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36. The Mortar Method with Approximate Constraint

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1. Introduction. The *Mortar* method is a non conforming approach for solving PDEs in domain decomposition. It consists in imposing weak continuity across the interfaces by requiring that the jump of the solution along two adjacent subdomains is orthogonal to a suitable *Multiplier* space. This method is particulary well suited for choosing different kinds of discretizations in each subdomain.
We will consider here the case of coupling finite elements with wavelets, which will allow us to overcome the limit of application of wavelet basis to tensor product domains, using FEM for more complicated shapes.
The constraint operator, which is used to impose weak continuity, leads to the problem of computing integrals of product of functions of different type and this can be extremely technical or even impossible. This is the case of wavelet/finite elements coupling, where such integral can not be computed exactly due to the particular nature of wavelets, which are not known in closed form. We will propose here to approximate it by a technique that is particulary well suited for the case we are treating. Moreover we will show that the use of such a technique allows to easily integrate new type of functions in existing codes, without the need of providing specific tools for computing the integrals of the product of a function of the new type with all functions of each of the types already present in the code.
The paper is organized as follows: in Section 2 we introduce the general context in the case of a simple splitting of the domain into two subdomains, introducing the approximate constraint and analizing the error estimate. In Section 3 we consider the particular case of coupling Wavelet and Finite Element disctretiations by studing the explicit form of the approximate constraint in both the cases of Wavelet type discretization in the Master subdomain and Finite Element discretization in the Slave one, and viceversa. Finally, Section 4 is devoted to a brief overwiev of the C++ code implemented for such an approach.

2. The mortar method with approximate constraint. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and consider the model problem: given $f \in L^2(\Omega)$ find $u : \Omega \to \mathbb{R}$ s.t.

$$-\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{2.1}$$

where for simplicity we assume that the matrix a is constant symmetric positive definite. We consider here a very simple example of non conforming domain decomposition. More precisely consider a splitting of Ω in two subdomains as $\bar{\Omega} = \bar{\Omega}_+ \cup \bar{\Omega}_-$, with $\gamma = \partial\Omega_+ \cap \partial\Omega_-$.

Denote by V_h^+ and V_h^-

$$V_h^+ \subset H^1_\Gamma(\Omega_+) = \{u \in H^1(\Omega_+) : \quad u = 0 \text{ on } \Gamma \cap \underline{\Omega_+}\} \tag{2.2}$$

$$V_h^- \subset H^1_\Gamma(\Omega_-) = \{u \in H^1(\Omega_-) : \quad u = 0 \text{ on } \Gamma \cap \underline{\Omega_-}\} \tag{2.3}$$

the two discrete spaces chosen for approximating u in Ω_+ and Ω_- respectively, and let $M_h \subset H^{-1/2}(\gamma)$, with $dim(M_h) = dim(V_h^-|_\gamma)$ be a suitable multiplier space — which in the mortar method is obtained from a subspace of $V_h^-|_\gamma$ with suitable modifications at the vertices of γ ([1]), or which coincides, in a more general formulation, with a suitable "dual space" of $V_h^-|_\gamma$ ([3], [5]). In the classical formulation of the mortar method, the approximation of the solution of (2.1) is sought in the constrained space $\mathcal{X}_h$ defined as

$$\mathcal{X}_h = \{u : \quad u|_{\Omega_+} \in V_h^+, \quad u|_{\Omega_-} \in V_h^-, \quad \int_\gamma [u]\lambda = 0 \quad \forall \lambda \in M_h\},$$

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where introducing the notation $u^+ = u|_{\Omega_+}$ and $u^- = u|_{\Omega_-}$, $[u] = u^+|_\gamma - u^-|_\gamma$ denotes the jump of the function u across the interface γ. The solution u to problem (2.1) is approximated by looking for u_h in $\mathcal{X}_h$ such that for all $v_h \in \mathcal{X}_h$ it holds

$$\int_{\Omega_+} a\nabla u_h \nabla v_h + \int_{\Omega_-} a\nabla u_h \nabla v_h = \int_\Omega f v_h.$$

In the solution of the linear system resulting from such problem the need arises eventually of computing the integrals appearing in the constraint

$$\int_\gamma (u_h^+ - u_h^-)\lambda = 0, \qquad \forall \lambda \in M_h. \tag{2.4}$$

Since in the mortar method the multiplier space M_h is strongly related to the "slave" space V_h^-, it is not reasonable to assume that the integrals of the products $u_h^-\lambda$ are computable in practice. This is not necessarily the case of the products $u_h^+\lambda$ where functions originating from totally unrelated spaces are involved. We will concentrate here on approximating this term in the constraint.

In order to do that, let us introduce two auxiliary spaces $U_\delta^- \subset L^2(\gamma)$ and $U_\delta^+ \subset L^2(\gamma)$ depending on a parameter δ, which we assume to have the same finite dimension and to satisfy

$$\inf_{\zeta \in U_\delta^-} \sup_{\eta \in U_\delta^+} \frac{\int_\gamma \zeta\, \eta}{\|\zeta\|_{H^{1/2}_{00}(\gamma)} \|\eta\|_{H^{-1/2}(\gamma)}} \geq \alpha > 0. \tag{2.5}$$

Assume that the two auxiliary spaces are chosen in such a way that the integrals of the form $\int_\gamma \zeta\, \eta$ are computable provided either $\zeta \in V_h^+|_\gamma$ and $\eta \in U_\delta^+$ or $\zeta \in V_h^-|_\gamma$ and $\eta \in U_\delta^-$. For all $\zeta \in L^2(\gamma)$ let $P^-(\zeta) \in U_\delta^-$ be the unique element in U_δ^- such that

$$\int_\gamma P^-(\zeta)\, \eta = \int_\gamma \zeta\, \eta, \qquad \forall \eta \in U_\delta^+. \tag{2.6}$$

We propose here to approximate the integral of the product $u_h^+\lambda$ with the integral of $P^-(u_h^+)\lambda$ (where, by abuse of notation we will write u_h^+ instead of $u_h^+|_\gamma$). The constraint (2.4) is then replaced by the approximated constraint

$$\int_\gamma (P^-(u_h^+) - u_h^-)\, \lambda = 0, \qquad \forall \lambda \in M_h, \tag{2.7}$$

which corresponds to defining a new constrained space as

$$\mathcal{X}_h^* = \{u : \quad u|_{\Omega_+} \in V_h^+, \quad u|_{\Omega_-} \in V_h^-, \quad \int_\gamma (P^-(u^+) - u^-)\lambda = 0 \;\; \forall \lambda \in M_h\}, \tag{2.8}$$

and approximating the solution to (2.1) by the solution of the following discrete problem: find $u_h \in \mathcal{X}_h^*$ such that for all $v_h \in \mathcal{X}_h^*$ it holds

$$\int_{\Omega_+} a\nabla u_h \nabla v_h + \int_{\Omega_-} a\nabla u_h \nabla v_h = \int_\Omega f v_h. \tag{2.9}$$

Denoting by $\|\cdot\|_{1,*} = \|\cdot\|_{H^1(\Omega_+)} + \|\cdot\|_{H^1(\Omega_-)}$ the broken H^1 norms, we can prove the following bound [2]

Theorem 2.1 *Let the multiplier space M_h be chosen in such a way that the following assumptions are satisfied:*

(A1) *there exists a bounded projection* $\pi : L^2(\gamma) \to V_h^-|_\gamma$, *such that for all* $\eta \in H_{00}^{1/2}(\gamma)$ *and for all* $\lambda \in M_h$ *it holds that*

$$\int_\gamma (\eta - \pi(\eta))\ \lambda = 0, \qquad \text{and} \qquad \|\pi\eta\|_{H_{00}^{1/2}(\gamma)} \lesssim \|\eta\|_{H_{00}^{1/2}(\gamma)}. \tag{2.10}$$

(A2) *there exists a discrete lifting* $R_h : V_h^-|_\gamma \to V_h^-$ *such that for all* $\eta \in V_h^-|_\gamma$, $\|R_h\eta\|_{H^1(\Omega_-)} \lesssim \|\eta\|_{H_{00}^{1/2}(\gamma)}$.

Moreover let the two auxiliary spaces U_δ^+ *and* U_δ^- *be chosen in such a way that the following Jackson type inequality holds for some* $\tilde{R}, R \geq 1/2$: *for all* r, $1/2 \leq r \leq R$ *(resp. for all* $\tilde{r}$, $-1/2 \leq \tilde{r} \leq \tilde{R}$*)*

$$\forall \eta \in H_0^r(\gamma), \qquad \inf_{\eta_\delta \in U_\delta^-} \|\eta - \eta_\delta\|_{H^{1/2}(\gamma)} \lesssim \delta^{r-1/2}\|\eta\|_{H^r(\gamma)}, \tag{2.11}$$

$$\forall \eta \in H^{\tilde{r}}(\gamma), \qquad \inf_{\eta_\delta \in U_\delta^+} \|\eta - \eta_\delta\|_{H^{-1/2}(\gamma)} \lesssim \delta^{\tilde{r}+1/2}\|\eta\|_{H^{\tilde{r}}(\gamma)}, \tag{2.12}$$

Then, if u_h *is the solution of problem (2.9), and if the solution* u *of problem (2.1) verifies* $u \in H^s(\Omega)$ *for some* s, $2 \leq s \leq \min\{\tilde{R} + 3/2, R + 1/2\}$, *the following error estimate holds:*

$$\begin{aligned} \|u - u_h\|_{1,*} \quad \lesssim \quad & \delta^{s-1}\|u\|_{H^s(\Omega)} + \inf_{\lambda \in M_h} \|\partial_{\nu_a} u - \lambda\|_{H^{-1/2}(\gamma)} \\ & + \inf_{v_h \in V_h^+} \|u - v_h\|_{H^1(\Omega_+)} + \inf_{v_h \in V_h^-} \|u - v_h\|_{H^1(\Omega_-)} \end{aligned} \tag{2.13}$$

where ∂_{ν_a} *denotes the trace on* γ *of outer co-normal derivative to the subdomain* Ω_+.

Remark 2.1 *The extremely simple configuration considered (only two subdomains), hides some of the issues related to the analysis of the mortar method in more general configurations — namely the treatment of cross points. However, the approach used and the results obtained in this paper carry over to more complex cases (with the presence of cross-points), with, in the worse case, a loss of a logarithmic factor in the error estimate.*

3. Wavelet/FEM Coupling.

Let us now consider the case of Wavelet/FEM coupling. In order to get two suitable auxiliary spaces, we will in such a case need a couple of biorthogonal multiresolution analyses $\{V_j\}_{j \geq j_0}$ and $\{\tilde{V}_j\}_{j \geq j_0}$ of $L^2(\gamma)$ with the following characteristics ([4]).

- $V_j \subset H^1(\gamma)$ is the subspace of P1 finite elements on the uniform grid $\mathcal{G}_j$ obtained by splitting γ into 2^j equal segments;
- $\tilde{V}_j \subset H^1(\gamma)$ is a subspace having the same dimension as V_j, which is *biorthogonal* to V_j in the following sense: denoting by $e_{j,k}$ ($k = 0, \ldots, 2^j$) the nodal basis function in V_j corresponding to the k-th point in the grid $\mathcal{G}_j$, the space $\tilde{V}_j$ has a Riesz's basis $\{\tilde{e}_{j,k},\ k = 0, \ldots, 2^j\}$ which satisfies $\int_\gamma e_{j,k}\ \tilde{e}_{j,k'} = \delta_{kk'}$, $\forall k, k' = 0, \ldots, 2^j$;
- the functions $\tilde{e}_{j,k}$ can be obtained as linear combination of the restriction to γ (identified through a suitable mapping with the interval $(0,1)$), of the translates and contracted (with a contraction factor 2^j) of a compactly supported function $\tilde{e}$, which we assume to be *refinable*, to verify, for suitable values of the coefficients h_k, $\tilde{e}(s) = \sum_{k=0}^N h_k\ \tilde{e}(2s - k)$;

- $\tilde{V}_j$ satisfies a Strang-Fix condition of order M, that is it contains polynomials up to degree $M-1$, while, of course, V_j contains polynomials of order 1.

Let $V_j^0 = V_j \cap H_0^1(\gamma)$ and $\tilde{V}_j^0 = \tilde{V}_j \cap H_0^1(\gamma)$, it is possible (see [3]) to construct two subspaces $V_j^* \subset V_j$ and $\tilde{V}_j^* \subset \tilde{V}_j$ satisfying $dim(V_j^*) = dim\tilde{V}_j^0$, $dim(\tilde{V}_j^*) = dimV_j^0$, and

$$\inf_{\eta \in V_j^0} \sup_{\zeta \in \tilde{V}_j^*} \frac{\int_\gamma \eta\, \zeta}{\|\eta\|_{H_{00}^{1/2}(\gamma)} \|\zeta\|_{H^{-1/2}(\gamma)}} \geq \alpha_1, \qquad \inf_{\eta \in \tilde{V}_j^0} \sup_{\zeta \in V_j^*} \frac{\int_\gamma \eta\, \zeta}{\|\eta\|_{H_{00}^{1/2}(\gamma)} \|\zeta\|_{H^{-1/2}(\gamma)}} \geq \alpha_2,$$

in such a way that they satisfy a Strang-Fix condition with the same order as V_j and $\tilde{V}_j$ respectively. Moreover it is possible to construct Riesz's bases $e_{j,k}^*$ and $\tilde{e}_{j,k}^*$ for V_j^* and $\tilde{V}_j^*$ respectively in such a way that the two following biorthogonality relations hold:

$$\int_\gamma e_{j,k}\, \tilde{e}_{j,k'}^* = \delta_{k,k'}, \qquad \int_\gamma \tilde{e}_{j,k}\, e_{j,k'}^* = \delta_{k,k'}, \qquad \forall k, k' = 1, \dots, 2^j - 1. \tag{3.1}$$

Thanks to the *refinable* property of the function $\tilde{e}$, it is well known that it is possible to compute integrals of the product of a wavelet type function times any function in $\tilde{V}_j$ (and therefore in $\tilde{V}_j^0$ and in $\tilde{V}_j^*$), while the product of a function in V_j, V_j^0 and V_j^* with a finite element type function can be computed by standard techniques, already implemented in the mortar method for finite elements with non-matching grids.

For using such spaces for coupling wavelets and finite elements in the mortar method we distinguish two cases.

Case 1. FEM master / Wavelet slave. In this case we set V_h^+ to be a finite element space on an unstructured, non uniform grid while V_h^- and M_h are two wavelet type spaces. The approximate integration is done by setting $U_\delta^+ = V_j^0$ and $U_\delta^- = \tilde{V}_j^*$.

Case 2. Wavelet master / FEM slave. In this case we set V_h^+ to be a wavelet type space, while V_h^- and M_h are two finite element spaces defined on unstructured, non uniform grid, the grid for M_h being the trace on γ of the grid for V_h^-. The approximate integration is this time performed by setting $U_\delta^+ = \tilde{V}_j^0$ and $U_\delta^- = V_j^*$.

Once $P^-(v_h^+)$ is known, the space U_δ^- is chosen in both cases in such a way that the integrals of the product $\lambda_h P^-(v_h^+)$ can be computed. We then only need to compute the $P^-(v_h^+)$. This can be done by taking advantage of the biorthogonality property (3.1). In fact it is not difficult to see that, depending on which of the two cases we are in, we have

$$\text{Case 1: } P^- u = \sum_{k=1}^{2^j-1} \left(\int_\gamma u\, e_{j,k}^* \right) \tilde{e}_{j,k}, \qquad \text{Case 2: } P^- u = \sum_{k=1}^{2^j-1} \left(\int_\gamma u\, \tilde{e}_{j,k}^* \right) e_{j,k}.$$

Again, in both cases the two auxiliary spaces have been chosen in such a way that the two integrals defining the projectors are computable. Moreover, biorthogonality implies that *no linear system* has to be solved in order to compute the auxiliary projector.

By applying Theorem 2.1 we can finally estimate the effect of using the approximate integration technique proposed in the previous section. In both cases we get the following bound: if $u \in H^s(\Omega)$ with $2 \leq s \leq T$ it holds

$$\begin{aligned} \|u - u_h\|_{1,*} \quad &\lesssim \quad 2^{-j(s-1)} \|u\|_{H^s(\Omega)} + \inf_{\lambda \in M_h} \|\partial_{\nu_a} u - \lambda\|_{H^{-1/2}(\gamma)} \\ &\quad + \inf_{v_h \in V_h^+} \|u - v_h\|_{H^1(\Omega_+)} + \inf_{v_h \in V_h^-} \|u - v_h\|_{H^1(\Omega_-)}. \end{aligned}$$

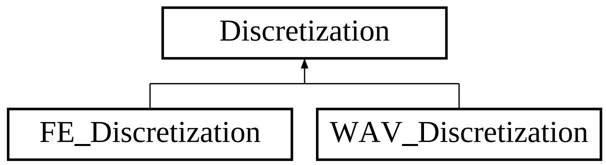

Figure 4.1: Inheritance diagram for Discretization

where, depending on the choice of the master and slave, the limit T in the bound is respectively $T = \min\{7/2, M + 1/2\}$ for the 'FEM master' case and $T = \min\{5/2, M + 3/2\}$ for the 'Wavelet master' case. The effect of approximating the constraint is contained in the first term on the r.h.s. which, by suitably choosing j can be tuned up in such a way it is comparable to the other three terms.

4. The implementation. The idea of replacing the classical Mortar method with the new approximate constraint, allows not only to overcome the problem of integrating functions of different kind, but gives also an advantage from the implementation point of view. In the first case, in fact, the introduction of a new discretization space in an existing code would require to provide specific tools for computing the integrals of the product of a function of the new type with all functions of each of the types already present in the code. On the other hand, the use of a projection on an auxiliary space to approximate the above integral reduces such a problem to the one of compute only the integral of an new type function with an auxiliary function.

We are now going to give a brief and schematic idea of the domain decomposition C++ code we implemented to couple Finite element and Wavelet discretitazions in the Mortar method. Without going into detailed descriptions, we will just give a brief overview to the two main classes defined in the code: the Class Discretization and the Class Mortar.

The Class Discretization It is a virtual class, from which the FE_Discretization (Finite Element Discretization) and the WAV_Discretization (Wavelet Discretization) classes are derived (Figure 4). It is associated to each subdomain of the global domain and provides the following main methods:

- **Trace_X_AuxBasis**: returns the integrals of a trace function with the auxiliary basis.
- **Get_Trace**: given a function, returns the trace of the function on an edge of the corresponding subdomain.
- **Set_Trace**: given a trace function f, sets the trace of the global function of the corresponding subdomain equals to f.
- **local_Stiff_x_u**: returns the Matrix-vector multiplication of the subdomain stiffness matrix with a vector u.

The Class Mortar The class Mortar is the class which allows to couple different kinds of discretization, in the sense that it is the way two Discretization classes comunicate with each other. It takes the traces of the functions of two adjacent subdomains and applies the approximate constraint operator, making use of the following methods:

- **Paux**: computes the projection of a trace function onto the auxiliary space.

- **Mortar_Projection**: returns the projection of a master edge function onto the multiplier space.

- **Local_Constraint**: applies the local Constraint operator.

In Figure (4.2), we show the numerical solution obtained by applying the approximate constraint to the Laplace equation

$$-\Delta u = 1 \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma = \partial\Omega,$$

REFERENCES

[1] C. Bernardi, Y. Maday, and A. T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In H. Brezis and J.-L. Lions, editors, *Collège de France Seminar*. Pitman, 1994. This paper appeared as a technical report about five years earlier.

[2] S. Bertoluzza, S. Falletta, and V. Perrier. Wavelet/fem coupling by the mortar method. *Lecture Notes in Computational Science and Engineering*, 2001. (to appear).

[3] S. Bertoluzza and V. Perrier. The mortar wavelet method. *Mathem. Mod. and Numer. nal.*, 35:647–674, 2001.

[4] A. Cohen, I. Daubechies, and P. Vial. Wavelets on the interval and fast wavelet transform. *ACHA*, 1:54–81, 1993.

[5] B. Wohlmuth. A mortar finite element method using dual spaces for the Lagrange multiplier. *SIAM J. Numer. Anal.*, 38:989–1012, 2000.

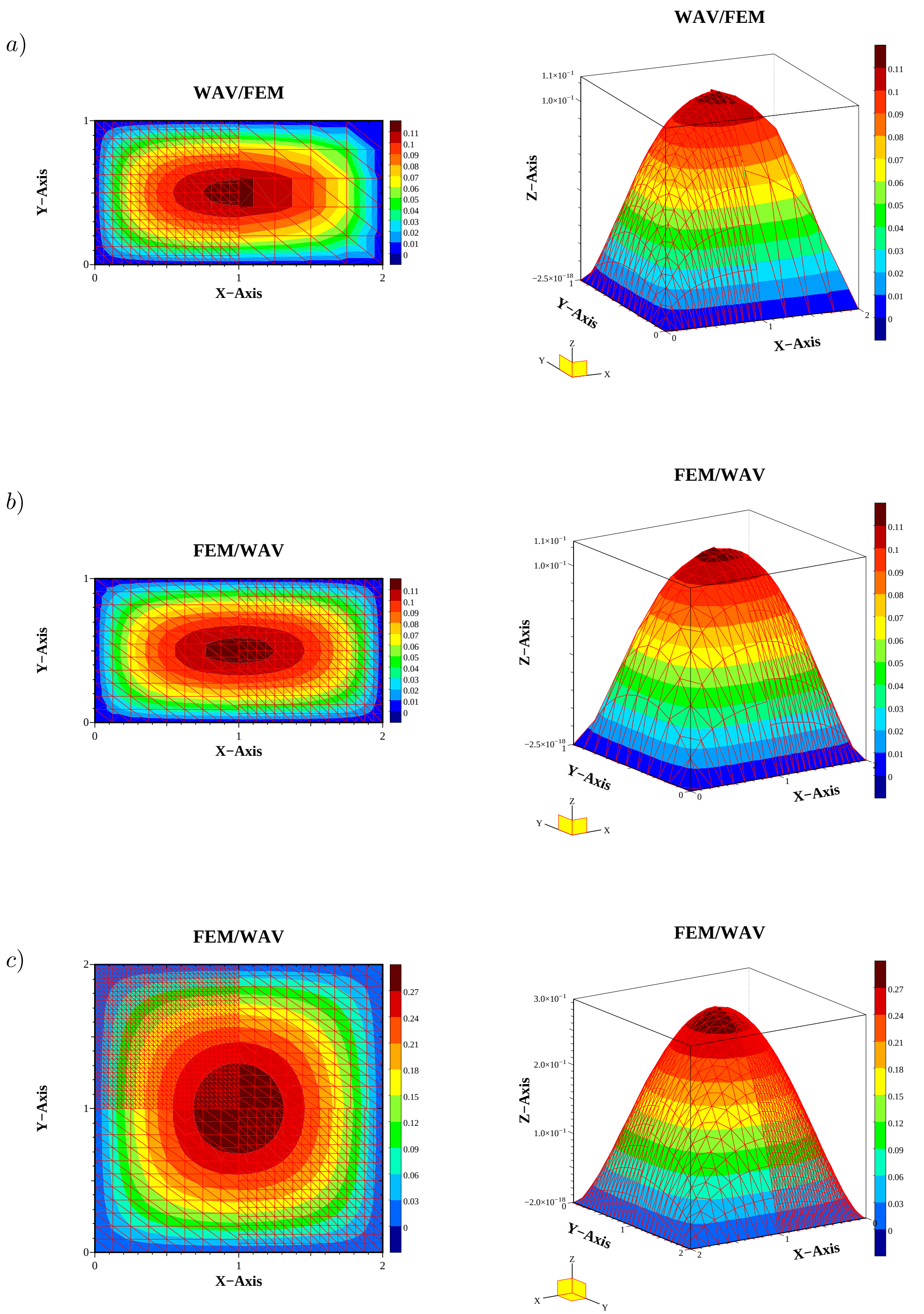

Figure 4.2: a):the case WAV master/FEM slave. b): the case FEM master/WAV slave c): the solution with mixed choice of discretizations and the presence of crosspoints.