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25 Building preconditioners for incompressible Stokes equations from saddle point solvers of smaller dimensions

L. F. Pavarino[1], O. B. Widlund[2]

Introduction

Balancing Neumann-Neumann methods are introduced and studied for incompressible Stokes equations discretized with mixed finite or spectral elements with discontinuous pressures. After decomposing the original domain of the problem into nonoverlapping subdomains, the interior unknowns, which are the interior velocity component and all except the constant pressure component, of each subdomain problem are implicitly eliminated. The resulting saddle point Schur complement is solved with a Krylov space method with a balancing Neumann-Neumann preconditioner based on the solution of a coarse Stokes problem with a few degrees of freedom per subdomain and on the solution of local Stokes problems with natural and essential boundary conditions on the subdomains. This preconditioner is of hybrid form in which the coarse problem is treated multiplicatively while the local problems are treated additively. The condition number of the preconditioned operator is independent of the number of subdomains and is bounded from above by the product of the square of the logarithm of the local number of unknowns in each subdomain and a factor that depends on the inverse of the inf-sup constants of the discrete problem and of the coarse subproblem. Numerical results show that the method is quite fast; they are also fully consistent with the theory. This work is described in much more detail in [PW02], which contains a full proof of our result as well as many more references to the literature.

The Stokes System and Discretizations

We consider the incompressible Stokes equations on a polyhedral domain $\Omega \subset \mathbf{R}^d, d = 2, 3$,

$$\begin{cases} \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} dx - \int_\Omega \mathrm{div} \mathbf{v}\, p dx &= \int_\Omega \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in (H_0^1(\Omega))^d, \\ - \int_\Omega \mathrm{div} \mathbf{u}\, q dx &= 0 \qquad\qquad\quad \forall q \in L_0^2(\Omega), \\ \mathbf{u}_{|\partial\Omega} = \mathbf{g}, \end{cases} \tag{1}$$

where $\mathbf{f} \in (H^{-1}(\Omega))^d$, $\mathbf{g} \in (H^{1/2}(\partial\Omega))^d$, and $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} ds = 0$. We discretize this system with any pair of stable Stokes elements with discontinuous pressures, such as $Q_2(h) - Q_0(h)$ and $Q_2(h) - P_1(h)$ mixed finite elements (see Brezzi and Fortin [BF91]), or $Q_n - Q_{n-2}$ mixed spectral elements (see Maday, Meiron, Patera, and Rønquist [MMPR93]). This last

[1]Università di Milano, pavarino@mat.unimi.it
[2]Courant Institute of Mathematical Sciences, widlund@cims.nyu.edu

choice is not uniformly stable, since the inf-sup constant of the discrete problem decays as $\beta_n = Cn^{-(\frac{d-1}{2})}$.

The discrete system obtained has the form

$$K \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}. \tag{2}$$

Substructuring for Saddle Point Problems

The domain Ω is decomposed into open, nonoverlapping quadrilateral (hexahedral) subdomains Ω_i, of characteristic size H, and the interface Γ, i.e.,

$$\Omega = \cup_{i=1}^N \Omega_i \cup \Gamma.$$

Here $\Gamma = \left(\bigcup_{i=1}^N \partial\Omega_i\right) \setminus \partial\Omega$. Each Ω_i typically consists of one, or a few, spectral elements of degree n or of many finite elements. We denote by Γ_h and $\partial\Omega_h$ the set of nodes belonging to the interface Γ and $\partial\Omega$, respectively. The starting point of our algorithm is the implicit elimination (static condensation) of the interior degrees of freedom, i.e., the velocity component that is supported in the open subdomains and the interior pressure components with zero average over the individual subdomains. This process is carried out by solving decoupled local Stokes problems on each subdomain Ω_i with Dirichlet boundary conditions for the velocities given on $\partial\Omega_i$. We then obtain a saddle point Schur complement problem for the interface velocities and a constant pressure in each subdomain. This reduced problem will be solved by a preconditioned Krylov space iteration normally the preconditioned conjugate gradient method.

In order to eliminate the interior degrees of freedom, we reorder the vector of unknowns as

$$\begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} \quad \begin{array}{l} \text{interior velocities} \\ \text{interior pressures with zero average} \\ \text{interface velocities} \\ \text{constant pressures in each } \Omega_i. \end{array}$$

Then, after using the same permutation, the discrete Stokes system matrix can be written as

$$\begin{bmatrix} K_{II} & K_{\Gamma I}^T \\ & \\ K_{\Gamma I} & K_{\Gamma\Gamma} \end{bmatrix} = \left[\begin{array}{cc|cc} A_{II} & B_{II}^T & A_{\Gamma I}^T & 0 \\ B_{II} & 0 & B_{I\Gamma} & 0 \\ \hline A_{\Gamma I} & B_{I\Gamma}^T & A_{\Gamma\Gamma} & B_0^T \\ 0 & 0 & B_0 & 0 \end{array}\right].$$

Eliminating the interior unknowns $\mathbf{u}_I$ and p_I by static condensation, we obtain the saddle point Schur complement system

$$S \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}} \\ 0 \end{bmatrix}$$

where

$$S = K_{\Gamma\Gamma} - K_{\Gamma I} K_{II}^{-1} K_{\Gamma I}^T =$$

$$= \begin{bmatrix} A_{\Gamma\Gamma} & B_0^T \\ B_0 & 0 \end{bmatrix} - \begin{bmatrix} A_{\Gamma I} & B_{I\Gamma}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{\Gamma I}^T & 0 \\ B_{I\Gamma} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} S_\Gamma & B_0^T \\ B_0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \tilde{\mathbf{b}} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_\Gamma \\ 0 \end{bmatrix} - \begin{bmatrix} A_{\Gamma I} & B_{I\Gamma}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b}_I \\ 0 \end{bmatrix}.$$

One can show that S_Γ is positive definite. By using a second permutation that reorders the interior velocities and pressures subdomain by subdomain, we note that K_{II}^{-1} represents the solution of N decoupled Stokes problems, one for each subdomain and all uniquely solvable, in parallel, with Dirichlet data given on $\partial\Omega_i$, i.e., $K_{II}^{-1} = diag({K_{II}^{(i)}}^{-1})$.

The Schur complement S does not need to be explicitly assembled since only its action Sv on a vector v is needed in a Krylov iteration. This operation essentially only requires the action of K_{II}^{-1} on a vector, i.e., the solution of N decoupled Stokes problems. In other words, Sv is computed by subassembling the actions of the subdomain Schur complements $S^{(i)}$ defined for Ω_i, given by

$$S^{(i)} = K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)} (K_{II}^{(i)})^{-1} {K_{\Gamma I}^{(i)}}^T =$$

$$= \begin{bmatrix} S_\Gamma^{(i)} & {B_0^{(i)}}^T \\ B_0^{(i)} & 0 \end{bmatrix}.$$

Once $\begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix}$ is known, $\begin{bmatrix} \mathbf{u}_I \\ p_I \end{bmatrix}$ can be found by back-substitution.

This substructuring procedure can be described in terms of a space decomposition of the discrete spaces, in the spirit of the standard Schwarz framework; cf. [PW02].

A Neumann-Neumann Preconditioner

We will solve the saddle point Schur complement problem

$$S \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} S_\Gamma & B_0^T \\ B_0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ p_0 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}} \\ 0 \end{bmatrix}$$

by a preconditioned Krylov space method such as GMRES or PCG. We note that this Schur complement problem is positive definite on the benign subspace where the constraints hold.

We are therefore able to use the PCG because we will start and keep the iterates in the benign subspace $\mathbf{V}_{\Gamma,B} = Ker(B_0)$.

Our balancing Neumann-Neumann preconditioner is based on the solution of a coarse Stokes problem with a few degrees of freedom per subdomain and of local Stokes problems with natural and essential boundary conditions on each subdomain. This preconditioner is of hybrid form in which the coarse problem is treated multiplicatively while the local problems are treated additively; cf. [SBG96, p. 152]. It is closely analogous to the balancing Neumann-Neumann preconditioner for the positive definite case, except that the coarse and local problems are saddle point problems. For previous work on Neumann-Neumann methods for elliptic problems, see [Man93], [MB96], [LT94], [TMV98], [DW95], and the references in [PW02].

The matrix form of the preconditioner is

$$Q = Q_H + (I - Q_H S) \sum_{i=1}^{N} Q_i (I - S Q_H),$$

where the coarse operator Q_H and local operators Q_i are defined below. The preconditioned operator is then

$$T = QS = T_0 + (I - T_0) \sum_{i=1}^{N} T_i (I - T_0),$$

where $T_0 = Q_H S$ and $T_i = Q_i S$.

Coarse solver: Given a residual vector r, the coarse term $Q_H r$ is the solution of a coarse, global Stokes problem with a few velocity degrees of freedom and one constant pressure per subdomain Ω_i:

$$Q_H = R_H^T S_0^{-1} R_H,$$

where

$$R_H = \begin{bmatrix} L_0^T & 0 \\ 0 & I \end{bmatrix},$$

and

$$S_0 = R_H S R_H^T = \begin{bmatrix} L_0^T S_\Gamma L_0 & L_0^T B_0^T \\ B_0 L_0 & 0 \end{bmatrix}.$$

We will consider three choices for the matrix L_0, resulting in the coarse velocity spaces $\mathbf{V}_0^0, \mathbf{V}_0^1$, and $\mathbf{V}_0^2$, respectively. Some of the columns of L_0 are always defined in terms of the Neumann-Neumann counting functions μ_i associated with each subdomain Ω_i: μ_i is zero at the interface nodes outside $\partial\Omega_i$ while its value at any node on $\partial\Omega_i$ equals the number of subdomains shared by that node. Its pseudo inverse $\mu_i^\dagger$ is the function $1/\mu_i(x)$ for all nodes where $\mu_i(x) \neq 0$, and it vanishes at all other points of $\Gamma_h \cup \partial\Omega_h$. We note that we use the function $\mu_i^\dagger$ in all or almost all of the subdomains and for each velocity component. Then the columns of L_0 are defined by one of the following three choices:

0) the inverse counting functions $\mu_i^\dagger$,
1) the $\mu_i^\dagger$ and the continuous coarse piecewise bi- or tri-linear functions,
2) the $\mu_i^\dagger$ and the continuous coarse piecewise bi- or tri-quadratic functions.

In order to avoid linearly dependent $\mu_i^\dagger$ functions, and hence a singular coarse space problem, we might have to drop all of the components of these functions for one subdomain, depending on the coarse triangulation.

Local solver: The local operators Q_i will only be applied to residuals of velocity fields in the benign subspace $\mathbf{V}_{\Gamma,B}$ and thus the second residual component will vanish. Each local operator Q_i is based on the solution of a local Stokes problem on Ω_i with natural boundary condition. These local problems are nonsingular for all subdomains Ω_i the boundaries of which intersect $\partial\Omega$, but they are singular otherwise, i.e., for the *floating* subdomains. To avoid possible complications with singular problems, we modify the local Stokes problems on the floating subdomains, by adding ϵ times the velocity mass matrix to the local stiffness matrix $K^{(i)}$. It can be shown that after the coarse correction, that follows the local solvers, the iterates will be independent of the pressure field computed locally.

Given a residual vector with a first component r_Γ and a zero second component, $Q_i r$ is the weighted solutions of a local Stokes problem on subdomain Ω_i with a natural boundary condition on $\partial\Omega_i \setminus \partial\Omega$:

$$Q_i r = \begin{bmatrix} R_i^T D_i^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{\Gamma,\epsilon}^{(i)} & B_0^{(i)^T} \\ B_0^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} D_i^{-1} R_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_\Gamma \\ 0 \end{bmatrix}.$$

Here R_i are $0,1$ restriction matrices mapping r_Γ into r_{Γ_i} and D_i are diagonal matrices representing multiplication by the counting functions μ_i. Moreover,

$$S_\epsilon^{(i)} = \begin{bmatrix} S_{\Gamma,\epsilon}^{(i)} & B_0^{(i)^T} \\ B_0^{(i)} & 0 \end{bmatrix}$$

is the local saddle point Schur complement, associated with subdomain Ω_i, of the regularized local stiffness matrix

$$K_\epsilon^{(i)} = \begin{bmatrix} A_{II,\epsilon}^{(i)} & B_{II}^{(i)^T} & A_{\Gamma I,\epsilon}^{(i)^T} & 0 \\ B_{II}^{(i)} & 0 & B_{I\Gamma}^{(i)} & 0 \\ A_{\Gamma I,\epsilon}^{(i)} & B_{I\Gamma}^{(i)^T} & A_{\Gamma\Gamma,\epsilon}^{(i)} & B_0^{(i)^T} \\ 0 & 0 & B_0^{(i)} & 0 \end{bmatrix},$$

where

$$A_\epsilon^{(i)} = A^{(i)} + \epsilon M^{(i)}.$$

ϵ is a positive parameter, and $M^{(i)}$ is the local velocity mass matrix.

This balancing Neumann-Neumann preconditioner can be associated with a subspace decomposition of the interface space; cf. [PW02]. Our main result in that paper is:

Theorem 1 *On the benign subspace $\mathbf{V}_{\Gamma,B} \times U_0$ the balancing Neumann-Neumann operator T is symmetric positive definite with respect to the S bilinear form and*

$$cond(T) \leq C(1 + \frac{1}{\beta_0})\frac{1}{\beta^2}\, \alpha,$$

where

$$\alpha = \begin{cases} (1 + \log(H/h))^2 & \text{for finite elements} \\ (1 + \log n)^2 & \text{for spectral elements,} \end{cases}$$

β_0 and β are the inf-sup constants of the coarse problem and the original discrete Stokes problem respectively.

We note that $\mathbf{V}_0^0$ results in a poor constant β_0, while we can prove that $\mathbf{V}_0^2$ results in a constant β_0 uniformly bounded away from 0. $\mathbf{V}_0^1$ also gives satisfactory results.

Numerical Results with $Q_n - Q_{n-2}$ Spectral Elements in the Plane

We report, in this last section, results of some numerical experiments, carried out in Matlab 5.3 on Unix workstations, for a model Stokes problem on the unit square and with homogeneous Dirichlet boundary conditions. The problem was discretized with $Q_n - Q_{n-2}$ spectral elements and the domain Ω divided into $\sqrt{N} \times \sqrt{N}$ square subdomains. After the implicit elimination of the interior unknowns, the saddle point Schur complement is solved iteratively by PCG, starting and keeping the iterations in the benign subspace $\mathbf{V}_{\Gamma,B} \times U_0$. The initial guess is always zero, the right hand side is random and uniformly distributed, and the stopping criterion is $\|r_k\|_2/\|r_0\| \leq 10^{-6}$, where r_k is the residual at the $k-$th iterate. The singularity of the local Neumann solves for the floating subdomains is avoided by shifting the diagonal of the local velocity stiffness matrices by $\epsilon = 10^{-5}$.

The iteration counts are reported in Figure 1. These results show that PCG with our balancing Neumann-Neumann preconditioner is quasi-optimal and scalable, except with the first choice of coarse space $\mathbf{V}_0^0$. In fact, we have found the $\mathbf{V}_0^0$ coarse space not to be inf-sup stable and the iteration counts of PCG seem to grow linearly with N in that case.

The maximum eigenvalue of T is reported in Figure 2 (it can be established that the minimum eigenvalue is always close to 1). The left panel of Figure 2 shows the corresponding results for Poisson equation. We note that the iteration counts for the Stokes case are just slightly worse than for the Poisson case; see [PW02] for more complete results.

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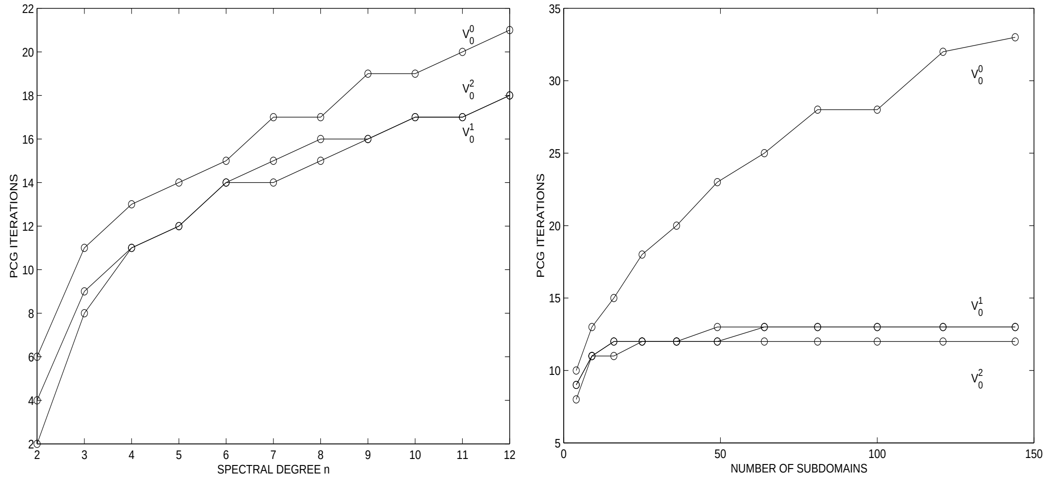

Figure 1: PCG iteration counts for the Stokes solver vs. spectral degree n when $N = 3 \times 3$ (left) and number of subdomains N when $n = 4$ (right)

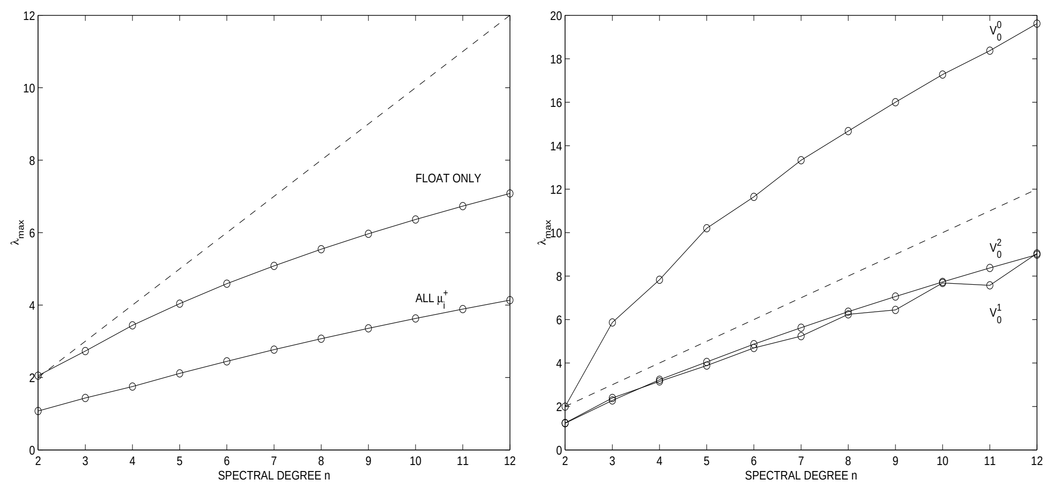

Figure 2: Maximum eigenvalue of the preconditioned operator vs. spectral degree n: Laplace solver (left) and Stokes solver (right)

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