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42 Comparison of domain decomposition methods for solving continuous casting problem

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Introduction

Two different kind of domain decomposition methods and algorithms to solve the continuous casting problem are presented and analyzed. The multiplicative Schwarz method with overlapping subdomains, and splitting iterative method with nonoverlapping subdomains are studied. Results considering convergence for both of these methods are presented and studied via numerical example. The finite element method with rectangular elements was used to discretize the problem. Advantages and disadvantages for both of these methods for this problem are discussed and analyzed.

The continuous casting problem can be stated mathematically as follows. Let $\Omega = \{0 < x_1 < L_{x_1}, 0 < x_2 < L_{x_2}\}$ be the rectangular domain with the boundary $\Gamma = \partial\Omega$ consisting of two parts: $\Gamma_1 = \{x \in \partial\Omega : x_2 = 0 \vee x_2 = L_{x_2}\}$, $\Gamma_2 = \{x \in \partial\Omega \setminus \Gamma_1\}$. We assume that the domain $\Omega \subset \mathbb{R}^2$ is occupied by thermodynamically homogeneous and isotropic steel. We denote by $H(x,t)$ the enthalpy related to unit mass and by $u(x,t)$ the temperature for $(x,t) \in \Omega\times]0,T[$. We have constitutive law

$$H = H(u) = \rho \int_0^u c(\Theta)d\Theta + \rho L(1 - f_s(u)) \text{ in } \Omega\times]0,T[,$$

where ρ is density, $c(u)$ is specific heat, L is latent heat and $f_s(u)$ is solid fraction.

Graph $H(u)$ is a increasing function $\mathbb{R} \to \mathbb{R}$ involving near vertical segments corresponding to the phase transition states, namely, for $u \in [T_L, T_S]$ where $0 < T_L < T_S$ are melting and solidification temperatures, correspondingly.

We study the following boundary-value problem: find $u = u(x,t)$ such that

$$\text{(P)} \begin{cases} \dfrac{\partial H(u)}{\partial t} + v\dfrac{\partial H(u)}{\partial x_2} - \Delta u = 0 \text{ for } x \in \Omega, t > 0, \\ u = z(x_1,t) > 0 \text{ for } x \in \Gamma_1, t > 0, \\ \partial u/\partial n + au + b|u|^3 u = g, a \geq 0, b \geq 0, g \geq 0 \text{ for } x \in \Gamma_2, t > 0, \\ u = u_0(x) > 0 \text{ for } x \in \bar{\Omega}, t = 0. \end{cases}$$

The existence and uniqueness of the weak solution for the problem (P) are proved in [RY90].

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To approximate the problem (P) we rewrite it as the integral equality for fixed $t > 0$. Let $V = H^1(\Omega), V^0 = \{u \in V : u(x) = 0 \text{ for } x \in \Gamma_1\}$ and $V^z = \{u \in V : u(x) = z \text{ for } x \in \Gamma_1\}$. The solution of the problem (P) for fixed $t > 0$ satisfies the following equality for all $\eta \in V^0, u(t) \in V^{z(t)}$:

$$\int_\Omega (\partial H/\partial t + v(t)\partial H/\partial x_2)\eta dx + \int_\Omega \nabla u \nabla \eta dx + \int_{\Gamma_2} (au + b|u|^3 u)\eta d\Gamma = \int_{\Gamma_2} g\eta d\Gamma$$

Let T_h be the triangulation of Ω in rectangular elements of dimensions $h_1 \times h_2$ and $V_h = \{u_h(x) \in H^1(\Omega) : u_h(x) \in Q_1 \text{ for all } \delta \in T_h\}$, where Q_1 is the space of bilinear functions. By Π_h we denote the local Q_1-interpolant. We also use the following notations: $V_h^0 = \{u_h(x) \in V_h : u_h(x) = 0, \text{ for all } x \in \Gamma_1\}$, $V_h^z = \{u_h(x) \in V_h : u_h(x) = z_h, \text{ for all } x \in \Gamma_1\}$ for the subsets of V_h. Here z_h is the V_h - interpolant of z on the boundary Γ_1. For any continuous function $v(x)$ we put

$$S_\delta(v) = \int_{\delta_h} \Pi_h(v)dx; S_\Omega(v) = \sum_{\delta \in T_h} S_\delta(v),$$

$$S_{\partial\delta}(v) = \int_{\partial\delta_h} \Pi_h(v)dx; S_{\Gamma_2}(v) = \sum_{\partial\delta_h \in T_h \cap \bar{\Gamma}_2} S_{\partial\delta}(v).$$

Let also $\omega_\tau = \{t_k = k\tau, 0 \le k \le M, M\tau = T\}$ be the uniform mesh in time on the segment $[0, T]$. To approximate the term $\left(\frac{\partial}{\partial t} + v(t)\frac{\partial}{\partial x_2}\right) H$ we use characteristics of this first order differential operator [Che91, JR82]. We use the notation

$$d_{\bar{t}}H = \frac{1}{\tau}(H(x,t) - \tilde{H}(x, t-\tau))$$

for the difference quotient approximating the term $\left(\frac{\partial}{\partial t} + v(t)\frac{\partial}{\partial x_2}\right) H$ in each mesh point on time level t by using characteristic method.

Then the approximation scheme can be written as follows: for all $t \in \omega_\tau, t > 0,$ find $u_h \in V_h^z$ such that

$$S_\Omega(d_{\bar{t}}H_h\eta_h) + S_\Omega(\nabla u_h \nabla \eta_h) + S_{\Gamma_2}((au_h + b|u_h|^3|u_h|)\eta_h) = S_{\Gamma_2}(g\eta_h) \text{ for all } \eta_h \in V_h^0. \tag{1}$$

Let $N_0 = \text{card } V_h^0$ and $u \in \mathbb{R}^{N_0}$ be the vector of nodal values for $u_h \in V_h^0$. Below we use the writing $u_h \Leftrightarrow u$ for this bijection. For the matrices $N_0 \times N_0$ we have the relations: for all $u_h \in V_h^0 \Leftrightarrow u \in \mathbb{R}^{N_0}, \eta_h \in V_h^0 \Leftrightarrow \eta \in \mathbb{R}^{N_0}$

$$(Au, \eta) = S_\Omega(\nabla u_h \nabla \eta_h) + S_{\Gamma_2}(au_h\eta_h); (Bu, \eta) = S_\Omega(1/\tau u_h \eta_h).$$

$$(Cu, \eta) = S_{\Gamma_2}(b|u_h|^3|u_h|\eta_h);$$

Similarly we define the vector f: $(f, \eta) = S_{\Gamma_2}(g\eta_h) + S_\Omega(1/\tau \tilde{H}_h\eta_h)$. Let now $\tilde{z}_h(x) \in V_h$ be the function which is equal to z_h in $\bar{\Gamma}_1$ and 0 for all nodes in ω, then f_0 is defined by the equality: $(f_0, \eta) = S_\Omega(\nabla\tilde{z}_h, \nabla\eta_h)$ for all $\eta_h \in V_h^0$. Finally we get $F = f + f_0$. In these notations the algebraic form for the mesh scheme (1) at fixed time level can be written as follows:

$$Au + BH(u) + Cu = F. \tag{2}$$

Here A, B are symmetric, positive definite M-matrices (moreover B is diagonal one) and $H(u)$ is vector with components $(H(u))_i = H(u_i)$. The operator C has the diagonal form: $Cu = (c_1(u_1), c_2(u_2), ..., c_N(u_N))^T$, where c_i are continuous non-decreasing functions.

Schwarz alternating methods

We study the convergence of multiplicative Schwarz alternating method (MSAM) and additive Schwarz alternating method (ASAM) for (2).

For the simplicity but without loss of generality we suppose that the domain Ω is decomposed into two overlapping subdomains Ω_1 and Ω_2, consisiting of the elements of triangulation T_h; any internal node of the grid in Ω is the internal node of at least one of the subdomains. We arrange the internal nodes of the mesh as follows. First, we enumerate the internal nodes lying in Ω_1, then the nodes in $\overline{\Omega_1 \cap \Omega_2}$ and at last the nodes in Ω_2. The vector $u \in \mathbb{R}^N$ takes the form $u = (u_{11}, u_{12}, u_{22})^T$ with subvector u_{ii} corresponding to the values of the mesh function $V_h \ni u_h \Leftrightarrow u$ in the nodes $x \in$ int Ω_i and subvector u_{12} corresponding to the values in $x \in \overline{\Omega_1 \cap \Omega_2}$.

This decomposition implies also the partitioning of the matricies and nonlinear operator C:

$$A = (A_{ij})_{ij=1}^3, B = (B_{ij})_{ij=1}^3, C = \text{diag}(C_1, C_2, C_3).$$

We need some more notations, namely:

$$A_0^1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B_0^1 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, A_1^1 = diag(0, A_{23}), B_1^1 = diag(0, B_{23});$$

$$A_0^2 = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}, B_0^2 = \begin{pmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{pmatrix}, A_1^2 = diag(A_{21}, 0), B_1^2 = diag(B_{21}, 0);$$

$$C^1 = diag(C_1, C_2), C^2 = diag(C_2, C_3),\ u_1 = (u_{11}, u_{12})^T, u_2 = (u_{12}, u_{22})^T$$

and similar for other vectors. (We note, that $A_{13}, A_{31}, B_{13}, B_{31}$ are zero matricies.)

Then MSAM can be written as follows:

$$\begin{cases} A_0^1 v_1^{k+1} + B_0^1 H(v_1^{k+1}) + C^1 v_1^{k+1} = f_1 - A_1^1 u_2^k - B_1^1 H(u_2^k) \\ v_{22}^{k+1} = u_{22}^k \\ \\ u_{11}^{k+1} = v_{11}^{k+1} \\ A_0^2 u_2^{k+1} + B_0^2 H(u_2^{k+1}) + C^2 u_2^{k+1} = f_2 - A_1^2 v_1^{k+1} - B_1^2 H(v_1^{k+1}) \end{cases} \tag{3}$$

and ASAM has the form:

$$\begin{cases} A_0^1 v_1^{k+1} + B_0^1 H(v_1^{k+1}) + C^1 v_1^{k+1} = f_1 - A_1^1 u_2^k - B_1^1 H(u_2^k) \\ A_0^2 w_2^{k+1} + B_0^2 H(w_2^{k+1}) + C^2 w_2^{k+1} = f_2 - A_1^2 u_1^k - B_1^2 H(u_1^k) \\ \\ u_{11}^{k+1} = v_{11}^{k+1}, u_{22}^{k+1} = w_{22}^{k+1}, u_{12}^{k+1} = \alpha v_{12}^{k+1} + (1-\alpha) w_{12}^{k+1} \end{cases} \tag{4}$$

Here $k = 0, 1, 2, ...$, initial guess $u^0 = (u_{11}^0, u_{12}^0, u_{22}^0)^T$ and $\alpha \in (0, 1)$.

Along with these methods we consider also the block variant of Jacoby method (BJM). Let $A^0 = \text{diag}(A_{11}, A_{22}, A_{33})$ be the block diagonal submatrix of $A, A^1 = A - A^0$ and $B = B^0 - B^1$ with similar splitting. Then A^0, B^0 are M - matricies and $A^1 \gg 0, B^1 \gg 0$. Moreover the iterative method (BJM) can be written in the form:

$$A^0 u^{k+1} + B^0 H(u^{k+1}) + C u^{k+1} = f - A^1 u^k - B^1 H(u^k). \tag{5}$$

Theorem 1 *Let A, B are $M-$matrices, where A is weakly diagonally dominant in columns, B is strictly diagonally dominant and C has the diagonal form $Cu = (c_1(u_1), c_2(u_2), ...,$ $c_N(u_N))^T$, where c_i are continuous non-decreasing functions. Let also there exist sub- and supersolutions for the problem (2). Then the iterative methods (3), (4) and (5) are correctly defined for any initial guess u^0 from ordered interval $< \underline{u}, \overline{u} >$. If the initial guess is supersolution then the sequences of iterations for all methods (3), (4) and (5) converge monotonically decreasing to the unique solution of the problem (2). Moreover, let the iterations of MSAM, ASAM and BJM be denoted by $u^k_{MSAM}, u^k_{ASAM}, u^k_{BJM}$. Then for any k the following inequalities hold:*

$$u^k_{MSAM} \ll u^k_{ASAM} \ll u^k_{BJM}.$$

If starting from subsolution, then the inequalities are vice versa and the iterative sequences converge monotonically increasing [LLP99].

Splitting iterative method

Let now Ω be divided into p nonoverlapping subdomains Ω_i with the interfaces $S_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$. We suppose that all interfaces as well as $\overline{\partial_1 \Omega}$ consist of the sides of $\delta \in T_h$.

The restrictions of functions from V_h^0 on subdomains Ω_i form the spaces $V_h^i, i = 1, 2, ..., p$. We also denote by $V_h = V_h^1 \times V_h^2 \times \cdots \times V_h^p$. It is easy to check that V_h^0 is isomorphic to the subspace K_h of V_h: $K_h = \{u_h = (u_h^1, u_h^2, ..., u_h^p) \in V_h : u_h^i(x) = u_h^j(x)$ for $x \in S_{ij}, i, j = 1, 2, ..., p\}$.

Let us put in the correspondence to the function $u_h^i \in V_h^i$ and the vector $u^i \in \mathbb{R}^{N_i}$ of its nodal values for nodes from $\bar{\Omega}_i \setminus \overline{\partial_1 \Omega}$ and denote this bijection by $u^i \Leftrightarrow u_h^i$. To $u_h \in V_h$ corresponds the vector $u \in \mathbb{R}^N, N = N_1 + N_2 + \cdots + N_p$. The subspace K_h corresponds to subspace of $\mathbb{R}^N$ which we denote by K. We have the following relations for $N_i \times N_i$ matrices: for all $V_h^i \ni u_h \Leftrightarrow u \in \mathbb{R}^{N_i}, V_h^i \ni \eta_h \Leftrightarrow \eta \in \mathbb{R}^{N_i}$

$$(A_i u_i, \eta_i)_i = S_{\Omega_i}(\nabla u_h \nabla \eta_h) + S_{\Gamma_2 \cap \partial\Omega_i}(a u_h \eta_h); (B_i u_i, \eta_i) = S_{\Omega_i}(1/\tau u_h \eta_h) \text{ and}$$

$$(c_i u_i, \eta_i)_i = S_{\Gamma_2 \cap \partial\Omega_i}(b|u_h|^3|u_h|\eta_h).$$

Similarly we define the vectors f_i, f_{0i}: $(f_i, \eta_i)_i = S_{\Gamma_2 \cap \partial\Omega_i}(g\eta_h) + S_{\Omega_i}(1/\tau \tilde{H}_h \eta_h)$ $(f_{0i}, \eta_i)_i = S_{\Omega_i}(\nabla \tilde{z}_h, \nabla \eta_h)$ for all $\eta_h \in V_h^i$. Finally we get $F_i = f_i + f_{0i}$.

Let further $A = diag(A_1, A_2, ..., A_p), B = diag(B_{01}, B_{02}, ..., B_{0p})$ and $F = (F_1, F_2, ..., F_p) \in \mathbb{R}^N$. Below we denote by $C(u) = BH(u) + cu + \partial I_K(u)$, where I_K is the indicator function of the subspace K. The operator A is bounded, hemicontinuous and uniformly monotone, C is maximal monotone operator. In these notations the algebraic form for the mesh scheme using DDM can be written (at fixed time level) as follows:

$$Au + Cu \ni F. \tag{6}$$

Due to the properties of A and C there exists unique solution u to the problem (6) [Bre73, Roc70].

We solve the inclusion (6) by splitting iterative method:

$$\begin{gathered} D_0^{-1}(u^{k+1/2} - u^k) + Au^k + Cu^{k+1/2} \ni F \\ D_1(u^{k+1} - u^k) = u^{k+1/2} - u^k. \end{gathered} \tag{7}$$

where D_0 and D_1 are some positive definite matrices. Due to the properties of D_0 and D_1 there exist the unique solutions $u^{k+1/2}$ and u^{k+1} for any k. For other examples of splitting methods see [Gab83, LS88, LM79].

For theoretical study of the convergence and rate of convergence for this splitting iterative method we can proof:

Theorem 2 *Let $V = V_1 \times V_2 \times ... \times V_p$, where V_i are Hilbert spaces with inner products $(.,.)_i$ and norms $||.||_i = (.,.)_i^{1/2}$ and let A be diagonal linear operator: $A = diag(A_1, A_2, ..., A_p)$ with $A_i : V_i \to V_i$ satisfying for all i the following assumptions: $m_i I_i \le A_i = A_i^* \le M_i I_i$ for all $i, m_i > 0$. Let also C be a maximal monotone operator and $z^k = u^k - u$, where u^k is the kth iteration and u is the exact solution.*

If $D_0 = diag(\lambda_1 I_1, \lambda_2 I_2, ..., \lambda_p I_p)$ and either $D_1 = I + D_0 A$ or $D_1 = 1/2(I + D_0 A)$ then the iterative method (7) converges for any $\lambda_i > 0$ and for the optimal choice of the iterative parameter $\lambda_i = 1/\sqrt{(m_i M_i)}$ the following estimate for rate of convergence is valid:

$$||D_0^{-1/2}(I + D_0 A^{(n)})z^n|| \le q^n ||D_0^{-1/2}(I + D_0 A^{(0)})z^0||, \tag{8}$$

with $q = q_1 = \max\limits_{1 \le i \le p} \dfrac{\sqrt{M}_i}{\sqrt{M}_i + \sqrt{m}_i}$ for the first choice of D_1 (corresponds to Douglas-Rachford scheme) and with $q = q_2 = \max\limits_{1 \le i \le p} \dfrac{\sqrt{M_i} - \sqrt{m_i}}{\sqrt{M_i} + \sqrt{m_i}}$ for for the second choice of D_1 (corresponds to Peaceman-Rachford scheme).

The iterative method (7) with, for example, $D_1 = I + D_0 A$ for DDM mesh scheme (6) leads to algorithm

$$D_0^{-1}(u^{k+1/2} - u^k) + Au^k + Cu^{k+1/2} \ni f \tag{9}$$

$$(I_i + \lambda_i A_i)(u^{i,k+1} - u^{i,k}) = u^{i,k+1/2} - u^{i,k}, i = 1, 2, ..., p, \tag{10}$$

$u^k = (u^{1,k}, u^{2,k}, ..., u^{p,k})$.

Linear equations (10) may be solved independently for $i = 1, 2, ..., p$. As for (9) then for coordinates of $u^{k+1/2}$ corresponding to internal nodes $x \in \Omega_i$ operator C has diagonal form: $C = \partial\theta$. It means that the system of non-coupled scalar nonlinear equations corresponds to these points. For nodes lying on the interfaces S_{ij} system (9) contains subsystems of two (if it is the interior node of the interface) or several (if it is a cross-point of several interfaces) coupled equations. These subsystems can be also reformulated as problems to minimise convex differentiable functions of two or several variables. To solve these subproblems we can use one of standard optimization method.

The assumptions of Theorem 2 are satisfied with $m_i = O(1), M_i = O(\tau h^{-2})$. If we choose $\lambda_i = O(h/\tau^{1/2})$ in method (7) with either $D_1 = I + D_0 A$ or $D_1 = 1/2(I + D_0 A)$, $D_0 = diag(\lambda_1 I_1, \lambda_2 I_2, ..., \lambda_p I_p)$, then $q_1 = 1 - O(h/\tau^{1/2}), q_2 = 1 - O(h/\tau^{1/2})$ and the number of iterations to achieve accuracy ϵ is $n(\epsilon) = O(\tau^{1/2} h^{-1} \ln 1/\epsilon)$.

Numerical results

To validate the numerical schemes described in sections 42 and 42 the following numerical example was considered.

Let $\Omega =]0,1[\times]0,1[$ with the boundary Γ divided in two parts such that $\Gamma_D = \{x \in \partial\Omega : x_2 = 0 \vee x_2 = 1\}$ and $\Gamma_N = \Gamma \setminus \Gamma_D$, moreover let $T = 1$. Let us consider the case where the phase change temperature $u_{SL} = 1$ and the latent heat $L = 1$. Let the phase change interval be $[u_{SL} - \varepsilon, u_{SL} + \varepsilon]$, $\varepsilon = 0.01$, and the velocity is $v(t) = \frac{1}{5}$. Our numerical example is

$$\begin{array}{rcll} \frac{\partial H}{\partial t} - \Delta K + v(t)\frac{\partial H}{\partial x_2} & = & f(x;t) & \text{on } \Omega, \\ u(x_1, x_2; t) & = & (x_1 - \frac{1}{2})^2 - \frac{1}{2}e^{-4t} + \frac{5}{4} & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} & = & 1 & \text{on } \Gamma_N, \\ u(x_1, x_2; 0) & = & (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 + \frac{1}{2} & \text{on } \Omega, \end{array}$$

where

$$K(u) = \begin{cases} u & \text{if } u < u_{SL} - \varepsilon, \\ \frac{3}{2}u - \frac{1-\varepsilon}{2} & \text{if } u \in [u_{SL} - \varepsilon, u_{SL} + \varepsilon], \\ 2u - 1 & \text{if } u > u_{SL} + \varepsilon, \end{cases}$$

and

$$H(u) = \begin{cases} 2u & \text{if } u < u_{SL} - \varepsilon, \\ \left(\frac{1+8\varepsilon}{2\varepsilon}\right)(u-1) + \frac{5+4\varepsilon}{2} & \text{if } u \in [u_{SL} - \varepsilon, u_{SL} + \varepsilon], \\ 6u - 3 & \text{if } u > u_{SL} + \varepsilon. \end{cases}$$

Furthermore

$$f(x;t) = \begin{cases} 4e^{-4t} + \frac{1}{5}(4x_2 - 2) - 4 & \text{if } u < u_M, \\ 12e^{-4t} + \frac{1}{5}(12x_2 - 6) - 8 & \text{if } u > u_M. \end{cases}$$

The exact solution of our problem is $u(x_1, x_2; t) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 - \frac{1}{2}e^{-4t} + 1$.

We split the enthalpy function $H(u)$ as follows: $H(u) = \alpha u + H_0(u)$, where α is the minimal slope of the enthalpy function. In our numerical example $\alpha = 2$.

For splitting iterative method the optimal iterative parameter $\lambda_i = \frac{1}{\sqrt{m_i M_i}}$, where $m_i = \alpha + \tau\mu^i_{min}(A_{00})$ and $M_i = \alpha + \tau\mu^i_{max}(A_{00})$, where $\mu^i_{min}(A_{00})$ is the smallest eigenvalue of the matrix $(A_{00})_i$, which is the approximation of the Laplacian operator and correspondingly $\mu^i_{max}(A_{00})$ is the biggest eigenvalue.

The numerical test was done such away that everything for different methods would be optimal. Numerical test were run in the computer Cedar in CSC, Espoo Finland, (128 RISC processors); mainly 4 processors were used. The stopping criterion was the norm of residual $||r|| \leq 10^{-4}$.

From the tables below **splitter** is splitting iterative method, **multi2** is multiplicative Schwarz with overlapping size $2h$ and **multi4** is multiplicative Schwarz with overlapping size $4h$. Moreover **proc** means the number of processors, **iter** the number of iterations and **S** is speedup.

Conclusions

Two different method was used to solve the problem (P). From Table 1 it can be seen that Splitting iterative method (SIM) is better (faster) than the Multiplicative Schwarz Alternating Method (MSAM) for the continuous casting problem. The speedups from the Table 1 show that (SIM) can be parallelized better than (MSAM).

	Splitter		**multi2**		**multi4**	
proc	Time [s]	**S**	Time [s]	**S**	Time [s]	**S**
1	466.4	–	259.4	–	259.4	–
2	166.8	2.8	212.6	1.22	177.5	1.46
4	124.6	3.74	174.9	1.48	157.4	1.65
6	106.7	4.37	140.3	1.85	131.9	1.97
8	85.4	5.46	119.3	2.17	109.6	2.37
10	70.9	6.58	95.4	2.72	92.7	2.80
12	59.3	7.87	85.6	3.03	85.2	3.04

Table 1: The comparison of calculation times and speedups when grid size is fixed to be 129×129 and 256 time steps. Number of processors are changed.

		Splitter		**multi2**		**multi4**	
grid	time steps	Time [s]	**iter**	Time [s]	**iter**	Time [s]	**iter**
17×17	32	0.45	24	0.68	6	0.49	4
33×33	65	1.75	25	1.44	7	1.31	4
65×65	128	12.3	26	14.2	8	12.6	5
129×129	256	124.6	29	174.9	9	157.4	6
161×161	320	188.2	29	391.8	9	350.1	6
257×257	512	1949.4	26	4425.2	9	3875.8	7

Table 2: The comparison of calculation times and number of iterations for different grid size and fixed number of processors; 4 processors.

From Table 2 it can be seen that when grid size increases the difference between calculation times for (MSAM) and (SIM) increases. Splitting iterative method is much more suitable for big continuous casting problems when we can use many processors and number of unknows are big, like in many real industrial application. For (SIM) we also know how to determine the optimal iterative parameter. The numerical experiments have shown that the theoretical optimal value for the iterative parameter is close to the practical optimal one.

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