u}_j^n, p_j^n) + (1-\theta)\Phi_k(\vec{u}_k^n, p_k^n) & \text{on } \Gamma_{jk}. \end{cases} \quad (13)$$

$\theta \in (0, 1]$ is again a relaxation parameter. The interface function is given by

$$\Phi_k(u, p) = \nu\vec{\nabla}\vec{u} \cdot \vec{n}_k - p\vec{n}_k + (-\frac{1}{2}\vec{a} \cdot \vec{n}_k + z_k)\vec{u} \quad (14)$$

with acceleration parameter z_k.
The corresponding parallel algorithm can be formulated (in weak form) similarly as for the scalar case. For this DD algorithm (and certain variants of it), a similar a-priori and a-posteriori analysis is available as briefly described for the scalar problem (6). In particular, the interface function z_k in (14) has the same structure as in (12). For details, we refer to [LMO01], [LMM00].

Application to room-air flow simulation

We applied our research code *Parallel NS* [Mue99] with piecewise linear ansatz functions for all unknowns ($l = r = s = 1$) on a triangular (resp. tetrahedral) mesh in 2D (resp 3D) to the numerical simulation of room-air flow.

Example 1. We present a stationary ventilated *laminar* flow with $Re = 66$, $Pr = 0.71$ through a cube $\Omega = (0, 1)^3$ with inlet zone $\Gamma_- = (0; 0.5) \times \{0\} \times (0; 0.5)$ and outlet zone $\Gamma_+ = (0.5; 1) \times \{1\} \times (0.5; 1)$, cf. Fig. 1. We impose $\tau_n = 0$, $\vec{u}|_{\Gamma_0} = \vec{0}$, $\theta(t = 0) = 293.15K$, $\theta_{in} = 283.15K$, $\theta|_{\Gamma_0} = 293.15K$. Furthermore, we used time step $\triangle_m = 1.0s$, a uniform mesh with 42^3 nodes and $max_{dlc} = 1$.

We studied the DDM on different macro partitions. Fig. 2 shows the reasonable convergence history (w.r.t. a mesh-dependent norm including H^1- and L^2-convergence of velocity and pressure, respectively) of the DD solution (with two subdomains) to the sequential discrete solution. ♣

Example 2. The application of the DDM to *turbulent* flows in 2D has been considered in [Mue99]. Here we present the natural convection for $Ra = 5.3 \cdot 10^{10}$, $Pr = 0.71$, $\dot{q}^V = 0$ in a cavity Ω of width $0.5m$ and height $H = 2.5m$. The flow is driven by a temperature difference of 45.8^o between the vertical walls and gravity. Further we impose $\vec{u}|_{\Gamma_0} = \vec{0}$ on $\Gamma_0 \equiv \partial\Omega$ and

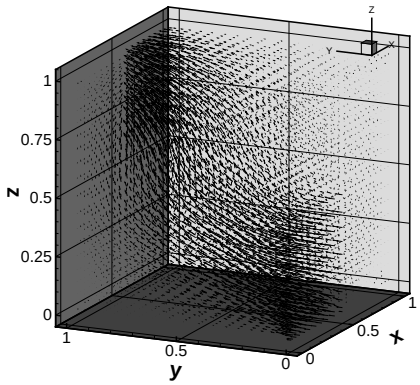

Figure 1: Laminar flow field in a ventilated room

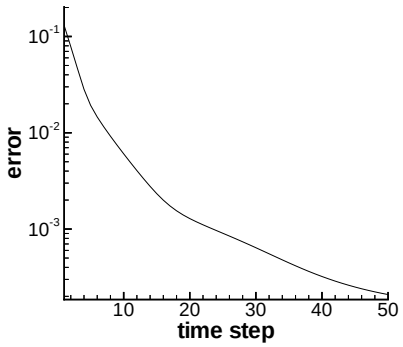

Figure 2: Convergence history of DD solution to discrete solution

adiabatic conditions for θ at the top and bottom of the flow domain Ω.

In Fig. 3 we compare the vertical velocity profiles at height $x_2 = 0.5H$ and $x_2 = 0.765H$

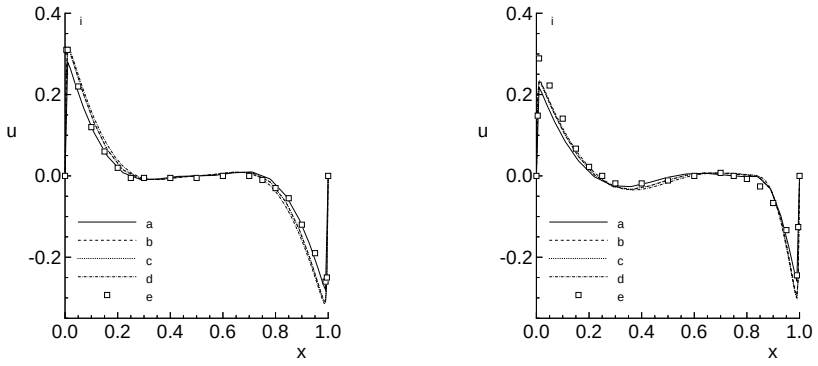

Figure 3: Vertical turbulent velocity at $x_2 = 0.5H$ and $x_2 = 0.765H$

for a DD solution with 10 subdomains and 3.432 finite elements (c) and the sequential solution on different grids with 2.728 (a), 3.432 (b) and 10.912 (d) elements. The results are in good agreement with measurements (e) by Cheesewright et.al (1986). ♣

The proposed method is currently applied at the Dresden University of Technology to the simulation of turbulent indoor air flows. Such calculations allow to predict certain parameters of the indoor-air climate over longer periods and to simulate different variants of ventilation or of heating systems. Results of this ongoing research will be presented elsewhere. Let us finally remark that the convergence of the method for the iteratively decoupled nonlinear problem (1) is rather sensitive w.r.t. different ingredients. A more robust implementation is probably given with an iterative substructuring method based on Dirichlet-Robin coupling, see [ATNV00], and with transformation to logarithmic variables in the $k - \epsilon$ equations.

References

[ATNV00]Y. Achdou, P. Le Tallec, F. Nataf, and M. Vidrascu. A domain decoposition preconditioner for an advection-diffusion problem. *Comp. Meth. Appl. Mech. Engng*, 184:145–170, 2000.

[CS99]Ramon Codina and Orlando Soto. Finite element implementations of two-equation and algebraic stress turbulence models for steady incompressible flow. *Intern. J. Numer. Meths. Fluids*, 90(3):309–334, 1999.

[KLGR00]Tobias Knopp, Gert Lube, Ralf Gritzki, and Markus Roesler. Simulation of indoor air movement using stabilized finite elements methods. Technical report, University of Goettingen, Math. Departm., 2000. accepted for Proc. Intern. Conf. FEM3D, Jyvaskyla 2000.

[LMM00]Gert Lube, Lars Mueller, and Hannes Mueller. A new non-overlapping domain decomposition method for stabilized finite element methods applied to the nonstationary Navier-Stokes equations. *Numer. Lin. Alg. Appl.*, 7:449–472, 2000.

[LMO00]Gert Lube, Lars Mueller, and Frank-Christian Otto. A non-overlapping domain decomposition method for the advection-diffusion problem. *Computing*, 64:49–68, 2000.

[LMO01]Gert Lube, Lars Mueller, and Frank-Christian Otto. A non-overlapping domain decomposition method for stabilized finite element approximations of the Oseen equations. *J. Comp. Appl. Math.*, 132:211–236, 2001.

[MP94]Bijan Mohammadi and Olivier Pironneau. *Analysis of the K-Epsilon Turbulence Model*. John Wiley and Sons, 1994.

[Mue99]Hannes Mueller. *A concept for the numerical simulation of incompressible flows using the discontinuous Galerkin method and a nonoverlapping DDM (in German)*. PhD thesis, University of Technology Dresden, 1999.