Fourteenth International Conference on Domain Decomposition Methods
Editors: Ismael Herrera , David E. Keyes, Olof B. Widlund, Robert Yates

23. Indirect Method of Collocation: 2^{nd} Order Elliptic Equations

M.A. Diaz[1], I. Herrera[2], R. Yates[3]

1. Introduction. These papers is part of a group of papers [7],[9],[4], [3],[10], included in these Proceedings, devoted to present and illustrate the applications of Herrera's Unified Theory of Domain Decomposition Methods (DDM). As an example of the applications of indirect -or Trefftz-Herrera- methods, in the present paper a new method of collocation - Trefftz-Herrera collocation- is developed applicable to any elliptic equation of second order, which is linear. The general problem considered is one with prescribed jumps for the function and its first order derivatives; actually, the *'fluxes'*, as its explained later in the sequel. Differential operators with discontinuous coefficients are included.

The collocation method based on the use of Hermite cubic polynomials has a good number of attractive features such as its high accuracy and the simplicity of its formulation [1],[2]. However, it suffers computationally from several drawbacks, such as a large number of degrees of freedom associated with each node of the discretized mesh. Also, the global matrix of the system of equations does not enjoy the property of being positive definite even when the differential operator itself has this property. Up to now, collocation has been applied by means of splines. However, a broader and more efficient formulation is obtained when collocation is applied using fully discontinuous functions by means of the indirect (or Trefftz-Herrera) domain decomposition methodology. In this paper Trefftz-Herrera indirect method, in combination with orthogonal collocation, is applied to a general boundary value problem with prescribed jumps to produce a family of "indirect collocation methods (Trefftz-Herrera collocation)". In particular, when the differential equation (or system of such equations) is positive definite the global matrix is also positive definite. Also, a dramatic reduction in the number of degrees of freedom associated with each node is obtained. Indeed, in the standard method of collocation that number is two in one dimension, four in two dimensions and eight in three dimensions, while for some of the new algorithms they are only one in all space dimensions. A final comment worth doing refers to the fact that the treatment of problems with prescribed jumps is just as easy as that without them; as a matter of fact, the global matrix is exactly the same for both problems.

2. Trefftz-Herrera Approach to Elliptic Equations (2^{nd} Order) . The general theory of Trefftz-Herrera DDM, presented in [9], is applied in this Section to elliptic equations of second order. The boundary value problem with prescribed jumps (BVPJ) for this case was given as an illustration in [9]; it is:

$$\mathcal{L}u \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla u) + \nabla \cdot (\underline{b} u) + cu = f_\Omega \equiv \mathcal{L}u_\Omega, \quad in \quad \Omega_i, \quad i = 1, ..., E \tag{2.1}$$

subjected to the boundary conditions

$$u = u_\partial; \quad on \quad \partial\Omega, \tag{2.2}$$

and the jump conditions

$$[u] = [u_\Sigma] \equiv j^0_\Sigma \quad and \quad [\underline{\underline{a}} \cdot \nabla u] \cdot \underline{n} = [\underline{\underline{a}} \cdot \nabla u_\Sigma] \cdot \underline{n} \equiv j^1_\Sigma; \quad on \quad \Sigma, \tag{2.3}$$

[1]Instituto de Geofísica Universidad Nacional Autónoma de México (UNAM), mdiaz@tonatiuh.igeofcu.unam.mx
[2]Instituto de Geofísica Universidad Nacional Autónoma de México (UNAM) , iherrera@servidor.unam.mx
[3]Instituto de Geofísica Universidad Nacional Autónoma de México (UNAM) , yates@altcomp.com.mx

The notation is the same as that introduced in [9] and [4]. In particular, $u_\Omega \in \hat{D}_1$, $u_\partial \in \hat{D}_1$ and $u_\Sigma \in \hat{D}_1$ are any functions which satisfy the differential equation, the external boundary conditions and the jump conditions, respectively. and A partition of a domain Ω is being considered and the internal boundary is denoted by Σ (see [9] for further details).

The general theory introduces the following bilinear functionals:

$$\langle Pu, w\rangle \equiv \int_\Omega w\mathcal{L}udx; \quad \langle Qw, u\rangle \equiv \int_\Omega u\mathcal{L}^* wdx \tag{2.4}$$

$$\langle Bu, w\rangle \equiv \int_{\partial\Omega} \mathcal{B}(u, w)dx; \quad \langle Cw, u\rangle \equiv \int_{\partial\Omega} \mathcal{C}(w, u)dx \tag{2.5}$$

$$\langle Ju, w\rangle \equiv \int_\Sigma \mathcal{J}(u, w)dx; \quad \langle Kw, u\rangle \equiv \int_\Sigma \mathcal{K}(w, u)dx \tag{2.6}$$

$$\langle S_J u, w\rangle \equiv \int_\Sigma \mathcal{S}_J(u, w)dx; \quad \langle R_J u, w\rangle \equiv \int_\Sigma \mathcal{R}_J(u, w)dx \tag{2.7}$$

$$\langle Sw, u\rangle \equiv \int_\Sigma \mathcal{S}(w, u)dx; \quad \langle Rw, u\rangle \equiv \int_\Sigma \mathcal{R}(w, u)dx \tag{2.8}$$

Where $\mathcal{J}(u, w)$ and $\mathcal{K}(w, u)$, are given by Eq. (5.4) of Ref. [9]:

$$\mathcal{J}(u, w) \equiv -\underline{\mathcal{D}}([u], \dot{w}) \cdot \underline{n} \quad \text{and} \quad \mathcal{K}(w, u) \equiv \underline{\mathcal{D}}(\dot{u}, [w]) \cdot \underline{n} \tag{2.9}$$

where

$$\underline{\mathcal{D}}\,(u, w) \equiv u\,(\underline{a}_n \cdot \nabla w + b_n w) - w\underline{a}_n \cdot \nabla u \tag{2.10}$$

has the property that

$$w\mathcal{L}u - u\mathcal{L}^* w \equiv \nabla \cdot \underline{\mathcal{D}}\,(u, w) \tag{2.11}$$

Here

$$\mathcal{L}^* w \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla w) - \underline{b} \cdot \nabla w + cw; \tag{2.12}$$

Then, for the case considered in this Section, the bilinear functions occurring in the integrals of Eqs. (2.4) to (2.8) are defined by [9]:

$$\mathcal{B}\,(u, w) \equiv u\,(\underline{a}_n \cdot \nabla w + b_n w) \cdot \underline{n}, \quad \mathcal{C}\,(w, u) \equiv w\underline{a}_n \cdot \nabla u \tag{2.13}$$

$$\mathcal{J}\,(u, w) \equiv \dot{w}\,[\underline{a}_n \cdot \nabla u] - [u]\,\overline{(\underline{a}_n \cdot \nabla w + b_n w)}^{\,\cdot} \tag{2.14}$$

$$\mathcal{K}\,(w, u) \equiv \dot{u}\,[\underline{a}_n \cdot \nabla w + b_n w] - [w]\,\overline{(\underline{a}_n \cdot \nabla u)}^{\,\cdot} \tag{2.15}$$

$$\mathcal{S}_J\,(u, w) \equiv \dot{w}\,[\underline{a}_n \cdot \nabla u]\,, \quad \mathcal{R}_J\,(u, w) \equiv -\,[u]\,\overline{(\underline{a}_n \cdot \nabla w + b_n w)}^{\,\cdot} \tag{2.16}$$

$$\mathcal{S}\,(w, u) \equiv \dot{u}\,[\underline{a}_n \cdot \nabla w + b_n w] \quad \textit{and} \quad \mathcal{R}\,(w, u) \equiv -\,[w]\,\overline{(\underline{a}_n \cdot \nabla u)}^{\,\cdot} \tag{2.17}$$

Define $\tilde{N}_1 \equiv N_P \cap N_B \cap N_{R_J}$ and $\tilde{N}_2 \equiv N_Q \cap N_C \cap N_R$, then a function $v \in \tilde{N}_1$, if and only if

$$Pv = 0, \quad Bv = 0 \quad \textit{and} \quad R_J v = 0 \tag{2.18}$$

and $w \in \tilde{N}_2$, if and only if

$$Qw = 0, \quad Cw = 0 \quad \textit{and} \quad Rw = 0 \tag{2.19}$$

The result that is basic for deriving the kind of domain decomposition to be applied in the present article, is given by the Theorem of Section 10 of Ref. [9]:

Theorem 2.1 *Assume $\mathcal{E} \subset \tilde{N}_2$ is a system of weighting functions TH-complete for S^* [9]. Let $u_P \in \hat{D}_1$ be such that*

$$Pu_P = Pu_\Omega, \quad Bu_P = Bu_\partial \quad and \quad R_J u_P = R_J u_\Sigma \tag{2.20}$$

Then there exists $v \in \tilde{N}_1$ such that

$$-\langle S^* v, w\rangle = \langle S_J (u_P - u_\Sigma), w\rangle, \quad \forall w \in \mathcal{E} \subset \tilde{N}_2 \tag{2.21}$$

In addition, define $\hat{u} \in \hat{D}_1$ by $\hat{u} \equiv u_P + v$. Then $\hat{u} \in \hat{D}_1$ contains the sought information. Even more, $\hat{u} \equiv u$, where u is the solution of the BVPJ.

Observe that Eq.(2.21) can also be written as

$$-\langle S^* v, w\rangle = \langle S_J u_P, w\rangle - \left\langle j^1, w\right\rangle, \quad \forall w \in \mathcal{E} \subset \tilde{N}_2 \tag{2.22}$$

where $\left\langle j^1, w\right\rangle \equiv \int_\Sigma w j^1_\Sigma dx$.

3. Interpretation of the Algebraic Theory. According to the definitions given in Section 2, a function $v \in \tilde{N}_1$ if and only if

$$\mathcal{L}v \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla uv) + \nabla \cdot (\underline{b}v) + cv = 0, \quad v = 0, \quad on \quad \partial\Omega \quad and \quad [v] = 0, \quad on \quad \Sigma \tag{3.1}$$

In addition, a function $w \in \tilde{N}_2$, if and only if

$$\mathcal{L}^* w \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla w) - \underline{b} \cdot \nabla w + cw = 0, \quad w = 0, \quad on \quad \partial\Omega \quad and \quad [v] = 0, \quad on \quad \Sigma \tag{3.2}$$

i.e., such functions satisfy the homogenous adjoint equation, are continuous and vanish on the external boundary.

When $\mathcal{S}(w, u)$ is given by Eq.(2.17), the *sought information* is the average of the solution of the BVPJ on Σ. Even more, the choice of the pair of decompositions $\{S_J, R_J\}$ and $\{S, R\}$, is optimal [9], because the problem

$$(P - B - J)\,\hat{u} = Pu_\Omega - Bu_\partial - Ju_\Sigma \tag{3.3}$$

subjected to

$$S^* \hat{u} = S^* u_I \tag{3.4}$$

is well posed and local. Indeed, Eq.(2.3) corresponds to the following system of equations

$$\mathcal{L}\hat{u} \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla \hat{u}) + \nabla \cdot (\underline{b}\hat{u}) + c\hat{u} = f_\Omega \equiv \mathcal{L}u_\Omega, \quad in \quad \Omega_i, \quad i = 1, ..., E \tag{3.5}$$

subjected to the boundary conditions

$$\hat{u} = u_\partial; \quad on \quad \partial\Omega, \tag{3.6}$$

and the jump conditions

$$[\hat{u}] = [u_\Sigma] \equiv j^0_\Sigma; \quad on \quad \Sigma, \tag{3.7}$$

In addition, Eq.(2.4) corresponds to the condition

$$\dot{\hat{u}} = \dot{u}_I; \quad on \quad \Sigma, \tag{3.8}$$

i.e., the average across Σ, of $\hat{u}$, is prescribed. Therefore

$$\hat{u}_+ \equiv \dot{\hat{u}} + \frac{1}{2}[u] = \dot{\hat{u}}_I + \frac{1}{2} j^0_\Sigma \quad and \quad \hat{u}_- \equiv \dot{\hat{u}} - \frac{1}{2}[u] = \dot{\hat{u}}_I - \frac{1}{2} j^0_\Sigma \tag{3.9}$$

and it is seen that the system of equations (2.3) and (2.4), is equivalent to a family of well-posed local problems defined in each on of the subdomains of the partition.

The Eqs.(2.20), fulfilled by $u_P \in \hat{D}_1$, are

$$\mathcal{L}u_P \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla u_P) + \nabla \cdot (\underline{b} u_P) + c u_P = f_\Omega \equiv \mathcal{L}u_\Omega, \quad in \quad \Omega_i, \quad i = 1, ..., E \tag{3.10}$$

subjected to the boundary conditions

$$u_P = u_\partial; \quad on \quad \partial\Omega, \tag{3.11}$$

and the jump conditions

$$[u_P] = [u_\Sigma] \equiv j^0_\Sigma; \quad on \quad \Sigma, \tag{3.12}$$

This is the same as Eq. (2.3); i.e., the system of Eqs.(2.5) to (2.7). However, Eq.(2.3) is not imposed on u_P and, therefore, it is not uniquely determined. However, u_P is uniquely determined if it's average across Σ is specified. This can be chosen arbitrarily, except that it must be compatible with the external boundary conditions of Eq. (2.11). It must be observed that in a similar manner, elements of each one of the sets $\tilde{N}_1$ and $\tilde{N}_2$ are determined uniquely by the traces on Σ. A convenient manner of constructing such functions is, therefore, to specify their traces on Σ, and then solve each one of the well posed problems which in this manner are defined in the subdomains of the partition, as will be done numerically in the following Sections.

4. TH-Complete Systems of Test Functions. Discussions of TH-complete systems, in the context of the general theory, may be found in [6],[5]. Additional details in connection with applications to second order elliptic problems may be found in [8]. In what follows the traces on Σ, of the weighting functions, will be taken to be families of piecewise polynomials defined on Σ_{ij} (Fig. 4.1) . This kind of TH-complete families were first described in [5]. According to that figure, Σ_{ij} is the union of four intervals and using the numbering of internal boundaries of Fig. 4.1, associated with each node (x_i, y_j), five classes of weighting functions can be constructed [8]:

Class 0.- This is made of only one function, which is linear in each one of the four interior boundaries between the rectangles of Fig. 4.2, and such that $(x_i, y_j) = 1$.

Class 1.- The restriction to interval "1", of Fig. 4.2 is a polynomial which vanishes at the end points of interval "1". There is one such polynomial for each degree (G) greater than one.

Classes 2 to 4, are defined replacing interval "1" by the interval of the corresponding number in the definition of Class 1 [5].

5. The Numerical Implementation. In the theory that was presented in previous Sections, it is assumed that the exact local solutions are available. In numerical applications, they have to be produced by means of numerical methods and are, therefore only approximate solutions. Actually, the approximate nature of numerical solutions derived using TH-Domain Decomposition (TH-DD), stems from two sources: one of them is due to the approximate nature of the local solutions, which has just been mentioned, and the other one comes from the fact that TH-complete systems for problems in several dimensions constitute infinite families and in numerical implementations one can apply only finite sets of test functions. In particular, with reference to the families of functions introduced in the previous Section, one may construct algorithms in which only polynomials of degree less or equal to G, where G is a given number, are kept in each one of the Classes "1" to "4". In general, each choice of G will give rise to a different kind of algorithm.

In this Section the following notations are used, $H^0_i(x)$ is the one dimensional Hermite cubic polynomial with support in the interval (x_{i-1}, x_{i+1}), which takes the value 1 at node

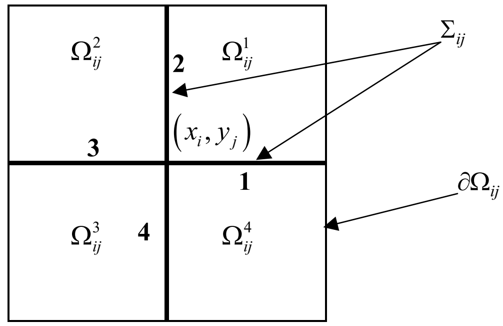

Figure 4.1: Subregion Ω_{ij} associated with the node (x_i, y_j).

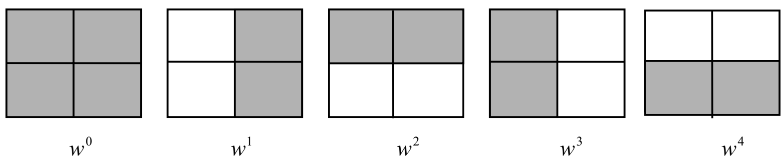

Figure 4.2: Supports of five classes of weighting functions.

x_i and zero at nodes x_{i-1} and x_{i+1}; and its first derivative is zero at all nodes x_{i-1}, x_i and x_{i+1}. Similarly, $H_i^1(x)$ - is the one dimensional Hermite cubic polynomial with support in the interval (x_{i-1}, x_{i+1}), which takes the value zero at nodes x_{i-1}, x_i and x_{i+1}; and its first derivative takes the value 1 at node x_i and zero at the other nodes x_{i-1} and x_{i+1}.

5.1. The Weighting Functions. In the numerical implementations reported in [8], two families of test functions were constructed:

$$\mathcal{F} \equiv \left\{w_{ij}^0, w_{ij}^1, w_{ij}^2\right\} \ \textit{and}\ \widehat{\mathcal{F}} \equiv \left\{\overset{0}{\widehat{w}}_{ij}, \overset{1}{\widehat{w}}_{ij}, \overset{2}{\widehat{w}}_{ij}, \overset{3}{\widehat{w}}_{ij}, \overset{4}{\widehat{w}}_{ij}\right\} \tag{5.1}$$

Here, $w_{ij}^0 \equiv \overset{0}{\widehat{w}}_{ij}$ is the unique function belonging to Class "0"-i.e., piecewise linear on Σ-, of Section 6, and $\overset{\alpha}{\widehat{w}}_{ij}$ is a function of Class "α", for each $\alpha = 1, ..., 4$, which fulfills, at interval "α", the boundary condition $\overset{\alpha}{\widehat{w}}_{ij}(x, y_j) = H_i^1(x)$, for $\alpha = 1, 3$, and $\overset{\alpha}{\widehat{w}}_{ij}(x_i, y) = H_j^1(y)$, for $\alpha = 2, 4$. In addition, one defines

$$w_{ij}^1(x, y) \equiv \overset{1}{\widehat{w}}_{ij}(x, y) + \overset{3}{\widehat{w}}_{ij}(x, y) \ \textit{and}\ w_{ij}^2(x, y) \equiv \overset{2}{\widehat{w}}_{ij}(x, y) + \overset{4}{\widehat{w}}_{ij}(x, y) \tag{5.2}$$

Observe that the supports of w_{ij}^1 and w_{ij}^2 are the whole rectangle Ω_{ij} . In addition, they fulfill the local boundary conditions $w_{ij}^1(x, y_j) = H_i^1(x)$ at the interval $x_{i-1} \leqslant x \leqslant x_{i+1}$ together with $w_{ij}^2(x_i, y) = H_j^1(y)$ at the interval $y_{j-1} \leqslant y \leqslant y_{j+1}$.

In Ref. [8], the family $\widehat{\mathcal{F}}$ was first constructed and the family $\mathcal{F}$ was then derived by application of Eq.(4.2). The family $\widehat{\mathcal{F}}$ was built by solving local boundary value problems in each one of the subregions $\left\{\Omega_{ij}^1, \Omega_{ij}^2, \Omega_{ij}^3, \Omega_{ij}^4\right\}$, separately. This was done introducing a set of functions $\left\{B_{ij}^0, B_{ij}^1, B_{ij}^2, B_{ij}^3, B_{ij}^4\right\}$, which satisfy the boundary conditions and adding to it a linear combination of a family of functions $\left\{N_{ij}^1, N_{ij}^2, N_{ij}^3, N_{ij}^4\right\}$ which vanish on the boundary of each one of the subdomains $\left\{\Omega_{ij}^1, \Omega_{ij}^2, \Omega_{ij}^3, \Omega_{ij}^4\right\}$, in order to fulfill the differential equation.

This leads to

$$\widehat{w}_{ij}^{\alpha}(x,y) = B_{ij}^{\alpha}(x,y) + \sum_{\beta=1}^{4} C_{ij}^{\alpha\beta} N_{ij}^{\beta}(x,y); \quad \alpha = 0, ..., 4 \tag{5.3}$$

The coefficients $C_{ij}^{\alpha\beta}$ are constant at each one of the subdomains $\left\{\Omega_{ij}^1, \Omega_{ij}^2, \Omega_{ij}^3, \Omega_{ij}^4\right\}$, but only piecewise constant in Ω_{ij} (Fig. 4.1). Therefore, each one of the functions $\widehat{w}_{ij}^{\alpha}(x,y)$ has different expressions at each one of the rectangles $\left\{\Omega_{ij}^1, \Omega_{ij}^2, \Omega_{ij}^3, \Omega_{ij}^4\right\}$. The same applies to the functions $\left\{B_{ij}^0, B_{ij}^1, B_{ij}^2, B_{ij}^3, B_{ij}^4\right\}$. The coefficients were obtained solving the system of collocation equations at four Gaussian points

$$\sum_{\beta=1}^{4} C_{ij}^{\alpha\beta} \mathcal{L}^* N_{ij}^{\beta}(x^p, y^p) = \mathcal{L}^* B_{ij}^{\alpha}(x^p, y^p); \quad p = 1, ..., 4 \tag{5.4}$$

5.2. Optimal Interpolation. According to the Theorem of Section 2, the approximate solution $\hat{u} \in \hat{D}_1$ is given by

$$\hat{u} = \hat{u}_P + v \tag{5.5}$$

The function fulfills Eqs.(2.9) to (2.11). As mentioned in Section 2, for its construction one can choose the average of this function arbitrarily, but compatible with the external boundary conditions of Eq.(2.11), and then solve the boundary value problems which are defined, when this specification is joined to the System of Eqs. (2.10) to (2.12), and Eq.(2.9) is also applied. These problems may be solved by any numerical method but in [8], orthogonal collocation was used and similar manner to that explained in the last Sub-Section.

The system of base functions used for building $v \in \hat{D}_1$ can be constructed in a similar manner. It is based on the fact that $v \subset \tilde{N}_1$. So those functions must fulfill the system of equations (2.1); i.e., the homogenous differential equation, and they be continuous and vanish on the external boundary. It is advantageous in many instances, to choose the traces on Σ of such base functions to be the same as those of the weighting functions, as was explained in the Sub-Section 5.1. In that case, Eq.(4.3) can also be applied for the construction the base functions, but to determine the coefficients $C_{ij}^{\alpha\beta}$ one has to replace in Eq.(4.4), the adjoint differential operator $\mathcal{L}^*$ by the differential operator $\mathcal{L}$, itself.

5.3. The Algorithms. To obtain the system of equations satisfied by the values of v on Σ (the values of v on both sides of Σ are the same since it is continuous), one has to apply Eq.(2.22). This is

$$-\int_{\Sigma} v\,[\underline{a}_n \cdot \nabla w]dx = \int_{\Sigma} w\,[\underline{a}_n \cdot \nabla u_P]dx - \int_{\Sigma} w j_{\Sigma}^1 dx \tag{5.6}$$

This form is simpler than that presented in [8], where additional details can be found.

In [8], two algorithms were developed. In **Algorithm 1**, both base and test functions are piecewise linear on Σ, while both of them are piecewise cubic on Σ in **Algorithm 2**.

6. Conclusions. This article illustrates the applications of Trefftz-Herrera methods to the derivation of new discretization procedures. In particular, in Trefftz-Herrera method, the order of approximation that is used in the internal boundary is independent of that used in the interior of the elements of the partition. Using this fact a non-standard method of collocation on Hermite cubics is presented which possesses many advantages over standard methods. Two algorithms are discussed, one in which the interpolation on Σ is piecewise linear and another in which it is piecewise cubic. Quadratic interpolation is also possible but was not discussed here. In this manner, a dramatic reduction in the number of degrees of freedom associated with each node is obtained: in the standard method of collocation that number is two in one dimension, four in two dimensions and eight in three dimensions, while for some of the new algorithms they are only one in all space dimensions -this is due to the relaxation in the continuity conditions required by indirect methods-. Also, the global matrix is symmetric and positive definite when so is the differential operator, while in the standard method of collocation, using Hermite cubics, this does not happen. In addition, it must be mentioned that the boundary value problem with prescribed jumps at the internal boundaries can be treated as easily as the smooth problem -i.e., that with zero jumps-, because the solution matrix and the order of precision is the same for both problems. It must be observed also that, when the indirect method is applied, the error of the approximate solution stems from two sources: the approximate nature of the test functions, and the fact that TH-complete systems of test functions -which are infinite for problems in several dimensions- are approximated by finite families of such functions. In particular, when Hermite cubics are used to approximate the local solutions, in the problems treated in this paper, the error is $O\left(h^4\right)$, if the test functions are piece-wise cubic on Σ, and it is $O\left(h^2\right)$ when the test functions are only piece-wise linear, on that interior boundary. Finally, the construction of the test functions is quite suitable to be computed in parallel.

REFERENCES

[1] B. Bialecki and M. Dryja. Multilevel Additive and Multiplicative Methods for Orthogonal Spline Collocation Problems. *Numer. Math.*, 77:35–58, 1997.

[2] B. Bialecki and G. Fairweather. Orthogonal Spline Collocation Methods for Partial Differential Equations. *Journal of Computational and Applied Mathematics*, 128:55–82, 2001.

[3] M. Diaz and I. Herrera. Indirect Method of Collocation for the Biharmonic Equation. In I. Herrera, D. E. Keyes, O. B. Widlund, and R. Yates, editors, *14th International Conference on Domain Decomposition Methods, Cocoyoc, Mexico*, 2002.

[4] F. Garcia-Nocetti, I. Herrera, R. Yates, E. Rubio, and L. Ochoa. The Direct Approach to Domain Decomposition Methods. In I. Herrera, D. E. Keyes, O. B. Widlund, and R. Yates, editors, *14th International Conference on Domain Decomposition Methods, Cocoyoc, Mexico*, 2002.

[5] I. Herrera. Trefftz-Herrera Domain Decomposition. *Advances in Engineering Software*, 24:43–56, 1995.

[6] I. Herrera. Trefftz Method: A General Theory. *Numerical Methods for Partial Differential Equations*, 16(6):561–580, 2000.

[7] I. Herrera. Unified theory of domain decomposition methods. In I. Herrera, D. E. Keyes, O. B. Widlund, and R. Yates, editors, *14th International Conference on Domain Decomposition Methods, Cocoyoc, Mexico*, 2002.

[8] I. Herrera, R. Yates, and M. Diaz. General Theory of Domain Decomposition: Indirect Methods. *Numerical Methods for Partial Differential Equations*, 18(3):296–322, 2002.

[9] I. Herrera, R. Yates, and M. Diaz. The Indirect Approach to Domain Decomposition. In I. Herrera, D. E. Keyes, O. B. Widlund, and R. Yates, editors, *14th International Conference on Domain Decomposition Methods, Cocoyoc, Mexico*, 2002.

[10] R. Yates and I. Herrera. Parallel Implementation of Indirect Collocation Method. In I. Herrera, D. E. Keyes, O. B. Widlund, and R. Yates, editors, *14th International Conference on Domain Decomposition Methods, Cocoyoc, Mexico*, 2002.