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26 Multigrid for the Mortar-type Nonconforming Element Method for Nonsymmetric and Indefinite Problems

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Introduction

The mortar finite element method has been used to deal with non-overlapping domain decompositions. It can handle the situation where the mesh on different subdomains need not align across interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. In [2], Bernardi, Maday and Patera introduced basic concepts of general mortar elements, including the coupling of spectral elements with finite elements. Recently, many works have been done in constructing efficient iteration solvers for the discrete system resulting from the mortar element method. In [4], Gopalakrishnan and Pasciak presented a variable V-cycle preceonditioner, while Braess, Dahmen and Wieners [3] established another kind of W-cycle multigrid based on a hybrid formulation which gives rise to a saddle point problem. However, there are only few papers that are concerned with nonconforming elements, e.g. Marcinkowski [5] presented a P_1 nonconforming mortar element, but only for symmetric and definite problem. Meanwhile, an optimal multigrid for this method was given in [7].

The purpose of this paper is twofold. First, a mortar-type nonconforming element method is suggested for nonsymmetric and indefinite problems together with optimal error estimates. Second, a multigrid algorithm is proposed for the mortar element method which gives an optimal convergence rate, independent of the mesh size and mesh level. Finally, we describe the construction of the basis of the mortar-type nonconforming element space.

A model problem and the mortar element method

Consider the following model problem

$$\begin{cases} \mathcal{L}u = -\nabla \cdot (a\nabla u + b \cdot \nabla u) + du & = f \quad \text{in} \quad \Omega, \\ u & = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded polygonal domain, $a(x) = (a_{ij})$ is a uniformly symmetric positive definite tensor on $\bar{\Omega}$, $a_{ij}(x) \in C^1(\bar{\Omega})$, $b(x) \in (C^1(\bar{\Omega}))^2$, $d(x) \in C^0(\bar{\Omega})$, and $f \in L^2(\Omega)$.

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The variational form of (1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where the bilinear form

$$a(u,v) = (a\nabla u, \nabla v) - (b \cdot \nabla u, v) + (\tilde{d}u, v),$$

where $\tilde{d} = d - \nabla \cdot b$.

Assume problem (1) has the following regularity.

(**H1**). *For any $f \in L^2(\Omega)$, it holds that*

$$\|u\|_2 \leq C\|f\|_0.$$

We now introduce a mortar finite element method for solving (1). First, we partition Ω into nonoverlapping polygonal subdomains such that

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j.$$

They are arranged so that the intersection of $\Omega_i \cap \Omega_j$ for $i \neq j$ is either an empty set, an edge or a vertex, i.e., the partition is geometrically conforming. The interface

$$\Gamma = \bigcup_{i=1}^{N} \partial\Omega_i \backslash \partial\Omega$$

is broken into a set of disjoint open straight segments $\gamma_m (1 \leq m \leq M)$ (that are the edges of subdomains) called mortars, i.e.

$$\Gamma = \bigcup_{m=1}^{M} \bar{\gamma}_m, \quad \gamma_m \cap \gamma_n = \emptyset, \text{ if } m \neq n.$$

We denote the common open edge to Ω_i and Ω_j by γ_m. By $\gamma_{m(i)}$ we denote an edge of Ω_i which is a mortar and by $\delta_{m(j)}$ an edge of Ω_j that geometrically occupies the same place called nonmortar.

Let $\mathcal{T}_1^i$ be the coarsest triangulation of Ω_i with the mesh size h_1. The triangulation generally does not align at the subdomain interface. Denote the global mesh $\cup_i \mathcal{T}_1^i$ by $\mathcal{T}_1$. We refine the triangulation $\mathcal{T}_1$ to produce $\mathcal{T}_2$ by joining the mid-points of the edges of the triangles in $\mathcal{T}_1$. Obviously, the mesh size h_2 in $\mathcal{T}_2$ is $h_2 = h_1/2$. Repeating this process, we get the l-time refined triangulation $\mathcal{T}_l$ with mesh size $h_l = h_1 2^{-l+1}$ $(l = 1, ..., L)$. Let CR *nodal points* denote the nonconforming nodal points, i.e. the midpoints of the edges of the elements in $\mathcal{T}_l$. Moreover, on each level l, the sets of CR *nodal points* belonging to $\overline{\Omega}_i$, $\partial\Omega_i$ and $\partial\Omega$ are denoted by $\Omega_{l,i}^{CR}$, $\partial\Omega_{l,i}^{CR}$ and $\partial\Omega_l^{CR}$, respectively.

Define

$$Z = \{v | v|_{\Omega_i} \in H^1(\Omega_i), \ \forall i = 1, ..., N, \ v = 0 \text{ on } \partial\Omega\}.$$

On each level l, we define the P1 nonconforming element space locally and introduce the space $V_{l,i}(\Omega_i)$ whose functions are piecewise linear on each triangle of $\mathcal{T}_l^i$ and are continuous at the CR nodes of $\Omega_{l,i}^{CR} \backslash \partial\Omega_{l,i}^{CR}$, and equal zero at the CR nodes of $\partial\Omega_l^{CR}$.

Let

$$\tilde{V}_l = \prod_{i=1}^{N} V_{l,i} = \{v_l | v_l|_{\Omega_i} = v_{l,i} \in V_{l,i}\}.$$

Of course, we have

$$\tilde{V}_1 \not\subseteq \cdots \not\subseteq \tilde{V}_L.$$

Moreover, the P1 linear continuous finite element space over the triangulation $\mathcal{T}_l^i$ is denoted by $W_{l,i}$, whose functions have zero trace on $\partial\Omega$. Let

$$\tilde{W}_l = \prod_{i=1}^{N} W_{l,i},$$

for all $l = 1, ..., L$.

Obviously,

$$\tilde{W}_1 \subseteq \cdots \subseteq \tilde{W}_L.$$

and

$$\tilde{W}_l \subseteq \tilde{V}_l.$$

For any interface $\gamma_m = \gamma_{m(i)} = \delta_{m(j)}$, $1 \le m \le M$, there are two different and independent 1D triangulations $\mathcal{T}_l(\gamma_{m(i)})$ and $\mathcal{T}_l(\delta_{m(j)})$. Moreover, there are two sets of CR nodes belonging to γ_m: the midpoints of the elements belonging to $\mathcal{T}_l(\gamma_{m(i)})$ and to $\mathcal{T}_l(\delta_{m(j)})$ denoted by $\gamma_{l,m(i)}^{CR}$ and $\delta_{l,m(j)}^{CR}$ respectively. Additionly, we need an auxiliary test space $S_l(\delta_{m(j)})$ which is defined by

$$S_l(\delta_{m(j)}) \hat{=} \{v | v \in L^2(\delta_{m(j)}) \text{ and } v \text{ is piecewise constant}$$
$$\text{on the element of the nonmortar triangulation } \mathcal{T}_l(\delta_{m(j)})\}.$$

The dimension of $S_l(\delta_{m(j)})$ is equal to the number of midpoints on the $\delta_{m(j)}$, i.e. the number of elements on $\delta_{m(j)}$.

For each nonmortar $\delta_{m(j)}$, define an L^2-projection operator $Q_{l,\delta_{m(j)}} : L^2(\gamma_m) \to S_l(\delta_{m(j)})$ by

$$(Q_{l,\delta_{m(j)}} v, w)_{L^2(\delta_{m(j)})} = (v, w)_{L^2(\delta_{m(j)})} \quad \forall w \in S_l(\delta_{m(j)}),$$

where $(\cdot,\cdot)_{L^2(\delta_{m(j)})}$ denotes the L^2 inner product over the space $L^2(\delta_{m(j)})$.

Now we can introduce the following mortar finite element space for P1 nonconforming element on each level l:

$$V_l = \{v_l | v_l \in \tilde{V}_l,\ Q_{l,\delta_{m(j)}}(v_l|_{\delta_{m(j)}}) = Q_{l,\delta_{m(j)}}(v_l|_{\gamma_{m(i)}}),\ \text{for } \forall \gamma_m = \gamma_{m(i)} = \delta_{m(j)} \in \Gamma\}.$$

Define

$$\|v\|_{l,i} \hat{=} \sum_{K \in \mathcal{T}_l^i} \int_K \nabla v \cdot \nabla v dx \ \ \forall v \in V_{l,i},$$

and let

$$\|v\|_l^2 \hat{=} \sum_{i=1}^{N} \|v\|_{l,i}^2.$$

We know that $\|\cdot\|_l$ is a norm over the space V_l (see [5] for details).

Then the mortar element approximation of the problem (2) is to find $u_l \in V_l$ such that

$$a_l(u_l, v_l) = (f, v_l) \quad \forall v_l \in V_l, \tag{3}$$

where

$$\begin{aligned} a_l(u_l, v_l) &= a_l^s(u_l, v_l) + b_l(u_l, v_l) \\ a_l^s(u_l, v_l) &= \sum_{i=1}^{N} \sum_{K \in \mathcal{T}_l^i} (a \nabla u_l, \nabla v_l)_K \\ b_l(u_l, v_l) &= \sum_{i=1}^{N} \sum_{K \in \mathcal{T}_l^i} -(b \cdot \nabla u_l, v_l)_K + (\tilde{d} u_l, v_l). \end{aligned}$$

we can prove the following result.

Theorem 1 *Assume that u is the solution of (2), and $u_l \in V_l$ is the solution of (3). Then if h_l is sufficiently small, we have*

$$\|u - u_l\|_l \leq C(\sum_{i=1}^{N} h_{l,i}^2 \|u\|_{2,\Omega_i}^2)^{\frac{1}{2}}.$$

Proof. We only give a brief sketch. First we can prove

$$\begin{aligned} \|u - u_l\|_l \quad &\leq \quad C\{\|u - u_l\|_0 + \inf_{v_l \in V_l} \{\|u - v_l\|_0 + \|u - v_l\|_l\} \\ &\quad + \sup_{w_l \in V_l} \frac{|a_l(u, w_l) - (f, w_l)|}{\|w_l\|_{1,l}}\}. \end{aligned}$$

Then we can show that there exists an element $v_l \in V_l$ such that

$$\begin{aligned} &\|u - v_l\|_0 \leq C(\sum_{i=1}^{N} h_{l,i}^4 \|u\|_{2,\Omega_i}^2)^{1/2}, \\ &\|u - v_l\|_l \leq C(\sum_{i=1}^{N} h_{l,i}^2 \|u\|_{2,\Omega_i}^2)^{1/2}, \\ &\sup_{w_l \in V_l} \frac{|a_l(u, w_l) - (f, w_l)|}{\|w_l\|_{1,l}} \leq C(\sum_{i=1}^{N} h_{l,i}^2 \|u\|_{2,\Omega_i}^2)^{1/2}. \end{aligned}$$

Finally, using the idea of Schatz in [6], we can complete the proof.

Multigrid method

Due to the nonnestedness of the mesh spaces, we first introduce an intergrid transfer operator in this section. Based on this operator, a multigrid iterative method is suggested for solving (3). Some preliminary results are given in this section, which will be used to derive the convergence results of the multigrid. In the following, we always assume that the mesh sizes

$h_{l,i}$ for all i are comparable. The reason is that the convergence of multigrid always requires similar mesh parameters.

Define the operator $A_l : V_l \to V_l$ as:

$$(A_l v, w) = a_l(v, w) \quad \forall v, w \in V_l.$$

and

$$(\hat{A}_l v, w) = a_l^s(v, w) \quad \forall v, w \in V_l.$$

$$(B_l v, w) = b_l(v, w) \quad \forall v, w \in V_l.$$

It is easy to check that

$$A_l = \hat{A}_l + B_l.$$

Then (3) can be written as

$$A_l u_l = f_l,$$

where $(f_l, v) = f(v), \quad \forall v \in V_l$.

Before describing the algorithm, we must define a suitable intergrid transfer operator for the nonnested mesh space V_l. First, we give an operator $J_l^i : V_{l-1,i} \to W_{l,i}$ (see [7] for details) as follows:

- Case 1. If $p \in \Omega_{l-1,i}^{CR}$,
$$(J_l^i v)(p) = v(p).$$
- Case 2. If $p \in \Omega_{l,i}^N \backslash \Omega_{l-1,i}^{CR}$ and $p \notin \partial\Omega$,
$$(J_l^i v)(p) = \frac{1}{N(p)} \sum_{K_i} v|_{K_i}(p)$$
where $\Omega_{l,i}^N$ is the set of the vertices of the triangulation $\mathcal{T}_l^i$ that are in $\bar{\Omega}_i$ and the sum is taken over all triangles $K_i \in \mathcal{T}_l^i$ with the common vertex p and $N(p)$ is the number of those triangles.
- Case 3. If $p \in \partial\Omega \cap \partial\Omega_{l,i}^N$, then
$$(J_l^i v)(p) = 0,$$
where $\partial\Omega_{l,i}^N$ is the set of the vertices of the triangulation $\mathcal{T}_l^i$ that are in $\partial\Omega_i$.

Remark 1 Note that for different p, the value of $N(p)$ may be different. For example, if p is the vertex of triangular substructure Ω_i (see Fig.1 in [7]), then $N(p) = 1$ and if $p \in \Omega_{l,i}^N \backslash \partial\Omega_{l-1,i}^{CR}$, but $p \notin \partial\Omega_{l,i}^N$, then $N(p) = 6$, and if $p \in \partial\Omega_{l,i}^N$, but is not the vertex of substructure Ω_i and $p \notin \partial\Omega$, then $N(p) = 3$ (see Fig. 1. in [7] for details).

For the operator J_l^i, we have[7]

Lemma 1 *For $v \in V_{l-1,i}$, it holds that*

$$(1). \|J_l^i v\|_{l,i} \le C\|v\|_{l-1,i}.$$
$$(2). \|J_l^i v - v\|_0 \le C h_l \|v\|_{l-1,i}.$$
$$(3). \|J_l^i v - v\|_{0,\gamma_m} \le C h_l^{1/2} \|v\|_{l-1,i}.$$

where γ_m is an edge of Ω_i.

Proof. Please refer to [7] for details.

Based on the operator J_l^i, we define an intergrid transfer operator $J_l : \tilde{V}_{l-1} \to \tilde{V}_l$ as follows:

For any $v = (v_1, ..., v_N) \in \tilde{V}_{l-1}$,

$$J_l v = (J_l^1 v_1, ..., J_l^N v_N) \in \tilde{V}_l.$$

Moreover, the operator $\Xi_{l,\delta_{m(j)}} : \tilde{V}_l \to \tilde{V}_l$ is defined by

$$(\Xi_{l,\delta_{m(j)}}(v))(m_l) = \begin{cases} (Q_{l,\delta_{m(j)}}(v|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}}))(m_l) & m_l \in \delta_{l,m(j)}^{CR}, \\ 0 & \text{otherwise.} \end{cases}$$

Based on above preparation, we now define an intergrid transfer operator $I_l : \tilde{V}_{l-1} \to V_l$ which will appear in the following multigrid algorithm. For any $v \in \tilde{V}_{l-1}$,

$$I_l v = J_l v + \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(J_l v) \in V_l.$$

Lemma 2 *For the operator I_l, we have*

$$(1).\|I_l v\|_l \leq C\|v\|_{l-1};$$
$$(2).\|v - I_l v\|_0 \leq Ch_l\|v\|_{l-1}, \quad \forall v \in V_l.$$

Proof. Please refer to [7] for the proof.

Similar as in [1], we now describe an l-level scheme. The l-level iteration with initial guess z_0 yields $MG(l, z_0, G)$ as an approximation solution to the following problem:

Find $z \in V_l$, such that

$$a_l(z, v) = G(v) \quad \forall v \in V_l, \text{ where } G \in V_l'.$$

For $l = 1$, $MG(1, z_0, G)$ is the solution obtained by a direct method. For $l > 1$, $MG(l, z_0, G) = z_n + I_l q_p$, where $z_n \in V_l$ is constructed recursively from z_0 and the equations

$$z_i = z_{i-1} - \lambda_l^{-1}(G - A_l z_{i-1}) \quad 1 \leq i \leq n,$$

where λ_l is the largest eigenvalue of the operator $\hat{A}_l$. The coarse grid correction $q_p \in V_{l-1}$ is obtained by applying the $l-1$-level iteration p times ($p \geq 2$)

$$q_0 = 0, \ \ q_i = MG(l-1, q_{i-1}, \bar{G}), \quad 1 \leq i \leq p,$$

where $\bar{G} \in V_{l-1}'$ is defined by

$$\bar{G}(v) \hat{=} G(I_l v) - a_l(z_n, I_l v) \quad \forall v \in V_{l-1}.$$

Note that $q_p \in V_{l-1}$ is the approximation of $\bar{q}_{l-1} \in V_{l-1}$ which satisfies

$$a_{l-1}(\bar{q}_{l-1}, v) = \bar{G}(v), \quad \forall v \in V_{l-1}.$$

The main result of this paper is the following theorem

Theorem 2 *Let $p \geq 2$. If the number of the smoothing steps is large enough, and the coarsest mesh size h_1 is sufficiently small, then there exists $\delta \in (0,1)$, independent of l, such that if*

$$\|\bar{q}_{l-1} - q_p\|_{l-1} \leq C\delta^p \|\bar{q}_{l-1}\|_{l-1},$$

then

$$\|z - MG(l, z_0, G)\|_l \leq \delta \|z - z_0\|_l.$$

Proof. Here we also only provide a brief sketch. First we introduce a projection $P_{l-1} : V_l \to V_{l-1}$ defined by

$$a_{l-1}(P_{l-1}v, w) = a_l(v, I_l w), \;\; \forall v \in V_l, \; w \in W_{l-1}.$$

Then we can prove

$$\|v - I_l P_{l-1} v\|_{1,l} \leq Ch_l \|A_l v\|_0, \;\; \forall v \in V_l, \qquad (4)$$

$$\|P_{l-1} v\|_{1,l-1} \leq C\|v\|_{1,l}, \;\; \forall v \in V_l. \qquad (5)$$

Note that $e_{n+1} = e_n - I_l q_p$, we have

$$\|e_{n+1}\|_{1,l} \leq \|e_n - I_l \bar{q}_{l-1}\|_{1,l} + \|I_l(\bar{q}_{l-1} - q_p)\|_{1,l} \equiv E_1 + E_2,$$

where $\bar{q}_{l-1} = P_{l-1} e_n$.

Finally, using Lemmas 1-2 and (4)-(5) we can get

$$E_1 \leq C \frac{1}{n^{1/2}} (1 + Ch_1)^{2n} \|e_0\|_l,$$
$$E_2 \leq C\delta^p (1 + Ch_1)^2 \|e_0\|_l.$$

Therefore, we can choose $\delta \in (0,1)$, and obtain the desired result for sufficiently small h_1.

Construction of the basis

Let $\{y_l^i\}$ denote the CR nodes of $\mathcal{T}_l$. Define operator $\varepsilon_{l,\gamma} : Z \to \tilde{V}_l$ by

$$\varepsilon_{l,\gamma} \tilde{v}(y_l^i) = \begin{cases} (Q_{l,\delta_{m(j)}}(\tilde{v}_\gamma^M - \tilde{v}_\gamma^{NM}))(y_l^i), & \text{if} \quad y_l^i \in \delta_{m(j)} = \gamma, \\ 0, & \text{otherwise}, \end{cases}$$

where $\tilde{v}_\gamma^M$ and $\tilde{v}_\gamma^{NM}$ denote the restriction of $\tilde{v} \in Z$ on mortar $\gamma_{m(i)} = \gamma$ and nonmortar $\delta_{m(j)} = \gamma$ respectively. It is easy to see that if $\tilde{v}$ is in $\tilde{V}_l$ then $v = \tilde{v} + \sum_{\gamma \in \Gamma} \varepsilon_{l,\gamma} \tilde{v}$ is an element of V_l.

Let $\{\tilde{\phi}_l^i : i = 1, \cdots, \tilde{N}_l\}$ be the basis of $V_{l,i}$. Then the basis of V_l consists of functions of the form

$$\phi_l^i = \tilde{\phi}_l^i + \sum_{\gamma \in \Gamma} \varepsilon_{l,\gamma}(\tilde{\phi}_l^i).$$

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