Fourteenth International Conference on Domain Decomposition Methods
Editors: Ismael Herrera , David E. Keyes, Olof B. Widlund, Robert Yates

2. Non conforming domain decomposition: the Steklov-Poincaré operator point of view

S. Bertoluzza[1]

1. Introduction. One of the common approaches to solve the linear system arising in the domain decomposition method is to formally reduce it, by a Schur complement argument, to a lower dimensional linear system whose unknown is the value of the (discrete) solution on the interface of the decomposition. Solving such reduced linear system by any iterative technique implies the need of solving, at each iteration, independent discrete Dirichlet problems in the subdomains. Such Dirichlet problems constitute the most relevant part of the computational cost of such an approach and therefore attention needs to be paid in reducing the actual computational cost of the subdomain solvers. A key observation in this respect is that what one expects as an output of the iterative procedure is a (correct order) approximation of the trace of the solution u on the interface. There is no direct need of solving correctly the Dirichlet problems in the subdomains. The precision with which such problems are solved is only as relevant as its influence on the error on the trace of u on the interface. Only once the trace of u on the interface has been computed correctly, one will actually need to retrieve the solution in some or all of the subdomains.

In order to take advantage of this observation it is useful to look at the Schur complement linear system as non conforming discretization of the Steklov-Poicaré operator, mapping a function φ defined on the interface, to the jump of the normal derivative of its harmonic lifting (computed subdomain-wise). The non-conformity stems from replacing the harmonic lifting with its discretization. If we look at the Schur complement system from this point of view, a straightforward application of the first Strang Lemma, shows that the discretization in the subdomains needs to be designed in order to provide a correct order approximation of outer normal derivative, while there is no direct need to actually provide a good approximation of the solution u in the interior of the subdomains.

The aim of this paper is to formalise the above considerations in the case in which the starting domain decomposition formulation is the *three fields formulation*, and to provide a rigorous error estimate for the trace of u on the the interface, showing that the mesh can actually be chosen to be sensibly coarser in the interior of the subdomains without affecting the precision of the interface approximation, resulting in a sensible reduction in computational cost of the subdomain solvers.

2. The three fields formulation and the Steklov-Poincaré operator. Here and in the following we will use the notation $A \lesssim B$ and $A \gtrsim B$ to indicate that the quantity A is bounded from above – resp. from below – by a positive constant times the quantity B, the constant being independent of any relevant parameter, like the mesh size. The expression $A \simeq B$ will stand for $A \lesssim B \lesssim A$.

[1]IMATI-CNR, Pavia (Italy), silvia.bertoluzza@ian.pv.cnr.it

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. We will consider the following simple model problem: given $f \in L^2(\Omega)$, find u satisfying

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{2.1}$$

To fix the ideas, we will consider consider the *three fields domain decomposition* formulation of such a problem [4]. We want to underline however that the general ideas presented here carry over to many other domain decomposition formulations, both conforming and non-conforming. Considering for simplicity a geometrically conforming decomposition $\Omega = \cup_k \Omega_k$, with Ω_k convex shape-regular polygons, $\Gamma_k = \partial\Omega_k$, and letting $\Sigma = \cup_k \Gamma_k$, we introduce the following functional spaces

$$V = \prod_k H^1(\Omega_k), \qquad \Lambda = \prod_k H^{-1/2}(\Gamma_k),$$
$$\Phi = \{\varphi \in L^2(\Sigma) : \text{there exists } u \in H^1_0(\Omega),\ u = \varphi \text{ on } \Sigma\} = H^1_0(\Omega)|_\Sigma,$$

respectively equipped with the norms:

$$\|u\|_V^2 = \sum_k \|u^k\|^2_{H^1(\Omega_k)}, \qquad \|\lambda\|^2_\Lambda = \sum_k \|\lambda^k\|^2_{H^{-1/2}(\Gamma_k)},$$

and (see [2])

$$\|\varphi\|^2_\Phi = \inf_{u \in H^1_0(\Omega): u = \varphi \text{ on } \Sigma} \|u\|^2_{H^1(\Omega)} \simeq \sum_k |\varphi|^2_{H^{1/2}(\Gamma_k)}.$$

Let $a^k : H^1(\Omega_k) \times H^1(\Omega_k) \to \mathbb{R}$ denote the bilinear form corresponding to the Laplace operator:

$$a^k(w, v) = \int_{\Omega_k} \nabla w \nabla v.$$

The continuous three fields formulation of equation (2.1) is the following ([4]): find $(u, \lambda, \varphi) \in V \times \Lambda \times \Phi$ such that

$$\begin{cases} \forall k, \quad \forall v^k \in H^1(\Omega_k), \quad \forall \mu^k \in H^{-1/2}(\Gamma_k) : \\ \quad a^k(u^k, v^k) \ - \int_{\Gamma_k} v^k \lambda^k \qquad\qquad = \int_{\Omega_k} f v^k, \\ -\int_{\Gamma_k} u^k \mu^k \qquad\qquad + \int_{\Gamma_k} \mu^k \varphi = 0, \\ \\ \text{and } \forall \psi \in \Phi : \\ \qquad\qquad \sum_k \int_{\Gamma_k} \lambda^k \psi \qquad = 0. \end{cases} \tag{2.2}$$

It is known that this problem admits a unique solution (u, λ, φ), where u is indeed the solution of (2.1) and such that $\lambda^k = \partial u^k / \partial \nu^k$ on Γ_k, and $\varphi = u$ on Σ, where ν^k denotes the outer normal derivative to the subdomain Ω_k.

After choosing discretization spaces $V_h = \prod_k V_h^k \subset V$, $\Lambda_h = \prod_k \Lambda_h^k \subset \Lambda$ and $\Phi_h \subset \Phi$, equation (2.2) can be discretized by a Galerkin scheme, yielding the following problem: find $(u_h, \lambda_h, \varphi_h) \in V_h \times \Lambda_h \times \Phi_h$ such that

$$
\begin{cases}
\forall k, \quad \forall v_h^k \in V_h^k, \quad \forall \mu_h^k \in \Lambda_h^k: \\
\quad a^k(u_h^k, v_h^k) \quad - \int_{\Gamma_k} v_h^k \lambda_h^k \qquad\qquad\qquad = \int_{\Omega_k} f v_h^k, \\
-\int_{\Gamma_k} u_h^k \mu_h^k \qquad\qquad\qquad\qquad + \int_{\Gamma_k} \mu_h^k \varphi_h = 0, \\
\\
\text{and } \forall \psi_h \in \Phi_h: \\
\qquad\qquad\qquad \sum_k \int_{\Gamma_k} \lambda_h^k \psi_h \qquad\qquad = 0.
\end{cases}
\tag{2.3}
$$

Existence, uniqueness and stability of the solution of the discretized problem rely on the validity of two *inf-sup* conditions,

$$
\inf_{\lambda_h \in \Lambda_h} \sup_{u_h \in V_h} \frac{\sum_k \int_{\Gamma_k} \lambda_h^k u_h^k}{\|u_h\|_V \|\lambda_h\|_\Lambda} \geq \beta_1 > 0, \qquad \inf_{\varphi_h \in \Phi_h} \sup_{\lambda_h \in \Lambda_h} \frac{\sum_k \int_{\Gamma_k} \lambda_h^k \varphi_h}{\|\varphi_h\|_\Phi \|\lambda_h\|_\Lambda} \geq \beta_2 > 0 \tag{2.4}
$$

respectively coupling V_h with Λ_h, and Λ_h with Φ_h. Provided (2.4) holds, it is well known ([3]) that we can derive the following error estimate:

$$
\|u-u_h\|_V+\|\lambda-\lambda_h\|_\Lambda+\|\varphi-\varphi_h\|_\Phi \lesssim \inf_{v_h \in V_h} \|u-v_h\|_V + \inf_{\mu_h \in \Lambda} \|\lambda-\mu_h\|_\Lambda + \inf_{\psi_h \in \Phi_h} \|\varphi-\psi_h\|_\Phi.
$$

The linear system stemming from such an approximation takes the form

$$
\begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{u}_h \\ \underline{\lambda}_h \\ \underline{\varphi}_h \end{pmatrix} = \begin{pmatrix} \underline{f} \\ 0 \\ 0 \end{pmatrix}, \tag{2.5}
$$

($\underline{u}_h$, $\underline{\lambda}_h$, and $\underline{\varphi}_h$ being the vectors of the coefficients of u_h, λ_h and φ_h in the bases chosen for V_h, Λ_h and Φ_h respectively). The usual approach to the solution of such linear system is to reduce it, by a Schur complement argument, to the solution of a system in the unknown $\underline{\varphi}_h$, which takes the form

$$
\mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T \, \underline{\varphi}_h = -\mathbf{C}\mathbf{A}^{-1} \begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}, \quad \mathbf{C} = [\ 0 \quad C\ ], \quad \mathbf{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}. \tag{2.6}
$$

The matrix $S = \mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T$ does not need to be assembled. The system (2.6) is then solved by an iterative technique (like for instance a conjugate gradient method), for which only the action of S on a given vector needs to be implemented. In particular, multiplying by S implies the need for solving a linear system with matrix $\mathbf{A}$. This reduces, by a proper reordering of the unknowns, to independently solving a discrete Dirichlet problem with Lagrange multipliers in each subdomain. A key observation is that the significant unknown that one is looking for is φ, that is the trace on Σ of the solution u of the equation considered. The actual value of the function u_h and of the multiplier λ_h is only needed at the end of the iterative procedure and possibly only in some of the subdomains, namely the ones in which the end user is actually interested in computing the solution. Along the iterations, the precisions with which u_h and λ_h approximate u and λ respectively is only as important as its effect on the precision with which φ is approximated. From this point of view it would for instance make sense to replace, along the iterations, the discretization spaces V_h and Λ_h with two other spaces V_h^* and Λ_h^* with $dim(V_h^* \oplus \Lambda_h^*) \ll dim(V_h \oplus \Lambda_h)$ – resulting in a reduction of CPU time in the solution of the discrete Dirichlet problems at each iteration – provided this does not reduce the precision of the approximation of the unknown φ. In this respect, the above mentioned error estimate is pessimistic. In order to obtain a sharper error estimate on the error $\|\varphi - \varphi_h\|_\Phi$ we can look at the linear system (2.6) as a non conforming discretization of the Steklov-Poincaré problem

$$\mathcal{S}\varphi = g \tag{2.7}$$

where we recall that the Steklov-Poincaré operator $\mathcal{S} : \Phi \to \Phi'$ is defined as

$$\langle \mathcal{S}\varphi, \psi\rangle = \sum_k \langle \partial_{\nu^k} \mathcal{L}_H^k \varphi, \psi\rangle$$

where $\mathcal{L}_H^k : H^{1/2}(\Gamma_k) \to H^1(\Omega_k)$ denotes the harmonic lifting:

$$-\Delta(\mathcal{L}_H^k\varphi) = 0, \quad \text{on } \Omega_k, \qquad \mathcal{L}_H^k\varphi = \varphi, \quad \text{on } \Gamma_k,$$

and where $g = g(f)$ is the jump along the interface of the normal derivative of the function u^f verifying $-\Delta u^f = f$ in each Ω_k and $u^f = 0$ on Σ.

The linear system (2.6) is indeed a discrete version of (2.7), the non conformity stemming from the fact that in the computation of the Steklov-Poincaré operator the Dirichlet problem is solved approximatively and the Lagrange multiplier is used to approximate the normal derivative. We can then introduce the notation

$$\mathcal{S}_h\varphi = \sum_k \langle \lambda_h^k(\varphi), \psi\rangle$$

where the $\lambda_h^k(\varphi)$'s are obtained by solving: find $u_h(\varphi) = (u_h^k(\varphi))_k \in V_h$, $\lambda_h(\varphi) = (\lambda_h^k(\varphi))_k \in \Lambda_h$ such that

$$\left\{\begin{array}{lcll} \forall k, \quad \forall v_h \in V_h^k, \quad \forall \mu_h \in \Lambda_h^k & & & \\ \int_{\Omega_k} \nabla u_h^k(\varphi)\nabla v_h & - & \int_{\Gamma_k} \lambda_h^k(\varphi) v & = & 0 \\ \int_{\Gamma_k} u_h^k(\varphi)\mu_h & & & = & \int_{\Gamma_k} \varphi\mu_h. \end{array}\right. \tag{2.8}$$

In order to give an estimate on the φ component of the error we can use the first Strang Lemma ([5]), which yields

$$\|\varphi - \varphi_h\|_\Phi \lesssim \inf_{\zeta\in\Phi_h} \left\{ \|\varphi - \zeta\|_\Phi + \sup_{\psi_h\in\Phi_h} \frac{\langle(\mathcal{S} - \mathcal{S}_h)\zeta, \psi_h\rangle}{\|\psi_h\|_\Phi} + \sup_{\psi_h\in\Phi_h} \frac{\langle g - g_h, \psi_h\rangle}{\|\psi_h\|_\Phi} \right\}.$$

Let us better analyse the first consistency error term: setting $\lambda^k(\varphi) = \partial_{\nu^k}\mathcal{L}_H^k\varphi$ we have

$$\langle(\mathcal{S} - \mathcal{S}_h)\zeta, \psi_h\rangle = \sum_k \langle \lambda^k(\zeta) - \lambda_h^k(\zeta), \psi_h\rangle \lesssim \left(\sum_k \|\lambda^k(\zeta) - \lambda_h^k(\zeta)\|_{-1/2,\Gamma} \right)^{1/2} \|\psi_h\|_\Phi$$

which yields

$$\sup_{\psi_h \in \Phi_h} \frac{\langle (\mathcal{S} - \mathcal{S}_h)\zeta, \psi_h \rangle}{\|\psi_h\|_\Phi} \lesssim \left(\sum_k \|\lambda^k(\zeta) - \lambda_h^k(\zeta)\|_{-1/2,\Gamma} \right)^{1/2}.$$

It is not difficult to check that a similar result holds also for the second of the two consistency terms. The error $\|\varphi - \varphi_h\|_\Phi$ is thus not directly influenced by the precision with which the unknown u is approximated. The subdomain meshes should not necessarily be chosen by aiming at a good approximation of the whole u but only to a good approximation of its outer conormal derivative λ.

3. The mono-domain problem: local estimates. Let us from now on concentrate on one of the subdomain problems. For the sake of simplicity we will omit the subscript/superscript k. Ω will then denote a polygonal subdomain, Γ its boundary, and, given $\varphi \in H^{1/2}(\Gamma)$ and $f \in L^2(\Omega)$ we will consider the problem of finding $u \in H^1(\Omega)$ and $\lambda \in H^{-1/2}(\Gamma)$ such that

$$\left\{ \begin{array}{lcl} \forall v \in H^1(\Omega), \quad \forall \mu \in H^{-1/2}(\Gamma) & & \\ \int_\Omega \nabla u \nabla v \quad - \quad \int_\Gamma \lambda v & = & \int_\Omega f v \\ \int_\Gamma u\mu & = & \int_\Gamma \varphi\mu. \end{array} \right. \tag{3.1}$$

Again, we consider a Galerkin discretization: letting $V_h \in H^1(\Omega)$, $\Lambda_h \in H^{-1/2}(\Gamma)$ be two finite dimensional subspaces we look for $u_h \in V_h$, $\lambda_h \in \Lambda_h$ such that

$$\left\{ \begin{array}{lcl} \forall v_h \in V_h, \quad \forall \mu_h \in \Lambda_h & & \\ \int_\Omega \nabla u_h \nabla v_h \quad - \quad \int_\Gamma \lambda_h v_h & = & \int_\Omega f v_h \\ \int_\Gamma u_h \mu_h & = & \int_\Gamma \varphi\mu_h. \end{array} \right. \tag{3.2}$$

For the reasons explained in the previous section we are interested in giving a sharp bound on the λ component of the error. Under the usual classical assumptions needed for stability of the discrete problem (see (A4) in the following), the standard techniques yield estimates of the form

$$\begin{array}{rcl} \|\lambda - \lambda_h\|_{-1/2,\Gamma} & \leq & \|\lambda - \lambda_h\|_{-1/2,\Gamma} + \|u - u_h\|_{1,\Omega} \\ & \lesssim & \inf_{\eta_h \in \Lambda_h} \|\lambda - \eta_h\|_{-1/2,\Gamma} + \inf_{w_h \in V_h} \|u - w_h\|_{1,\Omega}. \end{array}$$

Such estimate provides a bound for the error on the multiplier λ depending not only on the regularity of λ and the approximation properties of the space Λ_h, but also on the overall regularity of the solution u and on the overall approximation property of the discretization space V_h. If we however try to estimate the error on λ directly, using a very simple argument, we could write

$$\begin{array}{rcl} \|\lambda - \lambda_h\|_{-1/2,\Gamma} & = & \sup_{v \in H^{1/2}(\Gamma)} \dfrac{\int_\Gamma (\lambda - \lambda_h) v}{\|v\|_{1/2,\Gamma}} \\ & = & \sup_{v \in H^{1/2}(\Gamma)} \left\{ \dfrac{\int_\Gamma (\lambda - \lambda_h)(v - v_h)}{\|v\|_{1/2,\Gamma}} + \dfrac{\int_\Gamma (\lambda - \lambda_h) v_h}{\|v\|_{1/2,\Gamma}} \right\}, \end{array}$$

where $v_h \in V_h|_\Gamma$ is the (unique) element such that $\int_\Gamma \mu_h(v - v_h) = 0$ for all μ_h in Λ_h, which exists and depends continuously on v, provided the standard inf-sup condition needed for stability of problem (3.2) holds. We can then easily bound the two terms on the right hand side thanks to the following bounds

$$\int_\Gamma (\lambda - \lambda_h)(v - v_h) = \int_\Gamma (\lambda - \mu_h)(v - v_h) \leq \|\lambda - \mu_h\|_{-1/2,\Gamma} \|v\|_{1/2,\Gamma}$$

which yields, thanks to the arbitrariness of μ_h,

$$\int_\Gamma (\lambda - \lambda_h)(v - v_h) \lesssim \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_{-1/2,\Gamma} \|v\|_{1/2,\Gamma}.$$

The second term can be bound by observing that for all $w_h \in V_h$, Galerkin orthogonality yields

$$\int_\Gamma (\lambda - \lambda_h) w_h = \int_\Omega \nabla(u - u_h) \nabla w_h.$$

We can then choose any (fixed) subdomain $\Omega_0 \subset \Omega$ such that $\Gamma \subset \partial\Omega_0$, construct a lifting $w_h \in V_h$ of v_h verifying

$$w_h|_\Gamma = v_h, \qquad \mathit{supp} w_h \subset \Omega_0, \qquad \|w_h\|_{1,\Omega} \lesssim \|v_h\|_{1/2,\Gamma}$$

(the constant in the last bound naturally depending on the subdomain Ω_0), and we would get

$$\int_\Gamma (\lambda - \lambda_h) v_h \lesssim \|u - u_h\|_{1,\Omega_0} \|v_h\|_{1/2,\Gamma}.$$

Now, we recall that we are dealing with the Galerkin solution an elliptic problem. If Ω_0 was an interior subdomain ($\bar{\Omega}_0 \subset\subset \Omega$) and letting Ω_1 be an intermediate subdomain, by applying a result by Nitsche and Schatz ([7]) we could bound $\|u - u_h\|_{1,\Omega_0}$ as

$$\|u - u_h\|_{1,\Omega_0} \lesssim h^{s-1} \|u\|_{1,\Omega_1} + \|u - u_h\|_{-p,\Omega}. \tag{3.3}$$

h being the mesh size of the discretization *relative to the subdomain* Ω_1 and p being any positive integer, arbitrary but fixed. Again the constants in the bound depends on the two subdomains Ω_0 and Ω_1. Since the global mesh size enters only through a negative norm of the error, and therefore, under suitable assumptions, with an higher order, its influence on the local error on Ω_0 is reduced.

In order to apply such kind of reasoning to the estimate of the error on the multiplier we need then to provide an estimate of the form (3.3) in the case in which Ω_0 is roughly speaking a strip all along the boundary. It turns out (see [1]) that in proving such an estimate we will also directly prove an estimate on the error $\|\lambda - \lambda_h\|_{-1/2,\Gamma}$ without need of using the above argument.

Let e_i, $i = 1, \cdots, N$ be the edges of Γ and let θ_i, $i = 1, \cdots, N$ be the interior angles. Let $\theta_0 = \max_i \theta_i$ be the maximum angle, and recall that the polygon is convex, that is $\theta_0 < \pi$. Assume that the discretization spaces V_h and Λ_h satisfy

(A1) **Global Approximation for u.** Let $1 \le s \le k_1$, $0 \le \ell \le r_1$. For each $u \in H^\ell(\Omega)$, there exists an element $w \in V_h$ such that

$$\|u - w\|_{s,\Omega} \lesssim H^{\ell-s}\|u\|_{\ell,\Omega};$$

Let now $\Omega_1 \subset \Omega$ be an open subdomain of Ω such that

$$\Gamma \subset \partial\Omega_1, \qquad \partial\Omega_1 \setminus \Gamma \text{ is of class } C^\infty.$$

(see figure 3.1) and assume that the space V_h has, when restricted to Ω_1, better approximation properties. More precisely assume that for any two open subdomains $G_0 \subset G \subseteq \Omega_1$ satisfying

$$\Gamma = \partial G_0 \cap \partial G, \qquad \partial G \setminus \Gamma \text{ and } \partial G_0 \setminus \Gamma \text{ are of class } C^\infty, \qquad \partial G_0 \setminus \Gamma \subset G$$

there exists an h_0 such that if $h \le h_0$ then

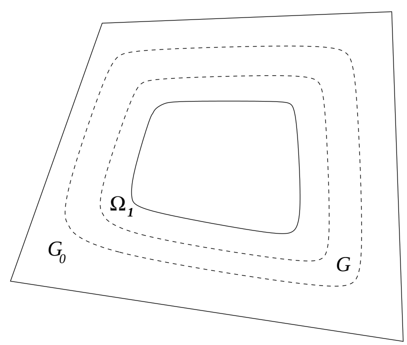

Figure 3.1: Subdomains $G_0 \subset G \subset \Omega_1$

(A2) **Local approximation for** u. Let $1 \le s \le k_1$, $s \le \ell \le r_1$. For each $u \in H^\ell(G)$, there exists an element $w \in V_h$ such that

$$\|u - w\|_{s,G} \lesssim h^{\ell-s}\|u\|_{\ell,G};$$

moreover if u is supported in G_0 then w can be chosen to be supported in G.

(A3) **Discrete commutator property**. Let $\omega \in C^\infty(G)$, $\omega = 0$ in $G \setminus G_0$, and let $v_h \in V_h$. Then there exists $w_h \in V_h$ such that $w_h = 0$ in $\Omega \setminus G_0$ and such that

$$\|\omega v_h - w_h\|_{1,G} \lesssim h\|v_h\|_{1,G}.$$

Remark 3.1 *Assumption A3 is a classical assumption that is usually made when some* localization *technique needs to be applied. It can be shown to hold under some standard assumptions, see [6]*

Finally, assume that the multiplier space Λ_h satisfies

(A4) **Stability conditions**. We have that

$$\inf_{\mu_h \in \Lambda_h} \sup_{v_h \in V_h} \frac{\int_\Gamma \mu_h v_h}{\|\mu_h\|_{-1/2,\Gamma} \|v_h\|_{1,\Omega}} \geq \alpha > 0.$$

and for all v_h in V_h such that $\int_\Gamma v_h \mu_h = 0$ forall $\mu_h \in \Lambda_h$, we have

$$\int_\Gamma |\nabla v_h|^2 \gtrsim \|v_h\|^2_{1,\Omega}.$$

(A5) **Approximation for** λ. Let $-1/2 < \ell \leq r_2$. For each $\lambda \in H^\ell(\Gamma)$, there exists an element $\mu \in \Lambda_h$ such that

$$\|\lambda - \mu\|_{-1/2,\Gamma} \lesssim h^{\ell+1/2} \sum_{i=1}^N \|\lambda\|_{\ell,e_i};$$

Let now $\Omega_0 \subset \Omega_1$ be an open subdomain satisfying

$$\Gamma \subset \partial\Omega_0, \qquad \partial\Omega_0 \setminus \Gamma \subset \Omega_1, \qquad \partial\Omega_0 \setminus \Gamma \text{ is on class } C^\infty.$$

Under the previous assumptions we can prove the following theorem.

Theorem 3.1 *Suppose that A1–A5 are satisfied. Assume that $u \in H^s(\Omega)$, Then, for t_0 positive arbitrary but fixed verifying $t_0 < s_0$, if h is sufficiently small the following bound holds*

$$\|u - u_h\|_{1,\Omega_0} + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \lesssim (h^\tau + H^{\sigma+t_0})\|u\|_{s,\Omega}.$$

with $\tau = \min\{s-1, r_1 - 1, r_2 + 1/2\}$ and $\sigma = \min\{s, r_1, r_2 + 3/2\}$. where the implicit constant in the inequality depends on Ω_0, Ω_1 and t_0.

Trivially this yields the following corollary

Corollary 3.1 *Under the same assumptions of theorem 3.1 it holds*

$$\|\lambda - \lambda_h\|_{-1/2,\Gamma} \lesssim (h^\tau + H^{\sigma+t_0})\|u\|_{s,\Omega}.$$

By applying such corollary, it is clear that choosing a discretization satisfying assumptions $A1 - -A5$ with

$$H = h^{\tau/(\sigma+t_0)}$$

yields the optimal error estimate

$$\|\lambda - \lambda_h\|_{-1/2,\Gamma} \lesssim h^\tau \|u\|_{s,\Omega}.$$

In particular, the above results implies that, as far as the approximation of the Lagrange multiplier λ is concerned it is possible to chose the mesh in the interior of the subdomain sensibly coarser than the mesh that would be needed to approximate the function u with the same accuracy.

4. Numerical results. Let us test the theoreical results of the previous section on a simple example. Let $\Omega =]-1,1[^2$ and consider the following model problem:

$$-\Delta u = 13\sin(2x)\cos(3x), \text{ in } \Omega, \qquad u = \sin(2x)\cos(3y), \text{ on } \Gamma. \tag{4.1}$$

It is not difficult to verify that the solution of such a problem is the function $u = \sin(2x)\cos(3y)$ (see Figure 4.1).

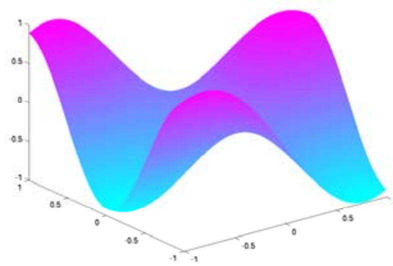

Figure 4.1: Solution of the model problem

In order to approximate u we consider a Lagrange multiplier formulation in the form (3.2) of the above problem, where V_h is chosen to be a P1 finite element space and Λ_h is defined as the trace of V_h on the boundary Γ. It is not difficult to check that if the triangulation on the boundary is quasi-uniform then assumptions A1–A5 are satisfied with $r_1 = 2$ and $r_2 = 1/2 - \varepsilon$ ($\varepsilon > 0$ arbitrary but fixed).

Letting $\delta \in]0,1[$ be a fixed parameter, we consider triangulations of Ω constructed in the following way: starting from a quasi uniform triangulation $\mathcal{T}_H$ of the whole Ω, set $\mathcal{T}_h^0 = \mathcal{T}_H$, and let $\mathcal{T}_h^j$ be obtained from $\mathcal{T}_h^{j-1}$ by "refining" (precisare) all those triangles T in $\mathcal{T}_h^{j-1}$ such that $suppT \cap \Omega \backslash]-1+\delta, 1-\delta[^2 \neq \emptyset$.

We compare the solution of problem (4.1) obtained with a quasi uniform triangulation of mesh-size $h = H/2^j$, with the one obtained using the triangulation $\mathcal{T}_H^j$ for $j = 1, \cdots, 4$ and for different values of the parameter δ. In the following figures we display both the $H^1(\Omega)$ and the $L^2(\Omega)$ norms of the error $u - u_h$, and the $L^2(\Omega)$ norm of the error $\lambda - \lambda_h$ (which for computational simplicity we prefer to the $H^{-1/2}(\Gamma)$). As one can expect, for the boundary refined triangulations, both the $H^1(\Omega)$ and the $L^2(\Omega)$ norms of the error on u are mainly influenced from coarse triangulations in the interior of Ω and do not sensibly vary as j increases, while they decrease with the expected rates when considering the quasi uniform mesh. Conversely, when considering the $L^2(\Gamma)$ norm of the error on λ, the boundary refined and the quasi uniform meshes display the same behaviour as j increases. However, the boundary refined meshes allows to get the same error with considerably less degrees of freedoms – and therefore with considerably lower computational cost.

REFERENCES

[1] S. Bertoluzza. Mesh design for the subdomain solvers arising in non conforming Steklov-Poincaré discretizations.

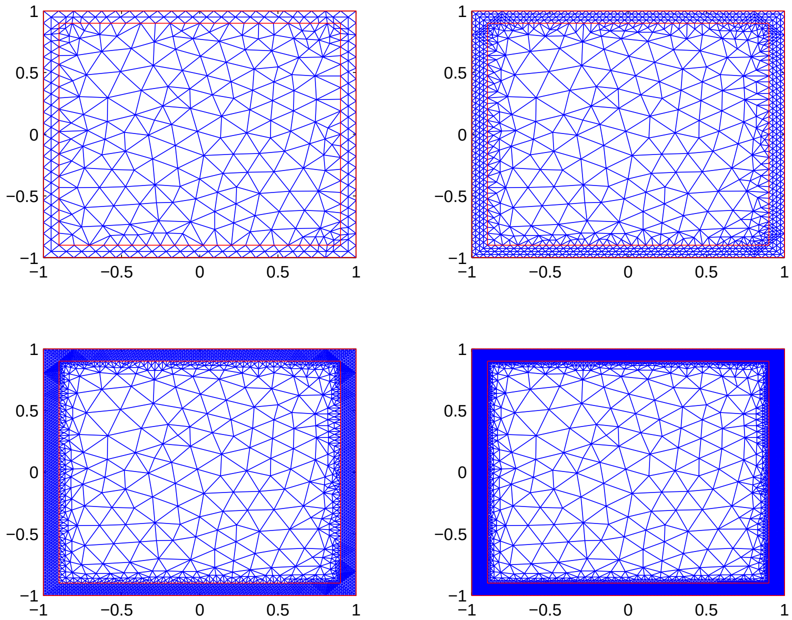

Figure 4.2: The triangulations used for the tests.

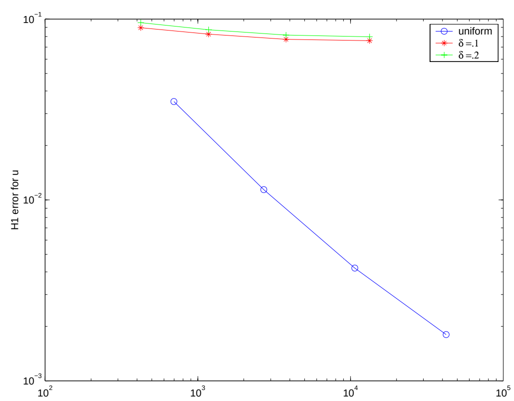

Figure 4.3: Error $\|u - u_h\|_{1,\Omega}$ vs. the number of degrees of freedom for the quasi uniform mesh and for the boundary refined mesh with δ resp. equals to .1 and .2

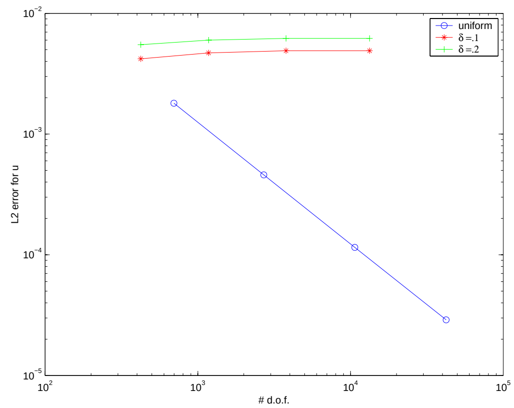

Figure 4.4: Error $\|u - u_h\|_{0,\Omega}$ vs. the number of degrees of freedom for the quasi uniform mesh and for the boundary refined mesh with δ resp. equals to .1 and .2

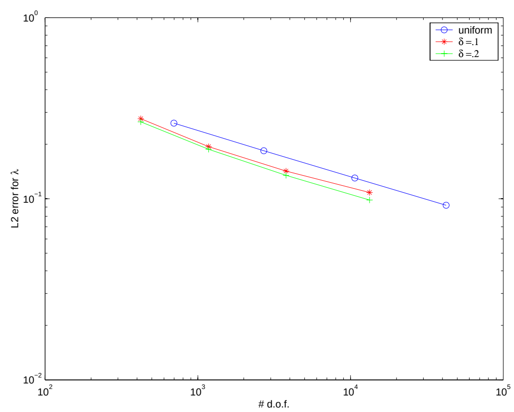

Figure 4.5: Error $\|\lambda - \lambda_h\|_{0,\Gamma}$ vs. the number of degrees of freedom for the quasi uniform mesh and for the boundary refined mesh with δ resp. equals to .1 and .2

[2] S. Bertoluzza. Analysis of a stabilized domain decomposition method. Technical report, I.A.N.-C.N.R. Pavia, 2000.

[3] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New-York, 1991.

[4] F. Brezzi and L. D. Marini. A three fields domain decomposition method. *Contemp. Math.*, 157:27–34, 1994.

[5] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

[6] J. A. Nitsche. Ein kriterium für die quasi-optimalitaet des Ritzschen verfahrens. *Numer. Math.*, 11:346–348, 1968.

[7] J. A. Nitsche and A. H. Schatz. Interior estimates for Ritz-Galerkin methods. *Math. Comp.*, 28:937–958, 1974.