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40. Indirect Method of Collocation for the Biharmonic Equation

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1. Introduction. Indirect methods of collocation (Trefftz-Herrera collocation), that were introduced in previous papers [5],[2], are formulated and applied to the biharmonic equation in two dimensions in combination with orthogonal collocation. The new approach allows relaxed continuity conditions. Two alternative procedures are considered and compared. The first one consists on the straight-forward application of the Trefftz-Herrera indirect collocation method to the biharmonic equation. From another hand, the second one uses the split formulation, also known as the mixed method of Ciarlet and Raviart, in which an auxiliary function is introduced and the biharmonic equation is rewritten as a coupled system of two Poisson equations. Then, to each one of these Poisson equations, Trefftz-Herrera indirect collocation method is applied. As illustration, some preliminary results of application of the last one approach to a numerical example are presented.

2. First Approach: Trefftz-Herrera Formulation for the Biharmonic Equation. In this Section, the general theory of Trefftz-Herrera DDM, presented in [6], will be applied to the biharmonic equation, when the problem is defined in a space of an arbitrary number of dimensions. The procedures are applicable to any kind of boundary conditions for which the problem is well-posed.

The notation is the same as that introduced in [6] and [3]. In particular, $u_\Omega \in \hat{D}_1$, $u_\partial \in \hat{D}_1$ and $u_\Sigma \in \hat{D}_1$ are any functions which satisfy the differential equation, the external boundary conditions and the jump conditions, respectively. and A partition of a domain Ω is being considered and the internal boundary is denoted by Σ (see [6] for further details).

Then, the boundary value problem with prescribed jumps (BVPJ) to be considered is

$$\Delta^2 u = f_\Omega, \quad in \quad \Omega \tag{2.1}$$

subjected to the boundary conditions

$$u = g_\partial^0 \quad and \quad \Delta u = g_\partial^2, \quad on \quad \partial\Omega \tag{2.2}$$

and the jump conditions

$$[u] = j_\Sigma^0, \quad \left[\frac{\partial u}{\partial n}\right] = j_\Sigma^1, \quad [\Delta u] = j_\Sigma^2, \quad \left[\frac{\partial \Delta u}{\partial n}\right] = j_\Sigma^3, \quad on \quad \Sigma \tag{2.3}$$

Since the biharmonic operator $\mathcal{L}$ of Eq.(2.1) is self adjoint, i.e., $\mathcal{L} \equiv \mathcal{L}^*$, then its formal adjoint operator $\mathcal{L}^*$ is given by:

$$\mathcal{L}^* w \equiv \Delta\Delta w; \tag{2.4}$$

Introducing the bilinear vector valued function $\underline{\mathcal{D}}(u, w)$

$$\underline{\mathcal{D}}(u, w) \equiv w\nabla\Delta u + \Delta w\nabla u - \Delta u\nabla w - u\nabla\Delta w \tag{2.5}$$

which satisfies the property that

$$w\mathcal{L}u - u\mathcal{L}^* w = \nabla \cdot \underline{\mathcal{D}}\,(u, w) \tag{2.6}$$

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where

$$\begin{aligned} w\mathcal{L}u &\equiv -\nabla w \cdot (\nabla \Delta u) + \nabla \cdot (w \nabla \Delta u) \\ &= \Delta w \Delta u + \nabla \cdot (w \nabla \Delta u - \nabla w \Delta u) \end{aligned} \tag{2.7}$$

and

$$u\mathcal{L}^* w \equiv \Delta u \Delta w + \nabla \cdot (u \nabla \Delta w - \nabla u \Delta w) \tag{2.8}$$

Recalling that

$$\underline{\mathcal{D}}\,(u,w) \cdot \underline{n} = \mathcal{B}\,(u,w) - \mathcal{C}^*\,(u,w) \tag{2.9}$$

$$-\left[\underline{\mathcal{D}}\,(u,w)\right] \cdot \underline{n} = \mathcal{J}\,(u,w) - \mathcal{K}^*\,(u,w) \tag{2.10}$$

where

$$\mathcal{J}\,(u,w) = -\underline{\mathcal{D}}\,([u]\,,\dot{w}) \cdot \underline{n}; \quad and \quad \mathcal{K}^*\,(u,w) = \underline{\mathcal{D}}\,(\dot{u},[w]) \cdot \underline{n} \tag{2.11}$$

Then, the bilinear functions $\mathcal{B}(u,w)$ and $\mathcal{C}(w,u)$ in according to the boundary conditions given by Eqs. (2.2) may be defined as follow:

$$\mathcal{B}(u,w) \equiv (\Delta w)\,\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\,(\Delta w)\,, \quad \mathcal{C}(w,u) \equiv (\Delta u)\,\frac{\partial w}{\partial n} - w\frac{\partial}{\partial n}\,(\Delta u) \tag{2.12}$$

and correspondingly $\mathcal{J}(u,w)$ and $\mathcal{K}(w,u)$ are defined as:

$$\mathcal{J}(u,w) \equiv \ [u]\dot{\overline{\frac{\partial \Delta w}{\partial n}}} - \left[\frac{\partial u}{\partial n}\right]\dot{\overline{\Delta w}} + [\Delta u]\dot{\overline{\frac{\partial w}{\partial n}}} - \left[\frac{\partial \Delta u}{\partial n}\right]\dot{w}, \tag{2.13}$$

$$\mathcal{K}(w,u) \equiv \ -\dot{u}\left[\frac{\partial \Delta w}{\partial n}\right] + \dot{\overline{\frac{\partial u}{\partial n}}}[\Delta w] - \dot{\overline{\Delta u}}\left[\frac{\partial w}{\partial n}\right] + \dot{\overline{\frac{\partial \Delta u}{\partial n}}}[w] \tag{2.14}$$

Introducing the weak decompositions $\{S_J, R_J\}$ and $\{S, R\}$ of J and K, respectively, as was defined in [6]:

$$\mathcal{S}_J(u,w) \equiv -\dot{w}\left[\frac{\partial \Delta u}{\partial n}\right] - \dot{\overline{\Delta w}}\left[\frac{\partial u}{\partial n}\right], \quad \mathcal{R}_J(u,w) \equiv [u]\dot{\overline{\frac{\partial \Delta w}{\partial n}}} + [\Delta u]\dot{\overline{\frac{\partial w}{\partial n}}} \tag{2.15}$$

$$\mathcal{S}^*(u,w) \equiv \ -\dot{u}\left[\frac{\partial \Delta w}{\partial n}\right] - \dot{\overline{\Delta u}}\left[\frac{\partial w}{\partial n}\right], \quad \mathcal{R}^*(u,w) \equiv [w]\dot{\overline{\frac{\partial \Delta u}{\partial n}}} + [\Delta w]\dot{\overline{\frac{\partial u}{\partial n}}} \tag{2.16}$$

Thus, the bilinear functionals P, B, J, S_J, R_J, Q^*, C^*, K^*, S^* and R^* are defined in the same fashion of Eqs. (5.6)-(5.8) given in Ref. [6], by means of corresponding integrals.

Define $\tilde{N}_1 \equiv N_P \cap N_B \cap N_{R_J}$ and $\tilde{N}_2 \equiv N_Q \cap N_C \cap N_R$, but $\tilde{N}_1 \equiv \tilde{N}_2 \equiv \tilde{N}$ since the biharmonic operator is self adjoint. Then, a function $\phi \in \tilde{N}$, if and only if

$$\begin{aligned} &\Delta\Delta\phi = 0, \quad in \quad \Omega_i \quad (i = 1, ..., E) \\ &\phi = \Delta\phi = 0, \quad on \quad \partial\Omega \\ &[\phi] = [\Delta\phi] = 0, \quad on \quad \Sigma \end{aligned} \tag{2.17}$$

Applying the Theorem of Section 10 of Ref. [6] a Trefftz-Herrera domain decomposition procedure can be obtained:

Assume $\mathcal{E} \subset \tilde{N}$ is a system of weighting functions TH-complete for S^* [6]. Let $u_P \in \hat{D}_1$ be such that

$$Pu_P = Pu_\Omega, \quad Bu_P = Bu_\partial \quad and \quad R_J u_P = R_J u_\Sigma \tag{2.18}$$

Then there exists $v \in \tilde{N}$ such that

$$-\langle S^{*}v, w\rangle = \langle S_J\left(u_P - u_\Sigma\right), w\rangle, \quad \forall w \in \mathcal{E} \subset \tilde{N} \tag{2.19}$$

In addition, define $\hat{u} \in \hat{D}_1$ by $\hat{u} \equiv u_P + v$. Then $\hat{u} \in \hat{D}_1$ contains the sought information. Even more, $\hat{u} \equiv u$, where u is the solution of the BVPJ.

The general outlines about the construction of a TH-complete system of weighting functions $\mathcal{E} \subset \tilde{N}$ for the biharmonic equation are given in [4]. As is known, TH-complete systems for such problems in several dimensions are constituted by infinite families, but, in numerical implementation, only one can use finite sets of test functions produced by means of numerical methods.

In particular, one may construct such systems of test functions solving local BVP problems of Eqs. (2.17) applying collocation method for families of piecewise polynomials of degree less or equal to G on the internal boundary Σ, where G is a given number, in the same manner as was developed in [5],[2] for the second order elliptic equation. In this sense, an indirect Trefftz-Herrera collocation method is obtained, which possesses the property that its global matrix is symmetric and positive definite.

3. Second Approach: Trefftz-Herrera Collocation for the Biharmonic Equation using the splitting formulation. A common approach for solving the biharmonic equation is to use the splitting principle in which an auxiliary function $v = \Delta u$ is introduced and the biharmonic equation is rewritten as a system of two Poisson equations in the form [7]:

$$\left\{\begin{array}{l} -\Delta u = -v; \\ -\Delta v = -f_\Omega; \end{array}\right. \quad in \quad \Omega \tag{3.1}$$

In the context of the finite element Galerkin method, this approach is known as the mixed method of Ciarlet and Raviart [1].

Using the splitting principle of Eq.(3.1), the boundary value problem with prescribed jumps (BVPJ) of the previous section Eqs. (2.1), (2.2) and (2.3), becomes one of solving sequentially two nonhomogeneous Dirichlet problems with prescribed jumps for Poisson's equation:

$$\left\{\begin{array}{l} -\Delta u = -v; \quad in \quad \Omega \\ u = g_1; \quad on \quad \partial\Omega \\ [u] = j^0, \quad \left[\frac{\partial u}{\partial n}\right] = j^1 \quad on \quad \Sigma \end{array}\right. \qquad and \qquad \left\{\begin{array}{l} -\Delta v = -f_\Omega; \quad in \quad \Omega \\ v = g_2; \quad on \quad \partial\Omega \\ [v] = j^2, \quad \left[\frac{\partial v}{\partial n}\right] = j^3, \quad on \quad \Sigma \end{array}\right. \tag{3.2}$$

The resulting coupled system of equations (3.2) can be solved applying the indirect Trefftz-Herrera collocation procedures, developed in [5],[2] for the second order elliptic equation, sequentially to each one of the BVPJs of Eq. 3.2).

In short, the algorithms reported in papers [5],[2] have the following features:

Algorithm I.- The family of test functions, using linear polynomials on the internal boundary Σ, contained only one member associated with each internal node. This leads to an algorithm in which only one degree of freedom is associated with each internal node. The resulting global matrix for each one of the BVPJs of Eqs. (3.2) is nine-diagonal, symmetric and positive definite.

Algorithm II.- Using cubic polynomials on the internal boundary Σ, a family of test functions is composed by three functions (or less, at those nodes in which some of the functions of this family do not satisfy the required zero boundary condition on the external boundary) associated with each node, including boundary nodes. This leads to an algorithm in which three, or less, degrees of freedom are associated with each node. The global matrix for each one of the BVPJs of Eqs. (3.2) is block nine-diagonal, with blocks 3x3, symmetric and positive definite.

Here, it is worth to point out, that the resulting systems in both previous algorithm are symmetric and positive definite and consequently they can be solved using a Conjugate Gradient Method. In contrast, hermite collocation method does not enjoy this property.

4. The Numerical Experiments. In this section, some preliminary results of application of the second approach to a numerical example are presented.

The numerical experiments were carried out for the following BVPJ of the biharmonic equation in two dimensions:

$$\Delta^2 u = f_\Omega; \quad in \quad \Omega = [0,1] \times [0,1] \tag{4.1}$$

where the right hand side term is $f_\Omega = 24\left(e^x + e^y\right) + \left(y^2-1\right)^2 e^x + \left(x^2-1\right)^2 e^y + 8\left[\left(3y^2-1\right)e^x + \left(3x^2-1\right)e^y\right]$ and the corresponding analytical solution has the expression:

$$u(x,y) = \left(y^2-1\right)^2 e^x + \left(x^2-1\right)^2 e^y \tag{4.2}$$

Consequently, the imposed boundary conditions implied by the analytical solution were:

$$u = \left(y^2-1\right)^2 e^x + \left(x^2-1\right)^2 e^y; \quad on \quad \partial\Omega \tag{4.3}$$

$$\Delta u = \left(y^2-1\right)^2 e^x + 4\left(3x^2-1\right)e^y + 4\left(3y^2-1\right)e^x + \left(x^2-1\right)^2 e^y; \quad on \quad \partial\Omega \tag{4.4}$$

and it was considered the continuous case, i.e, the jump conditions imposed were taken equal to zero.

The numerical results are summarized in Figures 4.1 and 4.2. Each one of the examples was solved in a uniform rectangular partition ($E = E_x = E_y$) of the domain using Algorithm I and, subsequently, Algorithm II, for which the weighting functions are piecewise linear and piecewise cubic, respectively, on Σ. The convergence rate of the error -measured in terms of the norm $\|.\|_\infty$ - is $O\left(h^2\right)$ and $O\left(h^4\right)$ respectively, as shown in those figures.

5. Conclusions. In the present article, the indirect approach to domain decomposition methods has been applied to the BVPJ for the biharmonic equation using two different approaches. In the first one, the Trefftz-Herrera indirect method has been applied in straightforward manner to the biharmonic equation without further elaboration, while, in the second one, a BVPJ for the biharmonic equation has been reduced to a system of two BVPJs for Poisson's equation. In both cases, when the numerical procedure which is used for producing the local solutions is collocation, a non-standard method of collocation is obtained which possesses several attractive features. Indeed, a reduction with respect to other collocation methods, in the number of degrees of freedom associated with each node is obtained. This is due to the relaxation in the continuity conditions required by indirect methods-. Also, the global matrix is symmetric and positive definite when so is the differential operator, while in the standard method of collocation, using Hermite cubics, this does not happen. In addition, it must be mentioned that the boundary value problem with prescribed jumps at the internal boundaries can be treated as easily as the smooth problem -i.e., that with zero jumps-, because the solution matrix and the order of precision is the same for both problems. It must be observed also that, when the indirect method is applied, the error of the approximate solution stems from two sources: the approximate nature of the test functions, and the fact that TH-complete systems of test functions -which are infinite for problems in several dimensions- are approximated by finite families of such functions.

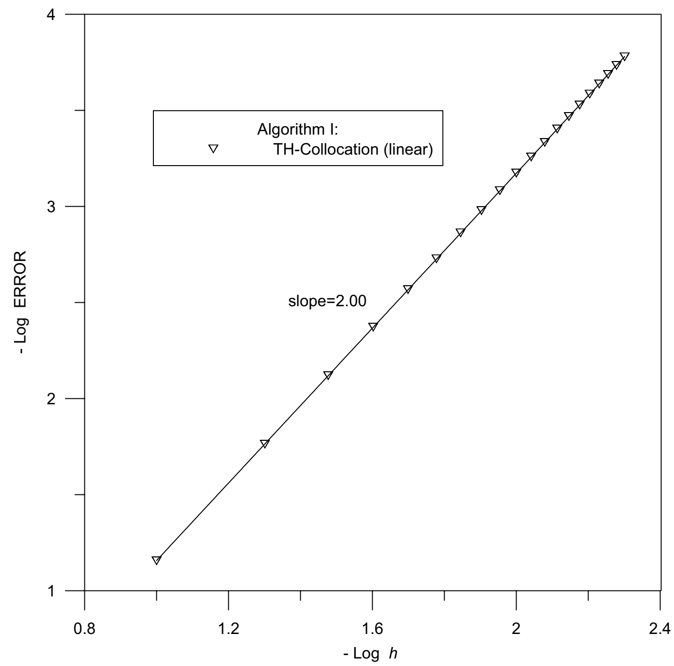

Figure 4.1: Convergence rate of Trefftz-Herrera collocation method for Algorithm I (using linear weighting functions).

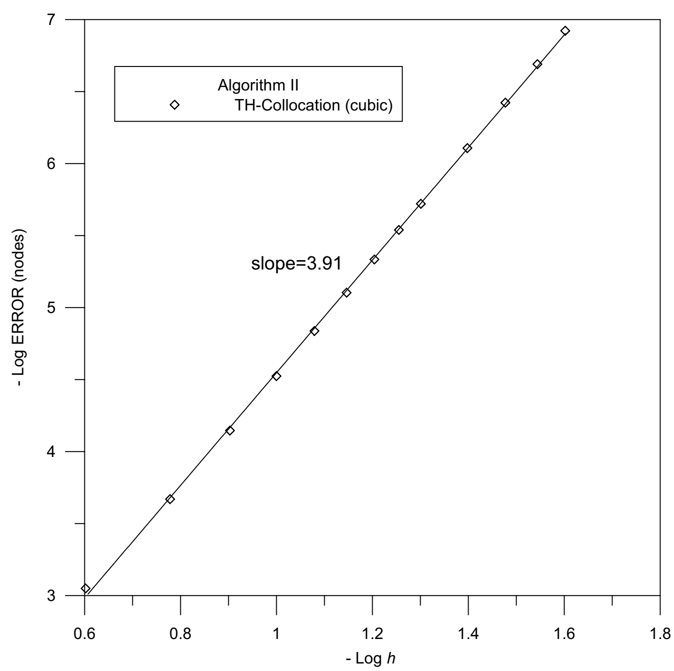

Figure 4.2: Convergence rate of Trefftz-Herrera collocation method for Algorithm II (using cubic weighting functions).

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