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44 A Preconditioner for Linear Elasticity Problems

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Introduction

We consider the linear elasticity problem for homogeneous and isotropic material with mixed boundary conditions. The traditional formulation of the problem reads [NH80]

$$\begin{cases} -2\mu\nabla\cdot\varepsilon(\vec{u}) - \lambda\nabla(\nabla\cdot\vec{u}) = \vec{f} & \text{in } \Omega\subset\mathbb{R}^d, \\ \vec{u} = 0 & \text{on } \Gamma_0\subset\partial\Omega, \\ [2\mu\varepsilon(\vec{u}) + \lambda(\nabla\cdot\vec{u})I]\cdot\vec{n} = 0 & \text{on } \Gamma_1 = \partial\Omega\setminus\Gamma_0, \end{cases} \tag{1}$$

where $\vec{n}$ is the outward unit normal vector, $\varepsilon(\vec{u})$ is the strain tensor and the Lamé coefficients μ and λ are defined by the Young modulus E and the Poisson ratio ν as follows

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

Hereafter, it is assumed that the Poisson ratio ν satisfies $0\le\nu\le 1/2$, although in theoretical considerations it is assumed that $\nu\le\hat{\nu}<1/2$, where $\hat{\nu}$ is a constant. The measures of the boundaries Γ_0 and Γ_1 are assumed to be positive. The drawback of formulation (1) is that for (nearly) incompressible materials the parameter λ approaches infinity and the problem becomes ill conditioned. One remedy for this problem is to alter the formulation [BF91], [NH80]. We define a scalar function $p = -\lambda(\nabla\cdot\vec{u})$. This definition is added to the problem as a second equation. Then, we divide the equation by λ and get the following set of equations (see, for example, [Kob94] and references therein)

$$\begin{cases} -2\mu\nabla\cdot\varepsilon(\vec{u}) + \nabla p = \vec{f} & \text{in } \Omega, \\ -\nabla\cdot\vec{u} - \lambda^{-1}p = 0 & \text{in } \Omega, \\ \vec{u} = 0 & \text{on } \Gamma_0, \\ [2\mu\varepsilon(\vec{u}) - pI]\cdot\vec{n} = 0 & \text{on } \Gamma_1. \end{cases} \tag{2}$$

For the Poisson ratio $\nu = 1/2$ the latter equation of (2) is $-\nabla\cdot\vec{u} = 0$, which is exactly the incompressibility constraint. There are other possibilities to treat the elasticity problem for almost incompressible material such as hp-methods [SS96], nonconforming methods [Fal91] and reduced integration rules [ZT89].
Our purpose is to develop an efficient method for the numerical solution of discretized counterpart of the partial differential system (2). Our tools for this are a block diagonal preconditioner, a fictitious domain method and distributed Lagrange multipliers. The idea of the fictitious domain method is to extend the problem with complicated geometry to a larger, simple

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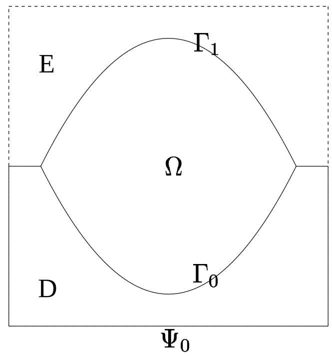

Figure 1: Rectangular domain Π with subdomains.

domain where an efficient solver can be used. This procedure can be justified with extension theorem for finite element functions [Wid87] on which the spectral optimality of the fictitious domain preconditioning is based [Ast78]. The incorporation of distributed Lagrange multipliers in fictitious domain method has been proposed in [GK98],[GPH$^+$99], for example. The advantage of the distributed Lagrange multipliers compared to the boundary Lagrange multipliers is the ease of preconditioning for both two-dimensional and three-dimensional problems.

Weak formulation of the elasticity problem

For the finite element discretization of the problem (2) we present a corresponding weak formulation. We define spaces

$$V = \{\vec{v} \in \left[H^1(\Omega)\right]^d \;:\; \vec{v}\,|_{\Gamma_0} = \vec{0}\} \quad \text{and} \quad Q = L^2(\Omega).$$

Then, the problem is to find $\vec{u} \in V$ and $p \in Q$, such that

$$\begin{cases} \displaystyle\int_\Omega 2\mu\varepsilon(\vec{u})\varepsilon(\vec{v}) - (\nabla\cdot\vec{v})p\,dx = \int_\Omega \vec{f}\cdot\vec{v}\,dx & \forall \vec{v} \in V \\ \displaystyle\int_\Omega -(\nabla\cdot\vec{u})q - \lambda^{-1}pq\,dx = 0 & \forall q \in Q. \end{cases} \tag{3}$$

In practise, we use a formulation which is equivalent to (3), but allows the application of the fictitious domain method. Therefore, we assume that there is a simple domain $\Pi \subset \mathbb{R}^d$, such that $\Omega \subset \Pi$ and a domain $D \subset \Pi$, such that $\Gamma_0 \subset \partial D$, $\Gamma_1 \cap \partial D = \emptyset$ and $D \cap \Omega = \emptyset$ as in Figure 1. We assume also that $\partial\Pi = \Psi_0 \cup \Psi_1$ and the measures of the boundary Ψ_0 and the domains Ω, D and $\Pi \setminus \overline{(\Omega \cup D)}$ are all positive. Then, we define spaces U and Ξ as follows

$$U = \{\vec{v} \in \left[H^1(\Omega \cup D)\right]^d \;:\; \vec{v}\,|_{\Psi_0} = \vec{0}\} \quad \text{and} \quad \Xi = \left[H^1(D)\right]^d.$$

A problem equivalent to (3) is to find $\vec{u} \in U$, $p \in Q$ and $\xi \in \Xi$, such that

$$\begin{cases} 2\mu \int_{\Omega\cup D} \varepsilon(\vec{u})\varepsilon(\vec{v})\,dx - \int_\Omega (\nabla\cdot\vec{v})p\,dx + \langle\vec{\xi},\vec{v}\rangle_{H^1(D)} = \int_\Omega \vec{f}\cdot\vec{v}\,dx & \forall\vec{v}\in U \\ \int_\Omega -(\nabla\cdot\vec{u})q - \lambda^{-1}pq\,dx = 0 & \forall q\in Q \\ \langle\vec{u},\vec{\eta}\rangle_{H^1(D)} = 0 & \forall\eta\in\Xi, \end{cases} \tag{4}$$

where $\langle\cdot,\cdot\rangle_{H^1(D)}$ is the $[H^1(D)]^d$ inner product. We use finite element method with quadratic triangular elements for the displacement components and distributed Lagrange multipliers and linear triangular elements for the pressure, also known as the Taylor-Hood element combination [BP79]. In this way, the elements have a one-to-one correspondence and the implementation of the discretization process is straightforward. If the element spaces are selected this way the distributed Lagrange multipliers tie the displacement components node by node and the system is equivalent to the system arising from (3) where the Dirichlet boundary conditions are treated by elimination. For this reason, the the triangulation must be compatible with the boundary Γ_0. The finite element mesh is assumed to be regular.
The form of the discretized linear system is

$$\begin{pmatrix} \mu\mathbf{A} & \mathbf{B}^T & \hat{\mathbf{C}}^T \\ \mathbf{B} & -\lambda^{-1}\mathbf{M} & \mathbf{0} \\ \hat{\mathbf{C}} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \\ \hat{\boldsymbol{\xi}} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{5}$$

where $\hat{\mathbf{C}}$ has the form $\hat{\mathbf{C}} = (\hat{\mathbf{E}}\ \mathbf{0})$. By applying the change of variables $\boldsymbol{\xi} = \hat{\mathbf{E}}\hat{\boldsymbol{\xi}}$ and multiplying the last block row of (5) by $\hat{\mathbf{E}}^{-1}$, we get our final linear system

$$\begin{pmatrix} \mu\mathbf{A} & \mathbf{B}^T & \mathbf{C}^T \\ \mathbf{B} & -\lambda^{-1}\mathbf{M} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \\ \boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{6}$$

where $\mathbf{C} = (\mathbf{I}\ \mathbf{0}) = \hat{\mathbf{E}}^{-1}\hat{\mathbf{C}}$. Here, the matrix $\mathbf{C}$ is defined by $\mathbf{C}\mathbf{v} = \mathbf{v}_D$ for all $\mathbf{v} = (\mathbf{v}_\Omega, \mathbf{v}_D)$. The system matrix of (6), which we denote by $\mathcal{A}$, is symmetric but indefinite.

The construction of the preconditioner

We would like to solve the linear system (6) using the preconditioned MINRES-method. We will show that a good preconditioner $\mathcal{P}$ for the discrete problem, given by its inverse is

$$\mathcal{P}^{-1} = \begin{pmatrix} \mu^{-1}\mathbf{R}\mathbf{A}_\Pi^{-1}\mathbf{R}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \alpha^{-1}\mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mu\mathbf{A}_D \end{pmatrix}. \tag{7}$$

The constant α is defined by $\alpha = \lambda^{-1} + \mu^{-1}$ and the matrix $\mathbf{R}$ by $\mathbf{R}\mathbf{v} = \mathbf{v}_{\Omega\cup D}$, where $\mathbf{v} = (\mathbf{v}_{\Omega\cup D}, \mathbf{v}_E)$. The matrices $\mathbf{A}_\Pi$ and $\mathbf{A}_D$ correspond to the elliptic part $2\int\varepsilon(\vec{u})\varepsilon(\vec{v})\,dx$ discretized in the domains Π and D with the element space for the displacement components, respectively and $\mathbf{M}$ is the mass matrix discretized in the domain Ω with the element space for the pressure.

We define the spectral equivalency of the matrices $\mathcal{A}$ and $\mathcal{P}$ as in [Kuz00]: Let λ_i, $i = 1, \dots, m$ be the eigenvalues of the matrix $\mathcal{P}^{-1}\mathcal{A}$. If there exists positive constants c_1 and c_2 such that $c_1 \leq |\lambda_i| \leq c_2$ for all $i = 1, \dots, m$ then the matrices $\mathcal{A}$ and $\mathcal{P}$ are spectrally equivalent with the constants c_1 and c_2. The condition number of the matrix $\mathcal{P}^{-1}\mathcal{A}$ is then bounded by $\kappa(\mathcal{P}^{-1}\mathcal{A}) \leq c_2/c_1$.
Since the convergence of the MINRES depends on the condition number of the preconditioned system we can guarantee the convergence rate by showing that the proposed preconditioner is spectrally equivalent to the system matrix.

Theorem 1 *Let us assume, that the Poisson ratio ν satisfies $0 < \nu \leq \hat{\nu} < 1/2$, where $\hat{\nu}$ is a constant. Then, the preconditioner $\mathcal{P}$ is spectrally equivalent to the system matrix $\mathcal{A}$ with constants c_1 and c_2 which are independent of the mesh step size h and the Poisson ratio ν.*

We begin to proof this result by showing that the matrix $\mathcal{A}$ is spectrally equivalent to a block diagonal matrix. Results for generalized eigenvalue problems, resembling Lemma 1 have been considered in articles [Kla95], [Kuz00] and [SW94]. We denote

$$\mathbf{G} = \begin{pmatrix} \mathbf{B} \\ \mathbf{C} \end{pmatrix} \quad \text{and} \quad \mathbf{H} = \begin{pmatrix} \lambda^{-1}\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Notice, that the requirement $|\Gamma_0|, |\Gamma_1|, |\Psi_0| \geq \rho > 0$ is essential for the blocks $\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T$ and $\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T$ to be positive definite.

Lemma 1 *Let us assume that matrices $\mathbf{A}$ and $\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T$ are positive definite and matrix $\mathbf{H}$ is positive semidefinite. Then the eigenvalues θ of the generalized eigenvalue problem*

$$\begin{pmatrix} \mathbf{A} & \mathbf{G}^T \\ \mathbf{G} & -\mathbf{H} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \theta \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} + \mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \tag{8}$$

belong to the intervals $\left[-1, \frac{1-\sqrt{5}}{2}\right]$ and $\left[1, \frac{1+\sqrt{5}}{2}\right]$.

Proof The solution of the eigenvalue problem (8) satisfies the equations:

$$\mathbf{A}\mathbf{u} + \mathbf{G}^T\mathbf{v} = \theta\mathbf{A}\mathbf{u} \tag{9}$$

and

$$\mathbf{G}\mathbf{u} - \mathbf{H}\mathbf{v} = \theta\left(\mathbf{H} + \mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T\right)\mathbf{v}. \tag{10}$$

We assume that $\theta \neq 1$ and $\mathbf{v} \neq \mathbf{0}$. The vector $\mathbf{u}$ can be solved from (9) with respect to $\mathbf{v}$

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{G}^T\mathbf{v}/(\theta - 1).$$

Inserting this in the equation (10) gives us

$$\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T\mathbf{v}/(\theta - 1) - \mathbf{H}\mathbf{v} = \theta\mathbf{H}\mathbf{v} + \theta\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T\mathbf{v}. \tag{11}$$

We multiply the equation (11) from left by $\mathbf{v}^T$, collect the terms $\mathbf{v}^T\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T\mathbf{v}$ and divide the equation by it. Then, we have

$$\frac{1}{\theta - 1} - \theta - (\theta + 1)\frac{\mathbf{v}^T\mathbf{H}\mathbf{v}}{\mathbf{v}^T\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T\mathbf{v}} = 0.$$

We denote $\alpha(\mathbf{v}) = \frac{\mathbf{v}^T\mathbf{H}\mathbf{v}}{\mathbf{v}^T\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T\mathbf{v}}$. From the assumptions for the matrices it follows that $0 \le \alpha(\mathbf{v}) < \infty$. Now, the eigenvalue θ can be solved from the equation $-(1+\alpha(\mathbf{v}))\theta^2 + \theta + (1+\alpha(\mathbf{v})) = 0$, and this gives the intervals $\left[-1, \frac{1-\sqrt{5}}{2}\right]$ and $\left]1, \frac{1+\sqrt{5}}{2}\right]$. By including the value $\theta = 1$, which was excluded during the calculations, the final intervals are obtained. ■

We continue the proof of the Theorem 1 by showing that the Schur complement matrix $\mathbf{S}_1 = \mathbf{H} + \mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T$ is again spectrally equivalent to a block diagonal matrix. First, we need to assume the following:

From here on, we assume that there exists positive constants c_1 and c_2 such that the inequality

$$c_1\mathbf{p}^T\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T\mathbf{p} \le \mathbf{p}^T\mathbf{M}\mathbf{p} \le c_2\mathbf{p}^T\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T\mathbf{p}$$

holds for any $\mathbf{p}$, where the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{M}$ are discretized with the formulation (4).

Note that the assumption above holds if the element combination satisfies the LBB-condition with the given mixed boundary conditions [SEKW01].

Lemma 2 *For the Schur complements*

$$\mathbf{S}_1 = \lambda^{-1}\begin{pmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} + \mu^{-1}\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T \quad \textit{and}$$
$$\mathbf{S}_2 = \lambda^{-1}\begin{pmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} + \mu^{-1}\begin{pmatrix}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T & \mathbf{0}\\ \mathbf{0} & \mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T\end{pmatrix}$$

there exists positive constants β and γ such that the inequality

$$\gamma\mathbf{s}^T\mathbf{S}_2\mathbf{s} \le \mathbf{s}^T\mathbf{S}_1\mathbf{s} \le \beta\mathbf{s}^T\mathbf{S}_2\mathbf{s}$$

holds for any $\mathbf{s} = (\mathbf{p}, \boldsymbol{\xi})$, when $0 < \nu \le \hat{\nu} < 1/2$.

Proof We study the quadratic forms related to the matrices $\mathbf{S}_1$ and $\mathbf{S}_2$. We denote $\tilde{\mathbf{B}} = \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T$, $\tilde{\mathbf{C}} = \mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T$ and $\tilde{\mathbf{D}} = \mathbf{B}\mathbf{A}^{-1}\mathbf{C}^T$. It must be shown that the inequalities

$$\begin{aligned}&\beta\left(\mathbf{p}^T(\lambda^{-1}\mathbf{M} + \mu^{-1}\tilde{\mathbf{B}})\mathbf{p} + \mu^{-1}\boldsymbol{\xi}\tilde{\mathbf{C}}\boldsymbol{\xi}\right) \ge \\ &\quad\mathbf{p}^T(\lambda^{-1}\mathbf{M} + \mu^{-1}\tilde{\mathbf{B}})\mathbf{p} + \mu^{-1}\boldsymbol{\xi}\tilde{\mathbf{C}}\boldsymbol{\xi} + 2\mu^{-1}\mathbf{p}^T\tilde{\mathbf{D}}\boldsymbol{\xi},\end{aligned} \tag{12}$$

$$\begin{aligned}&\gamma\left(\mathbf{p}^T(\lambda^{-1}\mathbf{M} + \mu^{-1}\tilde{\mathbf{B}})\mathbf{p} + \mu^{-1}\boldsymbol{\xi}\tilde{\mathbf{C}}\boldsymbol{\xi}\right) \le \\ &\quad\mathbf{p}^T(\lambda^{-1}\mathbf{M} + \mu^{-1}\tilde{\mathbf{B}})\mathbf{p} + \mu^{-1}\boldsymbol{\xi}\tilde{\mathbf{C}}\boldsymbol{\xi} + 2\mu^{-1}\mathbf{p}^T\tilde{\mathbf{D}}\boldsymbol{\xi}\end{aligned} \tag{13}$$

are satisfied. Clearly, (12) holds for any $\beta \ge 2$, since $\mathbf{A}$ is positive definite.

For every $\mathbf{p}$ and $\boldsymbol{\xi}$

$$\begin{aligned}0 &\le \mu^{-1}\begin{pmatrix}(1+c_1\mu\lambda^{-1})^{1/4}\mathbf{p}\\ (1+c_1\mu\lambda^{-1})^{-1/4}\boldsymbol{\xi}\end{pmatrix}^T \mathbf{G}\mathbf{A}^{-1}\mathbf{G}^T \begin{pmatrix}(1+c_1\mu\lambda^{-1})^{1/4}\mathbf{p}\\ (1+c_1\mu\lambda^{-1})^{-1/4}\boldsymbol{\xi}\end{pmatrix}\\ &= \mu^{-1}(1+c_1\mu\lambda^{-1})^{-1/2}\left((1+c_1\mu\lambda^{-1})\mathbf{p}^T\tilde{\mathbf{B}}\mathbf{p} + \boldsymbol{\xi}^T\tilde{\mathbf{C}}\boldsymbol{\xi}\right) + 2\mu^{-1}\mathbf{p}^T\tilde{\mathbf{D}}\boldsymbol{\xi}\\ &\le (1+c_1\mu\lambda^{-1})^{-1/2}\left(\lambda^{-1}\mathbf{p}^T\mathbf{M}\mathbf{p} + \mu^{-1}\mathbf{p}^T\tilde{\mathbf{B}}\mathbf{p} + \mu^{-1}\boldsymbol{\xi}^T\tilde{\mathbf{C}}\boldsymbol{\xi}\right) + 2\mu^{-1}\mathbf{p}^T\tilde{\mathbf{D}}\boldsymbol{\xi}\end{aligned}$$

holds. Therefore,

$$-(1+c_1\mu\lambda^{-1})^{-1/2}\left(\lambda^{-1}\mathbf{p}^T\mathbf{M}\mathbf{p}+\mu^{-1}\mathbf{p}^T\tilde{\mathbf{B}}\mathbf{p}+\mu^{-1}\boldsymbol{\xi}^T\tilde{\mathbf{C}}\boldsymbol{\xi}\right)\leq 2\mu^{-1}\mathbf{p}^T\tilde{\mathbf{D}}\boldsymbol{\xi}.$$

By adding the term $\lambda^{-1}\mathbf{p}^T\mathbf{M}\mathbf{p}+\mu^{-1}\mathbf{p}^T\tilde{\mathbf{B}}\mathbf{p}+\mu^{-1}\boldsymbol{\xi}^T\tilde{\mathbf{C}}\boldsymbol{\xi}$ to both sides of the inequality, we get the following lower bound

$$\gamma = 1-\frac{1}{\sqrt{1+\frac{\mu}{\lambda}c_1}} = 1-\frac{1}{\sqrt{1+\frac{1-2\nu}{2\nu}c_1}} \geq 1-\frac{1}{\sqrt{1+\frac{1-2\hat{\nu}}{2\hat{\nu}}c_1}} > 0.$$

■

Now, we have shown that the system matrix $\mathcal{A}$ is spectrally equivalent to the matrix

$$\begin{pmatrix}\mu\mathbf{A} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \lambda^{-1}\mathbf{M}+\mu^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mu^{-1}\mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T\end{pmatrix}.$$

Since the domain is extended only over the boundary Γ_1 with natural boundary condition and due to the Korn inequality the matrix block $\mathbf{A}$ is spectrally equivalent to the discretized vector laplacian, the fictitious domain preconditioner $\mathbf{R}\mathbf{A}_{\Pi}^{-1}\mathbf{R}^T$ is optimal for $\mathbf{A}$ [Ast78]. Using the same principles, $\mathbf{A}_D$ is an optimal preconditioner for $\mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T$ [GK98]. It follows from the assumption above that $\lambda^{-1}\mathbf{M}+\mu^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T$ and $(\lambda^{-1}+\mu^{-1})\mathbf{M}$ are spectrally equivalent. This concludes the proof of Theorem 1.

Numerical example

In this numerical example the operator $\mathbf{A}_{\Pi}^{-1}$ is approximated with the multigrid method using one symmetric multigrid V-cycle with one pre-smooth and one post-smooth with forward and backward Gauss-Seidel, respectively [Hac85]. This approximation is accurate and can be computed efficiently. The coarsest level problem is not solved exactly, but ten symmetric Gauss-Seidel sweeps are used instead. While this does not give the smallest possible number of outer iterations, it is very economical in the sense of the total computing time. The multigrid method is based fully on linear triangular elements. Systems with the mass matrix $\mathbf{M}$ are solved using the conjugate gradient method in machine precision with the lumped mass matrix as a preconditioner.

The example problem is a mixed boundary value problem in a domain Ω bounded by two parabole; see Figure 1. Homogeneous Dirichlet boundary condition is imposed on the lower half of the boundary and natural bondary condition is satisfied on the upper half of the boundary. The force term, which is pulling the structure left is distributed evenly over the computational domain. The displaced mesh can be seen in Figure 2. The numbers of iterations (it) and CPU times in seconds (time) with respect to the degrees of freedom (d.o.f.) and Poisson ratio (ν) are presented in Table 1. The numbers of iterations show that the convergence of the method is independent of both the mesh step size and the Poisson ratio and the method works well even for Poisson ratio 1/2. The numerical tests are performed on a PC with 400MHz Celeron processor running Linux operating system. The programs are compiled with the gcc compiler.

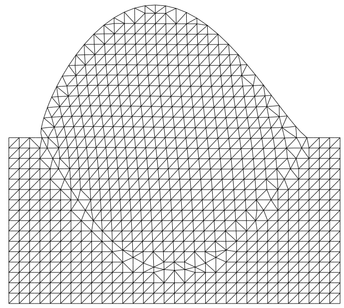

Figure 2: The finite element mesh with the displacements.

ν		0.1		0.2		0.3		0.4		0.5
d.o.f.	it	time	it	time	it	time	it	time	it	time
2253	108	2.7	105	2.6	102	2.6	99	2.5	99	2.5
8817	109	13.2	105	12.7	105	12.7	104	12.6	103	12.4
34889	108	57.0	105	55.6	103	54.6	102	54.2	103	54.8
138809	110	243.4	105	233.8	103	228.7	103	227.6	103	226.4

Table 1: The numbers of iterations and CPU times for the test problem.

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