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8 On Polynomial Reproduction of Dual FE Bases

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Introduction and abstract algebraic condition

We construct local piecewise polynomial dual bases for standard Lagrange finite element spaces which themselves provide maximal polynomial reproduction. By means of such dual bases for the Lagrange multiplier, extremely efficient realization of mortar methods on non-matching triangulations can be obtained without losing the optimality of the discretization errors. In contrast to the standard mortar approach, the locality of the constrained basis functions is preserved. The construction of dual bases and quasi-interpolants for univariate spline spaces is well-understood (see, e.g., [dB76, dB90, dBF73, Sch81]). However, the dual space is usually of a more complicated structure, and cannot be fixed beforehand, see also [DKU99, DS97, Ste00] for related research in the context of biorthogonal multiresolution analysis.

We start with an abstract framework. Let $P, V, W \subset X$ be subspaces of a real Hilbert space H. Furthermore, we assume that $q := \dim P \leq n := \dim V = \dim W \leq m := \dim X < \infty$. Let $\mathbf{P}$, Φ, Ψ and Θ be bases of P, V, W and X, respectively. All function systems are written as row vectors, the matrix notation used below will be consistent with this assumption. We also frequently use the notation $G_{\Psi_1,\Psi_2} := (\Psi_1^T, \Psi_2)_H$ for the Gram matrix associated with two finite systems $\Psi_1, \Psi_2 \subset H$. Note that $G_{\Psi_2,\Psi_1} = G^T_{\Psi_1,\Psi_2}$, and that $G^{-1}_{\Psi,\Psi}$ exists whenever the elements of Ψ are linearly independent. By our assumptions, there exist matrices A, $B \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times q}$, such that $\Phi = \Theta A$, $\Psi = \Theta B$ and $\mathbf{P} = \Theta D$. The sets of basis functions Φ and Ψ are called biorthogonal (or, equivalently, Ψ is dual to Φ) if

$$\mathrm{Id}_n = G_{\Psi,\Phi} \; . \tag{1}$$

The components of the function systems Φ and Ψ are denoted by ϕ_k and ψ_k, respectively. We introduce the dual operators $Q_W\nu = \sum_{k=1}^n (\phi_k, \nu)_H \, \psi_k$ and $Q_V\nu = \sum_{k=1}^n (\psi_k, \nu)_H \, \phi_k$, i.e., $(\nu, Q_W\mu)_H = (Q_V\nu, \mu)_H$. Assuming that Ψ and Φ are biorthogonal, we find that Q_W reproduces the subspace P, i.e.,

$$Q_W p = p \qquad \forall \, p \in P \; , \tag{2}$$

if and only if $P \subset W$.

In the rest of this section, we establish algebraic conditions on Ψ such that biorthogonality and subspace reproduction are satisfied for given choices of $\mathbf{P}$, Φ, and Θ.

Lemma 1 *Under the above assumptions, (1) and (2) hold if and only if*

$$\mathrm{Id}_n = A^T G_{\Theta,\Theta} B \; , \tag{3}$$

$$G_{\Theta,\mathbf{P}} = G_{\Theta,\Theta} B G_{\Phi,\mathbf{P}} \; . \tag{4}$$

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Proof Equation (3) is equivalent to (1) since

$$G_{\Phi,\Psi} = (\Phi^T, \Psi)_H = A^T(\Theta^T, \Theta)_H B = A^T G_{\Theta,\Theta} B \ .$$

In a second step, we establish (4). If $P \subset W$, then there exists a $C \in \mathbb{R}^{n\times q}$ such that $\mathbf{P} = \Psi C$. Assuming (1), we find $C = G_{\Phi,\mathbf{P}}$ and $G_{\Theta,\mathbf{P}} = G_{\Theta,\Psi}C = G_{\Theta,\Theta}BG_{\Phi,\mathbf{P}}$. On the other hand, since Θ is a basis in $X \subset H$, $G^{-1}_{\Theta,\Theta}$ exists. Thus, if (4) is satisfied, $P \subset W$ follows from $\mathbf{P} = \Theta G^{-1}_{\Theta,\Theta} G_{\Theta,\mathbf{P}} = \Theta B G_{\Phi,\mathbf{P}} = \Psi G_{\Phi,\mathbf{P}}$. ∎

Proposition 1 *For given subspaces $P, V \subset X \subset H$ and their bases $\mathbf{P}, \Phi, \Theta$ satisfying the above assumptions, there exists a subspace $W \subset X$ and its basis Ψ such that (1) and (2) are satisfied if and only if $G_{\mathbf{P},\Phi}$ has maximal rank $q = \dim P$.*

Proof We will use the result of Lemma 1. The necessity is obvious from (4). To proof the existence of W and Ψ we will find B from the system (3)-(4) using the SVD of A. Let $A = U(\Sigma, 0)^T Y^T$, where $U \in \mathbb{R}^{m\times m}$, $Y \in \mathbb{R}^{n\times n}$ are orthogonal, and $\Sigma \in \mathbb{R}^{n\times n}$ is diagonal and nonsingular. Obviously, B satisfies (3) if and only if it is of the form

$$B = G^{-1}_{\Theta,\Theta} U(\Sigma^{-1}, Z)^T Y^T \ , \qquad Z \in \mathbb{R}^{n\times(m-n)} . \tag{5}$$

It remains that the arbitrary matrix Z can be chosen such that (4) will be satisfied, too. Substituting the known factorizations for A and B, we obtain

$$G_{\mathbf{P},\Theta}U = G_{\mathbf{P},\Phi}B^T G_{\Theta,\Theta}U = G_{\mathbf{P},\Theta}AY^T(\Sigma^{-1}, Z) = G_{\mathbf{P},\Theta}U \begin{pmatrix} \mathrm{Id}_n & \Sigma Z \\ 0 & 0 \end{pmatrix} .$$

Thus, (4) holds if and only if the matrix equation $0 = G_{\mathbf{P},\Theta}U(Z^T\Sigma, -\mathrm{Id}_{m-n})^T$ is satisfied. Using the factorization of $G_{\mathbf{P},\Phi}$, the latter can be rewritten as

$$G_{\mathbf{P},\Phi} Y Z = G_{\mathbf{P},\Theta} U(0, \mathrm{Id}_{m-n})^T \ . \tag{6}$$

Since the $q \times n$ matrix $G_{\mathbf{P},\Phi}$ has maximal rank q, solutions Z exist. ∎

The above criterion does not depend on the particular choice of the bases Φ and $\mathbf{P}$ but rather on the choice of the spaces V and P themselves (it is equivalent to requiring that $(p, v)_H = 0$ for all $v \in V$ implies $p = 0$ for any $p \in P$). Equations (5)-(6) allow us to find all $\Psi \subset X$ dual to Φ, and such that the associated Q_W reproduces P. Whether this procedure is effective depends on the factorization of A, and the structure of $G_{\mathbf{P},\Phi}$ and $G^{-1}_{\Theta,\Theta}$. This is the place where the specific choices for $X \supset V, P$ and for the bases come in. In the subsequent sections, we specialize to the case $H = L_2(\Omega)$, to Lagrange finite element spaces V and corresponding spaces X of piecewise polynomials on a partition of $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, and to P coinciding with the space P_r of all polynomials of degree $\le r$. As we will see, the resulting matrices have then a simple, sparse structure, and can easily be computed, which in turn enables us to achieve additional properties of Ψ such as local support of all ψ_k.

1D construction

We consider the univariate finite element case. Let $\mathcal{T} = \{\Delta^n : n = 1, \ldots, N\}$ be the partition of a univariate interval $I = [a, b]$ into consecutive intervals Δ^n of length h_n. On the interval $[-1, 1]$, we define a special basis for P_r by

$$\Pi_r = [p_0, p_2, \ldots, p_r, p_1] \ ,$$

where $p_0(t) = (1-t)/2$, $p_1(t) = (1+t)/2$. The remaining polynomials $p_k(t)$ of degree $k = 2, \dots, r$ are supposed to vanish at $t = \pm 1$ and form an orthonormal system on $[-1,1]$ with respect to the $L^2(-1,1)$-scalar product. We note that $\dim P_r = r+1$ in 1D. Obviously, a basis in $X := \{v \in L^2(\Omega) \mid v_{|\Delta^n} \in P_s,\ n = 1, \dots, N\}$, $s \geq r$, is given by

$$\Theta = [\Theta_{\Delta^1}, \dots, \Theta_{\Delta^N}] \,,$$

where $\Theta_{\Delta^n} = [\theta_{0,n}, \dots, \theta_{s,n}]$ is the unscaled transformation of the system Π_s from $[-1,1]$ to Δ^n. For further reference, let $\Theta'_{\Delta^n} = [\theta_{1,n}, \dots, \theta_{s-1,n}]$. We note that Θ'_{Δ^n} is empty if $s = 1$. We restrict ourselves to the case that the conforming finite element space $V := X \cap H_0^1(I)$ satisfies homogeneous Dirichlet boundary condition. This is the interesting case for mortar finite elements. To obtain optimal results, the Lagrange multiplier space W which has by construction the same dimension as V has to be associated with the interior nodes on the interface. We found it convenient to use

$$\Phi = [\Theta'_{\Delta^1}, \theta_{s,1} + \theta_{0,2}, \Theta'_{\Delta^2}, \dots, \theta_{s,N-1} + \theta_{0,N}, \Theta'_{\Delta^N}]$$

as basis in V. In this form, it is sometimes called hierarchical finite element basis. Using the notation of the previous section, we can write $\Phi = \Theta\ U(\Sigma\ ,\ 0)^T$, where $\Sigma = \mathrm{diag}(\mathrm{Id}_{s-1}, \sqrt{2}, \mathrm{Id}_{s-1}, \sqrt{2}, \dots, \sqrt{2}, \mathrm{Id}_{s-1})$ and

$$U^T = \begin{pmatrix} 0 & \mathrm{Id}_{s-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_{s-1} & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & & 0 & 0 & 0 \\ & & \vdots & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathrm{Id}_{s-1} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} . \qquad (7)$$

We note that U is a $(s+1)N \times (s+1)N$ matrix, $U_1 \in \mathbb{R}^{sN-1 \times (s+1)N}$, $U_2 \in \mathbb{R}^{N+1 \times (s+1)N}$ whereas Σ is a $(sN-1) \times (sN-1)$ diagonal matrix. The dimension of X and V is $m = (s+1)N$ and $n = sN - 1$, respectively. If $s = 1$, Id_{s-1} formally stands for a matrix of zero size. Having in mind Proposition 1, it is sufficient to find a subsystem $\Phi_r \subset \Phi$ of size $r+1$ such that $\det G_{\mathbf{P}_r, \Phi_r} \neq 0$ to guarantee the existence of a dual system Ψ satisfying $P_r \subset W$.

The remaining part of this section is devoted to the construction of suitable subsystems Φ_s for the case $r = s$ of maximal possible degree of polynomial reproduction, i.e., $P_s \subset W$. This is a little bit more than required in the mortar context, where optimal *a priori* error estimates can be obtained with a Lagrange multiplier space satisfying $P_{s-1} \subset W$. We assume $N \geq 3$ (the case $N \leq 2$, $s \geq 2$, can be dealt with as a simple linear algebra problem). Then, automatically, $\dim V \geq \dim P_s$. Consider any three consecutive intervals $\Delta^{n-1}, \Delta^n, \Delta^{n+1}$. Without loss of generality, after a suitable linear coordinate transform we can assume that Δ^n coincides with $[-1,1]$ and that the new intervals left and right to $[-1,1]$ have lengths $h = 2h_{n-1}/h_n$ and $h' = 2h_{n+1}/h_n$, respectively. We choose the system

$$\Phi_s = [\theta_{s,n-1} + \theta_{0,n}, \theta_{1,n}, \dots, \theta_{s-1,n}, \theta_{s,n} + \theta_{0,n+1}];, \qquad (8)$$

and denote its transformation to $[-h-1,-1]\cup[-1,1]\cup[1,1+h']$ again by Φ_s. By construction, the transformations of $\theta_{1,n},\dots,\theta_{s-1,n}$ have support in $[-1,1]$ and coincide with the functions $p_2,\dots,p_s$ from Π_s while the remaining two functions are scaled piecewise linear hat functions.

In order to prove $\det G_{\mathbf{P}_s,\Phi_s}\neq 0$, we have a free choice of the basis in P_s. We will choose $\mathbf{P}_s=[\tilde p_0,\tilde p_1,\tilde p_2,\dots,\tilde p_s]$ such that for $x\in[-1,1]$

$$\tilde p_0(x)=1-\sum_{k=2}^{s}\int_{-1}^{1}p_k(t)\,dt\cdot p_k(x)\ ,\quad \tilde p_1(x)=x-\sum_{k=2}^{s}\int_{-1}^{1}t\,p_k(t)\,dt\cdot p_k(x)\ ,$$

and $\tilde p_k(x)=p_k(x),\ k=2,\dots,s$. We note the following properties:

(P1) Restricted to $[-1,1]$, the basis $\mathbf{P}_s$ is orthogonal, $\tilde p_0$ and $\tilde p_1$ are not normalized.

(P2) $\tilde p_0,\tilde p_2,\tilde p_4,\dots$ are even, $\tilde p_1,\tilde p_3,\tilde p_5,\dots$ are odd.

(P3) All zeros of $\tilde p_0$, $\tilde p_1$ are in $(-1,1)$, i.e., $\tilde p_0(t)>0$ for $t\in\mathbb{R}\backslash[-1,1]$ and $\tilde p_1(t)<0$ for $t<-1$ and $\tilde p_1(t)>0$ for $t>1$.

(P1) and (P2) follow from the definition. To prove (P3) for $\tilde p_0$, assume that it has zeros outside $[-1,1]$. Since $\tilde p_0$ is even and of even degree $2q'\le s$, its zeros are symmetrically located with respect to the origin. Let $0<x_1<\dots<x_k<1$ be the zeros of odd multiplicity inside $(0,1)$. By assumption at least one pair of zeros is outside $[-1,1]$ and thus $k\le q'-1$. Recall also that by construction $\tilde p_0(-1)=\tilde p_0(1)=1$. Now, we define a polynomial

$$p(x):=(1-t^2)(t^2-x_1^2)\dots(t^2-x_k^2)\ ,\quad t\in[-1,1]\ .$$

which has the same sign as $\tilde p_0$ everywhere in $(-1,1)$, with the exception of zeros of even multiplicity. Thus, $\int_{-1}^{1}p(t)\tilde p_0(t)\,dt>0$. This contradicts the orthogonality property since from $\deg(p)\le 2q'\le s$ and $p(-1)=p(1)=0$ we conclude that $p\in\operatorname{span}[p_2,\dots,p_s]$. The same reasoning goes through for $\tilde p_1$, we leave this upon the reader. We are now in the position to show that $\det G_{\mathbf{P}_s,\Phi_s}\neq 0$ for the above Φ_s and $\mathbf{P}_s$. Equivalently, we show that orthogonality of

$$p=a_0\tilde p_0+a_1\tilde p_1+\sum_{k=2}^{s}a_k\tilde p_k$$

to all functions from Φ_s yields $a_k=0$ for all $k=0,\dots,s$. Testing with the translates of $\theta_{l-1,n}$ (which coincide with $\tilde p_l|_{[-1,1]}$) and using (P1) immediately gives $a_l=0$ for all $l=2,\dots,s$. Thus, only a_0 and a_1 can be different from zero. Now, we test $\tilde p_0$ and $\tilde p_1$ with the two remaining hat functions which will be denoted by $\hat p_1$ and $\hat p_2$. Let $\hat p_1$ be supported in $[-1-h,1]$ and $\hat p_2$ in $[-1,1+h']$. We recall that $\hat p_1$ and $\hat p_2$ are positive in $(-1-h,-1)$ and $(1,1+h')$, respectively. Moreover, in $[-1,1]$ they can be written as $\hat p_1=0.5(\tilde p_0-\tilde p_1)+\hat q_1$ and $\hat p_2=0.5(\tilde p_0+\tilde p_1)+\hat q_2$, where $\hat q_1,\hat q_2\in\operatorname{span}[p_2,\dots p_s]$. Then, due to (P1) and (P3), the 2×2 determinant

$$\det G_{[\tilde p_0,\tilde p_1],[\hat p_1,\hat p_2]}=\begin{vmatrix}>0 & >0\\ <0 & >0\end{vmatrix}>0\ , \tag{9}$$

is positive. This shows $a_0=a_1=0$, and concludes the verification of $\det G_{\mathbf{P}_s,\Phi_s}\neq 0$. As a by-product, we see that the inverse $G^{-1}_{\mathbf{P}_s,\Phi_s}$ continuously depends on h,h'.

Proposition 2 *For the above defined basis Φ in the space V of C^0 finite elements of degree $s \geq 1$ on a partition $\mathcal{T}$ of an interval, there exists a locally supported dual basis Ψ consisting of piecewise polynomial functions of degree s on the same $\mathcal{T}$ such that $P_s \subset W$, and the associated projections Q_W and Q_V possess L_p-norm bounds ($1 \leq p \leq \infty$) which depend only on s, and on the local meshsize ratio $\gamma(\mathcal{T}) := \max_{|i-k|=1} h_i/h_k$.*

Proof The existence of a dual basis with $P_s \subset W$ has already been established. To construct a locally supported basis, we specify Z as follows: Obviously, the k-th column in Z is naturally associated with the vertex $x_k := a + \sum_{l=1}^{k-1} h_l$, $1 \leq k \leq N+1$, of $\mathcal{T}$, see (6) and the explicit form of U given in (7). For each k we chose three consecutive intervals Δ^{n_k}, Δ^{n_k+1} and Δ^{n_k+2} such that $k \in \{n_k - 1, n_k, n_k + 1, n_k + 2\}$. Then, we take the special choice of $\mathbf{P}_s$ and Φ_s given by (8) associated with these three consecutive intervals and define $\xi_k \in \mathbb{R}^{s+1}$, $1 \leq k \leq N+1$, by

$$G_{\mathbf{P}_s,\Phi_s}\xi_k = G_{\mathbf{P}_s,\Theta}(U_2)_k^T \; , \tag{10}$$

where $(U_2)_k$ is the k-th row of the matrix U_2. In a next step, we set the k-th column in Z by associating the components of ξ_k to the position in this column by correspondence to the functions in the chosen Φ_s and leaving zeros in positions corresponding to ϕ_l not in Φ_s. Note that we work with different subsystems Φ_s for the vertices x_k. Each column of Z has thus $\leq \dim P_s$ nonzero entries, associated with ϕ_l whose support is close to x_k by construction. Since (10) implies (6), we conclude that

$$B = G_{\Theta,\Theta}^{-1} U(\Sigma^{-1}, Z)^T \tag{11}$$

indeed defines a locally supported dual basis reproducing polynomials, with the supports of the ψ_l close to the supports of ϕ_l for all $l = 1, \ldots, \dim V$.

Since all steps in the construction depend only on the local meshsize ratio, the uniform L_p-stability bounds for the projections Q_W and Q_V (as well as local L_p-error estimates for smooth functions) can be derived. Since this is standard, we will not go into details. ■

Dual systems with basis functions of small support have been constructed in [Woh01] for $s \leq 2$ and $r = s - 1$. As was mentioned in the introduction, for the mortar finite element applications polynomial reproduction of degree $r = s - 1$ would suffice. Our above proof implies that for this case the construction of an adequate dual basis can be based on $\Phi_{s-1} = [\theta_{s-1,n-1} + \theta_{0,n}, \theta_{1,n}, \ldots, \theta_{s-2,n}, \theta_{s-1,n} + \theta_{0,n+1}]$ and $\mathbf{P}_{s-1}$, $s \geq 2$.

Higher order mortar finite elements

In this section, we establish optimal a priori estimates for the discretization error of nonconforming mortar finite element methods. These domain decomposition techniques provide a more flexible approach than standard conforming formulations, and are of special interest for time dependent problems, rotating geometries, diffusion coefficients with jumps, problems with local anisotropies, corner singularities and when different terms dominate in different regions of the simulation domain. To obtain optimal a priori estimates, the interface between the different regions has to be handled appropriately. Very often suitable matching conditions at the interfaces can be formulated as weak continuity conditions. We assume that the bounded polygonal subdomain $\Omega \subset \mathbb{R}^2$ is geometrically conforming decomposed in non-overlapping polygonal subdomains Ω_k, $1 \leq k \leq K$. In particular, the situation with many crosspoints is included. Each subdomain is associated with a locally quasi-uniform simplicial or quadrilateral triangulation $\mathcal{T}_k$, and the discrete space of conforming finite elements of order s satisfying homogeneous Dirichlet boundary conditions on $\partial\Omega \cap \partial\Omega_k$ is denoted by $V_s(\Omega_k)$. On each interface $\gamma_{lk} := \partial\Omega_k \cap \partial\Omega_l$, we use the one-dimensional mesh inherited either from $\mathcal{T}_k$ or $\mathcal{T}_l$. The choice is arbitrary but fixed. Now, we replace the standard Lagrange multiplier space, see [BMP94], by our dual space. The basis functions of the Lagrange multiplier space $V_{s-1}(\gamma_{lk})$ on the interface γ_{lk} are defined as the scaled transformed dual basis functions $\psi_k \in \Psi$. Here Ψ is our locally supported basis consisting of piecewise polynomial functions of degree $s-1$ or s and reproducing polynomials of order $s-1$. Then, the constrained nonconforming mortar finite element space $Y_s \subset L^2(\Omega)$ is defined by

$$Y_s := \{v \in L^2(\Omega) \mid v_{|\Omega_k} \in V_s(\Omega_k), \int_{\gamma_{lk}} [v]\psi \, ds = 0, \, \psi \in V_{s-1}(\gamma_{lk}), \, 1 \leq k < l \leq K\}.$$

The analysis of the resulting jump terms across the interfaces plays an essential role for the a priori estimates of the discretization schemes. It is sufficient to analyze the jump term on the reference interface $I = [a, b]$. In particular, optimal methods can only be obtained if the consistency error is small enough compared with the best approximation error on the different subdomain. Indeed, in [Woh01, Conditions (Sa)–(Sd)], sufficient conditions for abstract Lagrange multiplier spaces are given to obtain a discretization error of order h^s and h^{s+1} in the H^1- and L^2-norm, respectively. For convenience, we briefly review the conditions. In a short form they read for dual spaces as: Locality of the support of the dual basis functions, polynomial reproduction of degree $s-1$, stability of the projections Q_V, Q_W and the existence of a well-defined stable operator $Q_{\hat{V}} : L^2(I) \to \hat{V}$. The projection $Q_{\hat{V}}$ will be defined by

$$\int_I Q_{\hat{V}} v\psi \, ds = \int_I v\psi \, ds, \quad \psi \in W.$$

Here, $\hat{V}$ is a subspace of $X \cap H^1(I)$ having the same dimension as V and satisfying a low order approximation property for all $v \in H^1(I)$. We point out that the required approximation property of $\hat{V}$ does not depend on the order s. For a more detailed discussion on the properties of $\hat{V}$, we refer to [Woh01]. We note that in the case of our locally supported dual basis Ψ, the best approximation property of the nonconforming space Y_s is automatically guaranteed. Since Q_V is by construction H_{00}-stable, no problem at the crosspoints occurs. The analysis of the consistency error requires the polynomial reproduction of degree $s-1$ and the existence of such a $Q_{\hat{V}}$. Of crucial importance is the weighted L^2-norm of the jump, $1/\sqrt{h}\|[v]\|_{0;\gamma_{lk}}$, $v \in Y_s$ across the interfaces γ_{lk}.

Lemma 2 *Replacing in the mortar finite element approach the standard Lagrange multiplier space on each interface γ_{lk} by our locally constructed dual space $V_{s-1}(\gamma_{lk})$ yields optimal a priori estimates for the discretization error in the L^2-norm (order h^{s+1}) and in the H^1-norm (order h^s). Moreover, the error in the Lagrange multiplier measured in a weighted L^2-norm, $\sqrt{h}\|[\cdot]\|_{0;\gamma_{lk}}$, is of order h^s.*

Proof Almost all required conditions, as locality, polynomial reproduction of degree $s-1$, and stability of Q_V, Q_W are satisfied by our above construction. Thus to establish the optimality, it is sufficient to define a suitable $\hat{V}$ and show that the corresponding projection $Q_{\hat{V}}$ is uniformly stable. The low order approximation property is, e.g., satisfied if $\tilde{V} :=$

span $\{\theta_{0,1}+\theta_{s,1}+\theta_{0,2},\theta_{s,2}+\theta_{0,3},\ldots,\theta_{s,N-2}+\theta_{0,N-1},\theta_{s,N-1}+\theta_{0,N}+\theta_{s,N}\}$ is a subspace of $\hat{V}$. Considering $\Phi = \Theta\, U(\Sigma,0)^T$ and adding $\theta_{0,1}$ to the first basis function and $\theta_{s,N}$ to the last one provides a new set Φ_1 of linear independent functions. The associated space $\hat{V}$ satisfies $\tilde{V} \subset \hat{V}$, has the same dimension as V and a locally supported basis. In algebraic notation, we can write Φ_1 as $\Phi_1 := \Theta\, U(\Sigma,\hat{Z})^T$ where $\hat{Z} \in \mathbb{R}^{n\times m-n}$ has only two nonzero entries, $\hat{z}_{1,1} = \hat{z}_{n,m-n} = 1$. To show that $Q_{\hat{V}}$ is well defined, it is sufficient to prove that G_{Ψ,Φ_1} is non-singular. Using (11), we find $G_{\Psi,\Phi_1} = \mathrm{Id}_n + Z\hat{Z}^T$. The special structure of $\hat{Z}$ yields that the first and last column of $Z\hat{Z}^T$ is the first and last column of Z, respectively, all other columns are zero. Our construction of Z shows that the first column of Z depends on $\mathbf{P}_s$ or $\mathbf{P}_{s-1}$ which is associated with Δ^1, Δ^2, Δ^3. Since we have assumed $N \geq 3$, we find $z_{n,1} = z_{1,m-n} = 0$. Therefore it is sufficient to show that $1+z_{1,1}$ and $1+z_{n,m-n}$ are nonzero. Using the same notation as before $\tilde{p}_0$ and $\tilde{p}_1$ are orthonormal polynomials on Δ^2 and extended to I. The coefficient $z_{1,1}$ is the first component of the solution x of $G_{[\tilde{p}_0,\tilde{p}_1],[\hat{p}_1,\hat{p}_2]}x = y$, where $y_1 = \int_I \tilde{p}_0\theta_{0,1}dx$ and $y_2 = \int_I \tilde{p}_1\theta_{0,1}dx$. By means of (P3), we find $y_1 > 0$ and $y_2 < 0$ which together with (9) yields that $z_{1,1} > 0$. The same reasoning holds for $z_{n,m-n}$, and we obtain $\det G_{\Psi,\Phi_1} \neq 0$. ∎

1D examples for $s \leq 3$

The aim of this section is to illustrate the above theoretical result, and to provide explicit formulas for $s \leq 3$, at least for the case of uniform partitions. We base the construction of our dual bases on (8). From now on we will assume that $\Omega = [0,N]$, $\Delta^n = [n-1,n]$, $n = 1,\ldots,N$, and $s \leq 3$. We refer to [OW00] for a detailed discussion and for an explicit representation of the matrices Z and B. For our convenience, we will fix the basis for Θ_{Δ^1} on $\Delta^1 = [0,1]$ and obtain the bases Θ_{Δ^n} by translation:

$$\Theta_{\Delta^1} = \begin{cases} [\theta_1^1,\theta_1^2] = [1-t,t]\,, & s=1\,,\\ [\theta_1^1,\theta_1^3,\theta_1^2] = [1-t,6t(1-t),t]\,, & s=2\,,\\ [\theta_1^1,\theta_1^3,\theta_1^4,\theta_1^2] = [1-t,6t(1-t),10t(1-t)(2t-1),t]\,, & s=3\,. \end{cases}$$

This will lead to the following n-independent formulas for the diagonal blocks $G^{-1}_{\Theta_{\Delta^n},\Theta_{\Delta^n}}$ of $G^{-1}_{\Theta,\Theta}$:

$$G^{-1}_{\Theta_{\Delta^n},\Theta_{\Delta^n}} = \begin{pmatrix} 4 & -2\\ -2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 9 & -5 & 3\\ -5 & 5 & -5\\ 3 & -5 & 9 \end{pmatrix}, \quad \begin{pmatrix} 16 & -5 & 7 & -4\\ -5 & 5 & 0 & -5\\ 7 & 0 & 7 & -7\\ -4 & -5 & -7 & 16 \end{pmatrix},$$

for $s = 1,2,3$, respectively. The explicit formulas for Σ and U only differ in the sizes of the identity matrices for different s. Thus, we have all ingredients ready for using (11), with the exception of the matrix Z. The construction of Z is described in the proof of Proposition 2, and depends on s and the desired degree of polynomial reproduction $r \leq s$. We refer to [OW00] for the calculation of the entries of Z. Although we provide explicit results only for uniform partitions, the construction can be used for non-uniform partitions without essential changes. For $s = 1$, the dual basis functions ψ_n obtained along the lines of the previous section are as follows. Away from the endpoints of Ω, we get

$$\psi_n(x) := \begin{cases} -\frac{2}{3}\theta_n^1 + \frac{4}{3}\theta_n^2, & x \in [n-1, n), \\ \frac{7}{3}\theta_{n+1}^1 - \frac{2}{3}\theta_{n+1}^2, & x \in [n, n+1], \\ -\frac{2}{3}\theta_{n+2}^1 + \frac{1}{3}\theta_{n+2}^2, & x \in (n+1, n+2], \end{cases} \qquad 3 \le n \le N-3\ ,$$

for the *interior dual basis functions* (here and below, we only show formula for the intervals in the support of ψ_n). For the *boundary dual basis functions* near the left and right endpoint of Ω, we obtain modified expressions:

$$\psi_1(x) := \begin{cases} 2\theta_1^1 + \theta_1^2, & x \in [0,1), \\ \theta_2^1, & x \in [1,2], \\ -\frac{2}{3}\theta_3^1 + \frac{1}{3}\theta_3^2, & x \in (2,3), \end{cases} \qquad \psi_2(x) := \begin{cases} -\theta_1^1, & x \in [0,1), \\ \theta_2^2, & x \in [1,2], \\ \frac{7}{3}\theta_3^1 - \frac{2}{3}\theta_3^2, & x \in (2,3], \\ -\frac{2}{3}\theta_4^1 + \frac{1}{3}\theta_4^2, & x \in (3,4], \end{cases}$$

ψ_{N-2}, ψ_{N-1} are defined in a similar way. Now, it is easy to verify that the locally supported basis functions ψ_n are biorthogonal with the standard hat functions. Furthermore, $\sum_{n=1}^{N-1} \psi_n = 1$ and $\sum_{n=1}^{N-1} n\, \psi_n = x$ and thus $P_1 \subset W$. We note that in [Woh00, Woh01], a dual basis with smaller support but only $P_0 \subset W$ has been constructed.

For $s = 2$, we introduce a dual basis satisfying $P_2 \subset W$. We distinguish between two different types of dual basis functions ψ_n^b and ψ_n^h which are associated with the bubble and hat functions of the finite element basis functions, respectively. The interior dual basis functions ψ_n^b with support on $\Delta_{n-1} \cup \Delta_n \cup \Delta_{n+1}$ ($n = 3, \ldots, N-2$) and ψ_n^h with support on $\Delta_n \cup \Delta_{n+1}$ ($n = 2, \ldots, N-2$) are defined by the corresponding θ_k^i, $1 \le i \le 3$, as follows:

$$\psi_n^b(x) := \begin{cases} -\frac{3}{2}\theta_{n-1}^1 + \frac{5}{6}\theta_{n-1}^3 - \frac{1}{2}\theta_{n-1}^2 \\ -3\theta_n^1 + \frac{10}{3}\theta_n^3 - 3\theta_n^2 \\ -\frac{3}{2}\theta_{n+1}^1 + \frac{5}{6}\theta_{n+1}^3 - \frac{1}{2}\theta_{n+1}^2 \end{cases} , \qquad \psi_n^h(x) := \begin{cases} \frac{3}{2}\theta_n^1 - \frac{5}{2}\theta_n^3 + \frac{9}{2}\theta_n^2 \\ \frac{9}{2}\theta_{n+1}^1 - \frac{5}{2}\theta_{n+1}^3 \frac{3}{2}\theta_{n+1}^2 \end{cases} .$$

The modifications for the boundary dual basis functions ψ_1^b, ψ_2^b, and ψ_1^h concern only their values on Δ_1, otherwise the above formula apply correspondingly. We define on Δ_1: $\psi_1^b := 33/8\theta_1^1 - 5/8\theta_1^3 - 5/8\theta_1^2$, $\psi_2^b := 17/8\theta_1^1 - 5/8\theta_1^3 - 5/8\theta_1^2$, and $\psi_1^h := -21/4\theta_1^1 - 5/4\theta_1^3 - 9/4\theta_1^2$. The modifications for ψ_{N-1}^b, ψ_N^b, and ψ_{N-1}^h on Δ_N are analogous. Note that the support of the dual basis functions ψ_n^b, ψ_n^h is contained in ≤ 3 neighboring Δ^n close to the support of the corresponding finite element basis functions. Moreover, we find by construction $P_2 \subset W$.

For $s = 3$, we do not specify the explicit formulas for the basis functions and refer to [OW00] for details. Figure 1 illustrates the interior dual basis functions, $s = 3$, for $P_2 \subset W$ and $P_3 \subset W$, respectively. We have three different types associated with hat functions, quadratic, and cubic bubbles, and supports consisting of two, three and one/two consecutive intervals, respectively.

2D results

The above approach generalizes to higher dimensions, as we demonstrate with the following example. We consider the space V of quadratic Lagrange C^0-elements, i.e., $s = 2$, with homogeneous Dirichlet boundary conditions on a triangulation $\mathcal{T}$ of a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, and show the existence of a dual basis of locally supported piecewise quadratics on $\mathcal{T}$ such that W reproduces linear polynomials locally, i.e., $P_1 \subset W$, under a certain

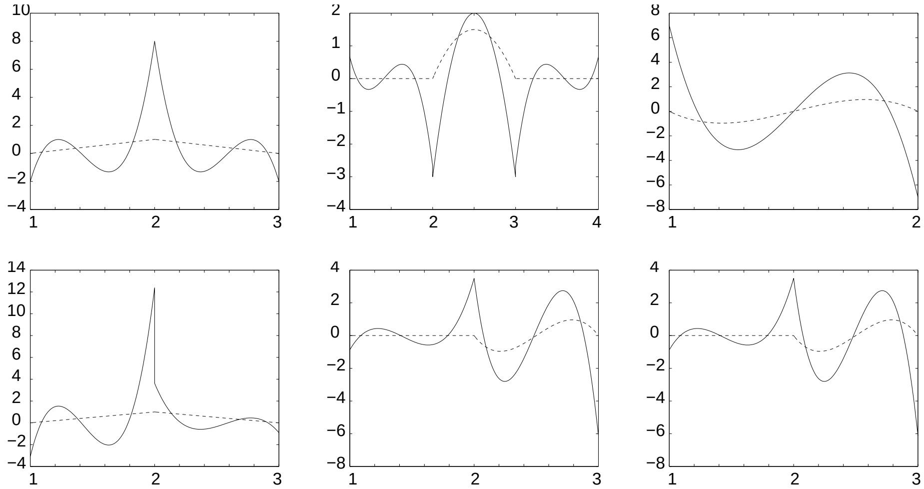

Figure 1: Interior dual basis functions $P_r \subset W$, $r = 2$ (above) and $r = 3$ (below)

regularity condition on $\mathcal{T}$. For lowest order finite elements, dual basis functions satisfying $P_0 \subset W$ have been constructed in [KLPV01, WK01, Woh01].

The basis Θ in X which is set to be the space of discontinuous piecewise quadratics is conveniently given by the collection of all elemental nodal shape functions $\theta_{\Delta,P}$, piecewise linear barycentric coordinate function for vertex P of triangle Δ, and $\theta_{\Delta,e}$, quadratic tent function associated with triangle Δ and its edge e. Each such function is supported on a single triangle, and there are 6 of them for each Δ. As before, an explicit, sparse factorization of A can be found (see [OW00]), and Proposition 1 can be applied. Following the considerations of Section 8, it is sufficient to find a locally defined subsystem Φ_1 such that $\det G_{\Phi_1,\mathbf{P}_1} \neq 0$. Let $\Delta \in \mathcal{T}$ be any triangle all edges of which are interior to Ω. We specify a basis $\mathbf{P}_1$ of P_1 by setting $\mathbf{P}_1 := [p_1, p_2, p_3]$ where p_k denotes the extension of the barycentric coordinate function associated with the vertex P_k of Δ to all of $\mathbb{R}^2$ which is defined by requiring $p_k \in P_1$ and $p_k(P_l) = \delta_{kl}$, $k, l = 1, 2, 3$. The subsystem Φ_1 is defined by $[\phi_{e_1}, \phi_{e_2}, \phi_{e_3}]$ where ϕ_{e_k} denote the conforming quadratic bubble functions associated with the edges e_k of the triangle Δ. Using the affine invariance of both Φ_1 and $\mathbf{P}_1$ we can without loss of generality assume that Δ is equilateral, with area $A = 1$. All the other notation can be found in the left of Figure 2. The area of the triangle Δ_k, attached to Δ along e_k, is denoted by A_k.

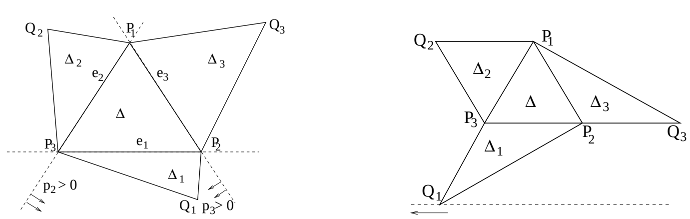

Figure 2: Notation for Lemma 3 (left) and counterexample (right)

Lemma 3 *Let the triangles in the left part of Figure 2 satisfy the following condition: For each $k = 1, 2, 3$, the diagonal P_kQ_k belongs to the closure of the corresponding quadrilateral $\Delta \cup \Delta_k$. Then the determinant of $G_{\mathbf{P}_1,\Phi_1}$ is positive and depends continuously on the location of the Q_k. If the additional geometric assumption is dropped, the matrix $G_{\mathbf{P}_1,\Phi_1}$ may become singular.*

Proof The proof is based on elementary calculations. We start by stating the formula

$$\int_\Delta \theta_{\Delta,e_1} \cdot \sum_{k=1}^{3} \alpha_k p_k \, dx = \frac{A}{15}(\alpha_1 + 2(\alpha_2 + \alpha_3)) \; ,$$

which holds, due to affine invariance of all functions involved, for all triangles. This allows us to compute all scalar products necessary for $G_{\Phi_1,\mathbf{P}_1}$. E.g.,

$$\int_\Omega \theta_{\Delta,e_1} \cdot p_1 \, dx = \int_\Delta \theta_{\Delta,e_1} \cdot p_1 \, dx + \int_{\Delta_1} \theta_{\Delta,e_1} \cdot p_1 \, dx = \frac{A}{15} + \frac{A_1}{15} p_1(Q_1) = \frac{1 - A_1^2}{15}$$

since $p_1(Q_1) = -A_1/A = -A_1$. Since $p_1 + p_2 + p_3 \equiv 1$, we have $p_2(Q_1) + p_3(Q_1) = 1 - p_1(Q_1) = 1 + A_1$ which leads to the ansatz

$$p_2(Q_1) = \frac{1 + A_1 - \epsilon_1}{2} \; , \quad p_3(Q_1) = \frac{1 + A_1 + \epsilon_1}{2} \; ,$$

where our geometric assumption implies that $\min(p_2(Q_1), p_3(Q_1)) \geq 0$ or, equivalently, $|\epsilon_1| \leq 1 + A_1$. With this at hand, we compute

$$\int_\Omega \theta_{\Delta,e_1} \cdot p_2 \, dx = \frac{2A}{15} + \frac{A_1}{15}(2 + p_2(Q_1)) = \frac{4 + 5A_1 + A_1^2 - A_1\epsilon_1}{30} \; ,$$

and, analogously,

$$\int_\Omega \theta_{\Delta,e_1} \cdot p_3 \, dx = \frac{4 + 5A_1 + A_1^2 + A_1\epsilon_1}{30} \; .$$

Applying the same analysis to the other rows of $G_{\Phi_1,\mathbf{P}_1}$ and observing that the rows almost completely divide by $(1 + A_k)$, we get the following explicit formula

$$\frac{30^3 \; \det G_{\Phi_1,\mathbf{P}_1}}{(1 + A_1)(1 + A_2)(1 + A_3)} = D \equiv \begin{vmatrix} 2 - 2A_1 & 4 + A_1 - \epsilon'_1 & 4 + A_1 + \epsilon'_1 \\ 4 + A_2 + \epsilon'_2 & 2 - 2A_2 & 4 + A_2 - \epsilon'_2 \\ 4 + A_3 - \epsilon'_3 & 4 + A_3 + \epsilon'_3 & 2 - 2A_3 \end{vmatrix} \; ,$$

where $|\epsilon'_k| = A_k|\epsilon_k|/(1 + A_k) \leq A_k$, $k = 1, 2, 3$, follows from our assumption.

A straightforward calculation reveals that

$$D = 10(3s_2(\mathbf{A}) + 4s_1(\mathbf{A}) + 4 + f(\mathbf{A}, \epsilon')) \; ,$$

where $s_1(\mathbf{x}) = x_1 + x_2 + x_3$, $s_2(\mathbf{x}) = x_1x_2 + x_2x_3 + x_3x_1$ for any $\mathbf{x} \in \mathbb{R}^3$, and

$$f(\mathbf{A}, \epsilon') = s_2(\epsilon') + \epsilon'_1(A_2 - A_3) + \epsilon'_2(A_3 - A_1) + \epsilon'_3(A_1 - A_2) \; .$$

The global minimum of f with respect to the cube $\epsilon'_k \in [-A_k, A_k]$, $k = 1, 2, 3$, is attained on the boundary of this cube, and can be determined easily:

$$f(\mathbf{A}, \epsilon') \geq s_2(\mathbf{A}) - 4\max(A_1A_2, A_2A_3, A_3A_1) \geq -3s_2(\mathbf{A})$$

holds for all ϵ'_k of interest. Substitution gives

$$D \geq 40(s_1(\mathbf{A}) + 1) > 40 \ . \tag{12}$$

since $A_k > 0$, $k = 1, 2, 3$. This shows the assertions of Lemma 3 under the geometric assumptions made. The continuous dependence of the determinant and thus the inverse of $G_{\mathbf{P}_1,\Phi_1}$ on the local topology is obvious.

It remains to provide a counterexample that shows that the above choice for Φ_1 may fail to guarantee the invertibility of $G_{\mathbf{P}_1,\Phi_1}$. The right part of Figure 2 contains the counterexample. We claim that if Q_1 is moved to the left, the determinant of $G_{\mathbf{P}_1,\Phi_1}$ will vanish at some point. Indeed, the specification of the example is such that $A = A_1 = A_2 = A_3 = 1$, both Δ and Δ_2 are equilateral (thus, $\epsilon'_2 = 0$), and $\epsilon'_3 = 1$ since Q_3 belongs to the extension of e_1. Thus, according to the above formula, $D = \alpha\epsilon'_1 + \beta$ is a linear function with respect to ϵ'_1, with slope $\alpha = 10(\epsilon'_2 + \epsilon'_3 + A_2 - A_3) = 10$ and $\beta = 250$ (since for $\epsilon'_1 = 0$ the geometric assumption is satisfied and therefore (12) is valid). Thus, moving Q_1 sufficiently far to the left or, equivalently, decreasing ϵ'_1, we finally hit a zero value for D. This proves our claim. ■

Remark 1 *One possible modification is to start the construction of dual bases with a finite element space X corresponding to a refined partition $\mathcal{T}'$ rather than with the space of non-smooth piecewise polynomials on the same $\mathcal{T}$. This could make the resulting W suitable for applications, where higher smoothness of the functions in the dual system is required. However, for use as Lagrange multiplier subspaces of $H^{-1/2}$ in the mortar finite element method this is not essential.*

Remark 2 *In contrast to constructions of biorthogonal wavelet systems [DKU99, DS97, Ste00], the spaces W obtained here are not refinable, i.e., if $\mathcal{T}'$ is a proper refinement of $\mathcal{T}$, we cannot expect to have $W' \supset W$. However, as suggested in a similar problem in [Osw99], we still have refinability $X' \supset X$ for the container spaces of piecewise polynomials which enables the use of our systems in a multilevel setup.*

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