Fourteenth International Conference on Domain Decomposition Methods
Editors: Ismael Herrera , David E. Keyes, Olof B. Widlund, Robert Yates

34. On Aitken Like Acceleration of Schwarz Domain Decomposition Method Using Generalized Fourier

J.Baranger[1], M.Garbey[2] and F.Oudin-Dardun[3]

1. Introduction. The idea of using Aitken acceleration [5] [11], on the classical Schwarz additive domain decomposition (dd) method [9] [7] [8] [6] has been introduced in [3]. For an elliptic operator with constant coefficients on a regular grid, this method is called Aitken-Schwarz (AS) procedure, and is a direct solver. This method has shown very good numerical performance, and has been used in more complex situations [4]. We have also extended the Aitken-Schwarz procedure to the case of a 2-D cartesian grid, *not necessarily regular,* with two subdomains [1].

In the present paper, we extend this method to more complex situations and give a general framework for our method. We first consider overlapping strip domain decomposition with P domains on a *non-uniform* Cartesian grid. The key idea is the replacement of the 1-D Fourier transform used on the regular space step discretization of the artificial interface grid by a transform using the eigenvectors of a suitable 1-D operator. We give a direct solver version of the Aitken-Schwarz algorithm for arbitrary number of subdomains, as well as an iterative version when acceleration is applied to dominant modes only. In this last case, the number of iterates is not sensitive to the number of subdomains with overlap of few mesh steps. Second, we consider non-matching grids to apply our method to non-trivial geometries. We present some experimental results for the Poisson and Helmholtz operator and comment on an adaptive version of our acceleration technique for incompressible unsteady flow in a channel past a disc. This paper is restricted to problems in two space dimensions, but most of the concepts introduced here can be extended to 3 space dimensions.

2. A general framework. We briefly describe a general framework for AS method. For more details see [2]. The AS method is built on three ideas:

- Schwarz's method is an iterative method on a trace transfer operator acting on functions defined on the interfaces. Sparsity of the Jacobian of this operator is related to the domain decomposition (**dd**).

- discretization and choice of the interface representation may in some cases, and if well chosen, increase this sparsity.

- for an operator with a sparse matrix, simple acceleration processes can be constructed. The Aitken process, for example, provides an exact solver in the linear case if the trace transfer operator can be diagonallized .

• **Trace transfer operator for Schwarz iterative method:** we consider a bounded domain Ω in $\mathbb{R}^N$ with a strip dd in P domains Ω_p, i.e Ω_p only intersects Ω_{p-1} and Ω_{p+1}, with obvious modifications for p=1 and P.

The boundary Γ_p of Ω_p is decomposed into three subsets: Γ_p^l (resp.Γ_p^r) included in Ω_{p-1} (resp.Ω_{p+1}) and the remaining part $\tilde{\Gamma}_p$.

Let (Π) be a boundary value problem (**bvp**) well posed in Ω. One step of the additive Schwartz dd method with Dirichlet-Dirichlet boundary conditions (**bc**) is: for all p, given the Dirichlet bc l_p (resp.r_p) on Γ_p^l (resp.Γ_p^r) solve the problem (Π_p) the restriction of (Π) to Ω_p with these bc and the one of (Π) on $\tilde{\Gamma}_p$.

(Π_p) is assumed to be well posed. We denote $\bar{r}_{p-1}$ (resp. $\bar{l}_{p+1}$) the trace of the solution of (Π_p) on Γ_{p-1}^r (resp. Γ_{p+1}^l). So, one step of Schwarz method is described by one application

[1]MCS-ISTIL-Universit Lyon1, baranger@mcs.univ-lyon1.fr
[2]Dept. of Computer Science-University of Houston, garbey@cs.uh.edu
[3]MCS-ISTIL-Universit Lyon1, foudin@mcs.univ-lyon1.fr

of the trace transfer operator

$$(\bar{l}_2, \bar{r}_1, \ldots, \bar{l}_P, \bar{r}_{P-1}) = T(l_2, r_1, \ldots, l_p, r_{p-1}, \ldots, l_P, r_{P-1})$$

acting on trace spaces of functions or distributions adapted to the bvp.

T has the special structure:

$$\bar{l}_2 = T_1^r(r_1), \quad \ldots, \quad \left\{ \begin{array}{l} \bar{r}_{p-1} = T_p^l(l_p, r_p) \\ \bar{l}_{p+1} = T_p^r(l_p, r_p) \end{array} \right\} p = 2 \text{ to } P-1, \quad \ldots, \quad \bar{r}_{P-1} = T_P^l(l_P)$$

Here $(\bar{r}_{p-1}, \bar{l}_{p+1}) = T_p(l_p, r_p)$ is composed of a local solver of the bvp (Π_p) and the trace operators on Γ_{p-1}^r and Γ_{p+1}^l. These operators can be exact or approximated.

Formally, the 2(P-1) Jacobian matrix of T has the pentadiagonal structure, pointed out for a special case in [3]:

$$\begin{pmatrix} 0 & \delta_1^{rr} & 0 & 0 & & & \\ \delta_2^{ll} & 0 & 0 & \delta_2^{lr} & 0 & & \\ \delta_2^{rl} & 0 & 0 & \delta_2^{rr} & 0 & 0 & \\ 0 & 0 & \delta_3^{ll} & 0 & 0 & \delta_3^{lr} & \ddots \\ & 0 & \delta_3^{rl} & 0 & 0 & \delta_3^{rr} & \ddots \\ & & 0 & 0 & \delta_4^{ll} & 0 & \ddots \\ & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.1}$$

with $\delta_p^{lr} = \partial T_p^l / \partial r_p(l_p, r_p)$. The derivatives are assumed to exist in some sense in the traces functional spaces.

• **Discretization and interface representation:** we introduce a discrete approximation of the traces. Each trace l_p (resp. r_p) is approximated by J numbers l_{pj} (resp. r_{pj}), $j = 1$ to J. These numbers may be point values, coefficients in a basis, and so on. J may vary with p if, for example, one has non-matching grids between subdomains. Retaining the previous notations, l_p and r_p are now J-vectors and $\delta_p = \begin{pmatrix} \delta_p^{ll} & \delta_p^{lr} \\ \delta_p^{rl} & \delta_p^{rr} \end{pmatrix}$ is a 2J square matrix. T is an application from $\mathbb{R}^{2J(P-1)}$ into itself with a sparse Jacobian matrix. For some problems, dd and meshes, a well-chosen change of unknowns $l_{pj} \to \hat{l}_{pj}$ may greatly increase the sparsity of the Jacobian of the transformed trace transfer operator $\hat{T}$. This idea which is the core of AS method has been introduced on a uniform mesh -using Fourier transform- in [3]. An extension to non-uniform rectangular meshes is given in the next section.

• **Acceleration process:** Schwarz method can be considered as an iterative method for the hat transform (associated with the interface representation) of T which map vectors of size 2J(P-1). Any acceleration process can be used. The AS method uses Aitken method, taking advantage of the sparsity coming on the one hand from the special dd, and on the other hand from the generalized Fourier transform.

In the next section, we describe a special situation in which a good choice of the interface representation leads to a very sparse Jacobian. For proofs and extensions of the content of this section, we refer the reader to[2].

3. Generalized Fourier transform and interface representation on a non-uniform rectangular mesh. We restrict ourselves to two space dimensions and a rectangular domain Ω with a strip dd into rectangles. The left (resp. right) boundary of Ω_p is $x = x_p^l$ (resp. x_p^r). (Π) is a homogenous Dirichlet bvp whose equation $Lu = f$ has a separable second order operator $L = L_1 + L_2$ with

$$L_1 = a_1 \partial_{xx} + b_1 \partial_x + c_1, \quad L_2 = a_2 \partial_{yy} + b_2 \partial_y + c_2.$$

a_1, b_1, c_1 are functions of x, and a_2, b_2, c_2 functions of y.

The grid is a tensorial product of the following two irregular meshes: x-mesh $x_i, i \in \mathbf{I}$, and y-mesh $y_j, j = 0$ to $J+1$. L_1^h (resp.L_2^k) are discretization of L_1 (resp. L_2) on the x (resp. y)-mesh. The unknowns u_{ij} are approximations of $u(x_i, y_j)$ with f_{ij} some given approximation of $f(x_i, y_j)$. We use the notation $U_j = (u_{ij})_{i \in \mathbf{I}}$.

Let [e,w]x[n,s] be the generic rectangular subdomain. The discrete approximation (Π_p^{hk}) of problem (Π_p) can be written

$$L_1^h U_j + L_2^k U_j = F_j,\ U_j(w) \text{ and } U_j(e) \text{ given},\ u_{i0} = u_{iJ+1} = 0, i \in \mathbf{I}$$

The following result is proved in [2]:

Theorem 3.1 *Assume that the eigenvalue problem*

$$L_2^k \Phi_m = \lambda_m \Phi_m,\ \ \Phi_{m0} = \Phi_{mJ+1} = 0 \qquad (E)$$

has J linearly independent real eigenvectors associated with real eigenvalues. We define the generalized Fourier transform:

$$u_{ij} = \sum_{m=1}^{J} \hat{u}_{im} \Phi_{mj},\ \ j = 1 \text{ to } J.$$

Then $(\hat{\Pi}_p^{hk})$ -the hat transform of (Π_p^{hk})- is a set of J uncoupled discrete one-dimensional linear problems:

$$[L_1^h + \lambda_m]\hat{U}_m = \hat{F}_m,\ \ m = 1 \text{ to } J,\ \hat{u}_{0m} \text{ and } \hat{u}_{I+1m} \text{ given}\ .$$

The hat trace transfer operator is affine on $\mathbb{R}^{2J(P-1)}$ with a block-diagonal matrix of J blocks. The m-th diagonal block has the form (2.1) and corresponds to the mode Φ_m and the operator $L_1^h + \lambda_m$.

We are going to apply this result to construct the AS algorithm.

3.1. Algorithm. We apply an Aitken-like acceleration procedure to each mode of the generalized Fourier transform of the interfaces values given by Schwarz dd method. It follows from theorem 3.1 that the method is an exact solver in this context. The algorithm is:

Step 1: compute the eigenvectors $\lambda_m, \Phi_m, m = 1$ to J solution of problem (E).

Step 2: given traces on the interfaces, perform 3 steps of the Schwarz method.

Step 3: take the generalized Fourier transform of the last 4 traces.

Step 4: apply the one-dimension Aitken acceleration formula to each mode of these transformed traces.

Step 5: recompose the physical traces from the result of step 4.

Step 6: from these traces, make one step of the Schwarz method.

We observe that Step 1, 3, 5 and 6 can be processed in parallel. Step 2 is the additive Schwarz algorithm that has, in general, poor numerical efficiency but scales very well on a so called MIMD architecture. Step 4 requires global communication of the hat transform of the traces but makes the numerical algorithm efficient.

In order to minimize the amount of global communications in the parallel algorithm and decrease the number of arithmetic operations, it is interesting to accelerate only the eigenvector components of the traces that correspond to dominant eigenvalues $\lambda_m,\ m = 1..J'$, with $J' < J$. As a matter of fact, eigenvector components that corresponds to small eigenvalues λ_m converge fast with the Schwarz method itself. In that case, steps 3 and 5 are modified and the direct and inverse hat transforms use only the $J' < J$ first modes. Further, we may have to iterate step 2 to step 6 until convergence. We call this variant of our method as the Steffensen-Schwarz method.

We are going to apply this result to the Poisson problem discretized by FE as done in [1] on a rectangular irregular grid.

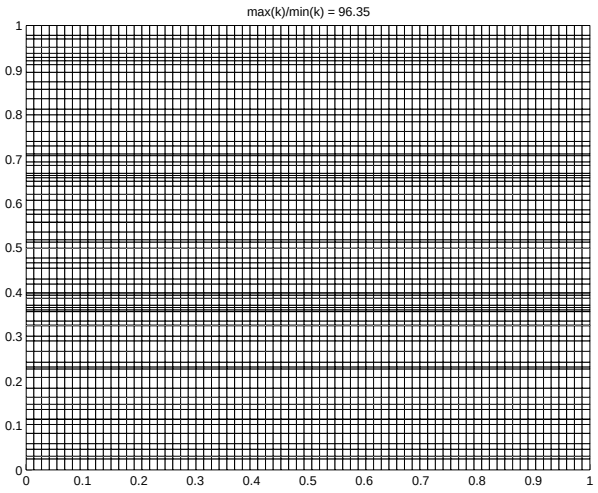

Figure 3.1: Mesh

3.2. Numerical experiments. We consider, on the domain $\Omega =]0,1[\times]0,1[$, the Poisson problem : $-(u_{xx}+u_{yy}) = f$, with $u = 0$ on $\partial\Omega$, such that the exact solution is : $u(x,y) = 150x(x-1)y(y-1)(y-1/2)$. We use a Cartesian grid of Ω with 73×73 elements, uniform in x, random in y (see Figure 3.1).

In Figure 3.2, we compare the error and the residual according to the number of subdomaims, and the number of modes that are accelerated. The error versus the exact discrete solution is then order 10^{-6}, after one Aitken acceleration, and becomes of order 10^{-5} with $J' = \frac{J}{2}$, *regardless of the number of subdomains used.*

Figure 3.3 shows error and residual at the first and second iteration, for different number of modes, and different sizes of overlap. We conclude that the larger the overlap, the better is the acceleration. These results suggest that one should adaptively select the minimum number of modes to accelerate as a function of the overlap and subdomain sizes. This is an essential feature of our method that may provide parallel scalability and should be the topic of further investigation.

We are going now to consider non-matching grids and application to CFD problems.

4. Experiments with Steffensen-Schwarz and Non-overlapping grids. We consider elliptic solvers with Dirichlet bc in a non-trivial geometric domain that are component of Navier Stokes incompressible flow simulations around obstacles. A good example is the two-dimensional test case proposed by Shäfer & Turek in [10] of incompressible flow in a straight channel around a disc. The domain Ω is $(0, L_x) \times (0, L_y)$ with a circular hole of radius R centered in (x_o, y_o). $\partial\Omega^R$ is the boundary of the rectangle and $\partial\Omega^C$ is the boundary of the disc. The linear elliptic solver corresponds either to the Poisson or the Helmholtz operator $-\epsilon\Delta + Id$. Figure 4.1 gives an illustration of the two non-matching grids that we do consider. This splitting of the domain is motivated by the physics for large Reynolds number. The boundary layer is approximated on the grid Ω^C in polar coordinates and the Cartesian grid Ω^R is used to approximate the main part of the flow. The overlap between subdomains is of the order of one mesh step of Ω^R.

We denote Γ^R (resp. Γ^C), the artificial boundary of the rectangular mesh Ω^R (resp. the mesh in polar coordinates Ω^C). If L^R (resp. L^C) represents the standard finite difference approximation of our linear operator on Ω^R in Cartesian coordinates, (resp. on Ω^C in polar

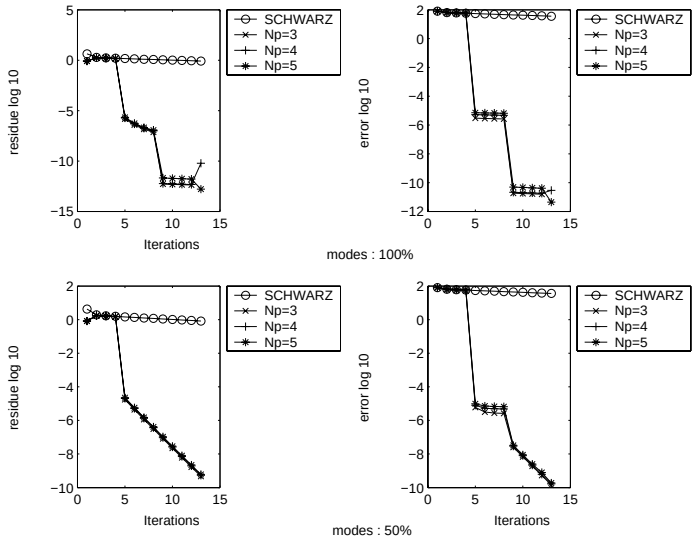

Figure 3.2: Error and residual - Np number of subdomains

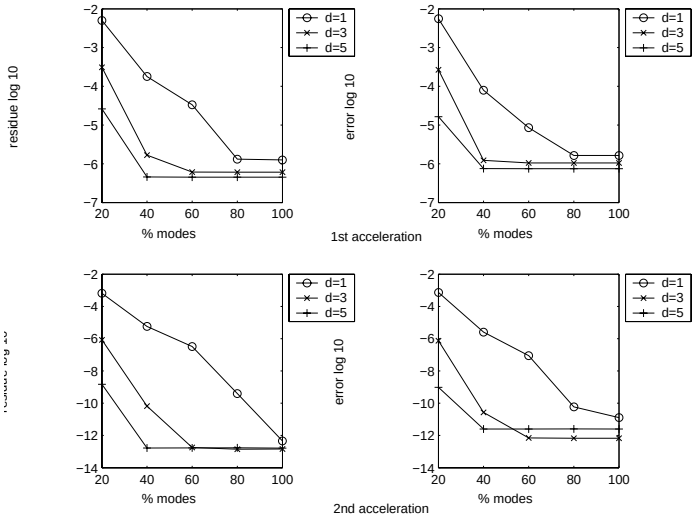

Figure 3.3: Error and residual - size d of overlap

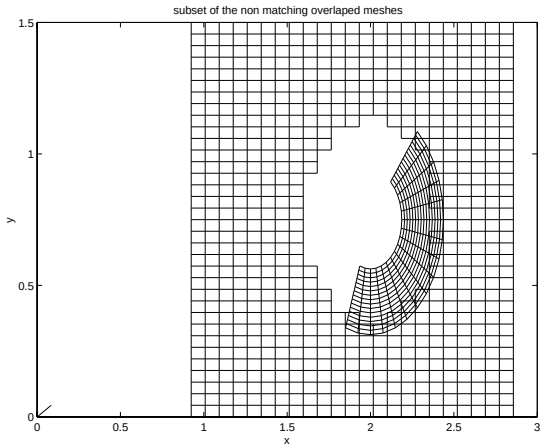

Figure 4.1: Representation of a subset of the overlapped non matching grids around the cylinder

coordinates), we close our discrete approximation problem by imposing:

$$I_R^C(U^R) = U^C \; on \; \Gamma^C, \; I_C^R(U^C) = U^R \; on \; \Gamma^R, \tag{4.1}$$

where I_C^R and I_R^C are linear second-order interpolation operators that satisfy a maximum principle. The discrete problem can be written as:

$$L^R[U^R] = f^R \; in \; \Omega^R, \; L^C[U^C] = f^C \; in \; \Omega^C, \tag{4.2}$$

with matching conditions (4.1), and Dirichlet bc on $\partial\Omega^R \bigcup \partial\Omega^C$.

The discrete solution process is the following alternate Schwarz iterative procedure:

$$L^R[U_n^R] = f^R, \; in \; \Omega^R, \; U_n^R = I_C^R(U_{n-1}^C) \; on \; \Gamma^R,$$

followed by $L^C[U_n^C] = f^C, \; in \; \Omega^C, \; U_n^C = I_R^C(U_n^R) \; on \; \Gamma^C$, using the corresponding Dirichlet bc on $\partial\Omega^R \bigcup \partial\Omega^C$, and an initial value for the artificial bc U_0^C.

From the maximum principle satisfied by the discrete operators L^R and L^C as well as the maximum principle satisfied by the interpolant operator I_C^R and I_R^C, one concludes the linear convergence of this iterative scheme to the unique solution of (4.2, 4.1), with Dirichlet bc on $\partial\Omega^R \bigcup \partial\Omega^C$. One applies then the Steffensen-Schwarz method described in [3] on the interface operator $U_{n\ |\Gamma^C}^C \rightarrow U_{n+1\ |\Gamma^C}^C$. To be more precise, let $\hat{U}^C = \sum_{k=-N/2,...,N/2} \hat{U}_k^C \; e^{ik\Theta}$ be the Fourier expansion of the discrete function U^C restricted to the circle Γ^C. The matrix P of the interface operator $(U_{n\ |\Gamma^C}^C \rightarrow U_{n+1\ |\Gamma^C}^C)$ in the set of basis function $e^{ik\Theta}, \; k = -N/2, ..., N/2$ satisfies

$$(\hat{U}_{n+1\ |\Gamma^C}^C - \hat{U}_{\infty\ |\Gamma^C}^C) = P \; (\hat{U}_{n\ |\Gamma^C}^C - \hat{U}_{\infty\ |\Gamma^C}^C).$$

One reconstructs a bandwidth approximation of P of size Z from the knowledge of the partial sequence $(\hat{U}_{0\ |\Gamma^C}^C, ..., \hat{U}_{n+Z+2\ |\Gamma^C}^C)$. The Aitken-like acceleration procedures can be written:

$$\hat{U}_{\infty\ |\Gamma^C}^C = (Id - P)^{-1} \; (\hat{U}_{n+1\ |\Gamma^C}^C - P \; \hat{U}_{n\ |\Gamma^C}^C). \tag{4.3}$$

We have observed that this Steffensen-Schwarz procedure is numerically most efficient with diagonal approximation of P. Each cycle of Steffensen-Schwarz algorithm requires two Schwarz

iterates to get from the sequence of interfaces, the diagonal approximation of P, and then one more Schwarz iterate to exploit the bc on the artificial interfaces (4.3). These Poisson or Helmholtz solvers have been used to solve an unsteady incompressible Navier Stokes (**NS**) equation written in Vorticity-Stream function ($\omega - \psi$) formulation, for the two-dimensional test case proposed by Shäfer & Turek in [10]. The main cost of the NS solution procedure corresponds to the Poisson problem for the stream function. The application of Steffensen-Schwarz procedure to ψ *at every time step* can take advantage of two interesting features. First, the initial guess for the trace of the stream function on the circle Γ^C in the iterative procedure is a second-order extrapolation in time of this trace value of ψ using the two previous time step's solution,i.e $\Psi_0^C = 2\Psi^C(t_n) - \Psi^C(t_{n-1})$. The diagonal approximation of the trace transfer operator T should be time-independent, but is in fact, with our approximation technique, solution-dependent. In practice, one can reuse the same diagonal approximation of P for O(10) time steps. The Steffensen-Schwarz cycle then reduces to two Schwarz iterates for those time steps that keep the same P approximation than the previous time step. For oscillatory flow with moderately large Reynolds number, time steps satisfying the CFL condition and grids of order 100×100, we can typically maintain the residual of order 10^{-6} with only one Steffensen-Schwarz cycle per time step.

5. Conclusion. We have presented a generalization of Aitken-Schwarz method [3] to grids that are tensorial products of one-dimensional grids with irregular mesh stepping and domain decomposition with non-matching grids. Our current work addresses the problem of the generalization of this method to unstructured meshes with Finite Volume approximation.

REFERENCES

[1] J. Baranger, M. Garbey, and F. Oudin-Dardun. Recent development on Aitken-Schwarz method. In N.Debit, M.Garbey, R.Hoppe, J.Periaux, D.Keyes, and Y.Kuznetsov, editors, *Domain Decomposition Methods in Science and Engineering*. CIMNE, UPS, Barcelone, 2001. to appear.

[2] J. Baranger, M. Garbey, and F. Oudin-Dardun. A generalisation of Aitken-Schwarz method on non uniform cartesian grids. in preparation, 2002.

[3] M. Garbey and D. Tromeur-Dervout. Operator splitting and domain decomposition for multi-clusters. In D. Keyes, A. Ecer, N. Satofuka, P. Fox, and J. Periaux, editors, *Proc. Int. Conf. Parallel CFD99*, pages 27–36. North-Holland, 2000.

[4] M. Garbey and D. Tromeur-Dervout. Two level domain decomposition for multi-clusters. In T. Chan and all editors, editors, *Domain Decomposition in Sciences and Engineering*, pages 325–339. DDM.org, 2001.

[5] P. Henrici. *Elements of Numerical Analysis*. John Wiley & Sons Inc, New York-London-Sydney, 1964.

[6] Y. A. Kuznetsov. Overlapping domain decomposition methods for fe-problems with elliptic singular perturbed operators. In R. Glowinski, Y. A. Kuznetsov, G. A. Meurant, J. Périaux, and O. Widlund, editors, *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1991. SIAM.

[7] P.-L. Lions. On the Schwarz alternating method. I. In R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 1–42, Philadelphia, PA, 1988. SIAM.

[8] P.-L. Lions. On the Schwarz alternating method. II. In T. Chan, R. Glowinski, J. Périaux, and O. Widlund, editors, *Domain Decomposition Methods*, pages 47–70, Philadelphia, PA, 1989. SIAM.

[9] H. A. Schwarz. *Gesammelte Mathematische Abhandlungen*, volume 2, pages 133–143. Springer, Berlin, 1890. First published in Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, volume 15, 1870, pp. 272–286.

[10] M. Shäfer and S. Turek. Benchmark computations of laminar flow around cylinder. In *Flow Simulation with High-Performance Computers II*. Vieweg, 1996.

[11] J. Stoer and R. Burlish. *Introduction to numerical analysis.* TAM 12 Springer, New York, 1980.