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3. A Generalized FETI - DP Method for a Mortar Discretization of Elliptic Problems

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1. Introduction. In this paper, an iterative substructuring method with Lagrange multipliers is proposed for discrete problems arising from approximations of elliptic problem in two dimensions on non-matching meshes. The problem is formulated using a mortar technique. The algorithm belongs to the family of dual-primal FETI (Finite Element Tearing and Interconnecting) methods which has been analyzed recently for discretization on matching meshes. In this method the unknowns at the vertices of substructures are eliminated together with those of the interior nodal points of these substructures. It is proved that the preconditioner proposed is almost optimal; it is also well suited for parallel computations.

We will consider a dual-primal FETI (FETI-DP) method, see [5], [9], and [6], for solving discrete problems arising from the approximation of the Dirichlet problem defined on a union of substructures Ω_i. Each substructure is the union of a number of elements of a coarse, shape-regular triangulation and the number of these triangles, which form such a substructure, is assumed to be uniformly bounded. The discretization is obtained by a mortar method on nonmatching meshes across the interface Γ; see [1], [2]. As in all other iterative substructuring methods, the unknowns corresponding to the interior nodal points are eliminated; in this dual-primal FETI method those at the vertices of Ω_i are eliminated as well. The remaining Schur complement system is solved by a FETI method; see Section 3 for details.

A full analysis of the convergence of several FETI-DP methods has been worked out for finite element approximations on matching meshes; see [9] for the two-dimensional case and [6] for three dimensions. This method, on nonmatching meshes and for the mortar discretizations in the 2-D case, was analyzed in [4]. The preconditioner used there is a standard one and the estimates are not optimal in the general case. In this paper, our analysis is extended to the preconditioner suggested in [7] for matching meshes. The results obtained for this method is better than those of [4]. The superiority of this method is consistent with the numerical results reported on in [11].

The remainder of this paper is organized as follows. In Section 2 differential and discrete problems are formulated while in Section 3 the dual-primal formulation is introduced. Sections 4 is are devoted to the analysis of the proposed preconditioner.

2. Differential and discrete problems. We will consider the following elliptic problem: find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \qquad v \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \qquad f(v) = \int_\Omega f v \, dx$$

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and Ω is a polygonal 2-D region which is a union of polygons Ω_i, $i = 1, \ldots, N$. These subregions form a coarse partitioning of Ω with subdomains with diameters on the order of H. In each Ω_i, we introduce a quasi-uniform, but otherwise arbitrary, triangulation of the subregion with a mesh parameter h_i; generally the resulting triangulations do not match across the edges of the Ω_i.

Let
$$W(\Omega) = W(\Omega_1) \times \cdots \times W(\Omega_N),$$
where $W(\Omega_i)$ are the finite element spaces of piecewise linear, continuous functions on the triangulation of Ω_i and which vanish on $\partial\Omega$ and let the interface be defined by $\Gamma = (\cup \partial\Omega_i) \backslash \partial\Omega$. We choose mortar and nonmortar edges of Γ, and denote them by $\gamma_{m(j)}$ and $\delta_{m(i)}$. In the analysis of the proposed preconditioner, we need a uniform bound on the ratios $h_{\gamma_{m(j)}}/h_{\delta_{m(i)}}$ where $h_{\gamma_{m(j)}}$ and $h_{\delta_{m(i)}}$ are the mesh parameters of $\gamma_{m(j)} \subset \partial\Omega_j$ and $\delta_{m(i)} \subset \partial\Omega_i$, $(\gamma_{m(j)} = \delta_{m(i)})$, respectively. The problem (2.1) is approximated in $X(\Omega)$, a subspace of $W(\Omega)$, of functions which satisfy the mortar condition, see [1], [2],

$$b(u, \psi) \equiv \sum_{i=1}^{N} \sum_{\delta_{m(i)} \subset \partial\Omega_i} \int_{\delta_{m(i)}} (u_i - u_j)\psi ds = 0, \qquad \psi \in M(\Gamma), \tag{2.2}$$

where $M(\Gamma) = \Pi_i \Pi_{\delta_{m(i)} \subset \partial\Omega_i} M(\delta_{m(i)})$ and $M(\delta_{m(i)})$ is the standard mortar space defined on $\delta_{m(i)}$, i.e., piecewise linear continuous functions which are constant on the elements which intersect $\partial\delta_{m(i)}$. Additionally, we assume that the functions of $X(\Omega)$ are continuous at the vertices of Ω_i, i.e., they take the same values, see [2]. In (2.2) $u_i \in W(\Omega_i)$ and $u_j \in W(\Omega_j)$ are the restrictions of u to $\delta_{m(i)}$ and $\gamma_{m(j)}$, respectively.

3. A dual-primal formulation of the problem. We will use some of the notations of [9], [6]. Let
$$K := diag_{j=1}^{N}(K^{(j)}), \tag{3.1}$$
where $K^{(j)}$ is the local stiffness matrix with respect to the standard basis functions of $W(\Omega_j)$. We eliminate the unknown variables corresponding to the interior nodal points and the vertices of Ω_i. A Schur complement $\tilde{S}$ results which is of the form:

$$\tilde{S} := K_{rr} - \begin{pmatrix} K_{ri} & K_{rc} \end{pmatrix} \begin{pmatrix} K_{ii} & K_{ic} \\ K_{ci} & K_{cc} \end{pmatrix}^{-1} \begin{pmatrix} K_{ir} \\ K_{cr} \end{pmatrix}. \tag{3.2}$$

Here,
$$\tilde{K} := \begin{pmatrix} K_{ii} & K_{ic} & K_{ir} \\ K_{ci} & K_{cc} & K_{cr} \\ K_{ri} & K_{rc} & K_{rr} \end{pmatrix},$$
where the rows correspond to the interior, vertex, and remaining (edge) nodal points, respectively. It is obtained from K by reordering the unknowns and taking into account that the functions of $X(\Omega)$ are continuous at the subdomain vertices.

Let
$$W(\Gamma) = W(\partial\Omega_1) \times \cdots \times W(\partial\Omega_N)$$

and let $W_r(\Gamma)$ denote the space of functions defined at the edge nodal points and which vanish at the vertices of Ω_i, and let $W_c(\Gamma)$ be the subspace of $W(\Gamma)$ of functions that are continuous at the vertices.

The dual-primal formulation of the mortar discretization of (2.1) is: find $u_r^* \in W_r(\Gamma)$ such that

$$J(u_r^*) = \min_{\substack{v_r \in W_r \\ Bv_r = 0}} J(v_r), \quad J(v) := 1/2\langle \tilde{S}v, v\rangle - \langle f_r, v\rangle, \tag{3.3}$$

where < , > means the scalar product in l_2. B is defined by the mortar condition (2.2) as follows: on $\delta_{m(i)} \subset \partial\Omega_i$, $\delta_{m(i)} = \gamma_{m(j)}$, the matrix form of (2.2) is

$$B_{\delta_{m(i)}} \underline{u}_{i|\delta_{m(i)}} - B_{\gamma_{m(j)}} \underline{u}_{j|\gamma_{m(j)}} = 0. \tag{3.4}$$

Here,

$$B_{\delta_{m(i)}} = \{(\psi_l, \varphi_p)_{L^2(\delta_{m(i)})}\}, \qquad l, p = 1, ..., n_{m(i)},$$

$$\varphi_p \in W_i(\partial\Omega_i)_{|\delta_{m(i)}}, \psi_l \in M(\delta_{m(i)}) \ ,$$

$$B_{\gamma_{m(j)}} = \{(\psi_l, \varphi_k)_{L^2(\delta_{m(i)})}\}, \quad l = 1, ..., n_{m(i)}, \ \ k = 1, ..., n_{m(j)},$$

and $\varphi_k \in W_j(\partial\Omega_j)_{|\gamma_{m(j)}}; n_{m(i)}$ and $n_{m(j)}$ are the number of interior nodal points of $\delta_{m(i)}$ and $\gamma_{m(j)}$, respectively. Condition (3.4) can be rewritten as

$$\underline{u}_{i|\delta_{m(i)}} - B^{-1}_{\delta_{m(i)}} B_{\gamma_{m(j)}} \underline{u}_{j|\gamma_{m(j)}} = 0, \tag{3.5}$$

since the matrix $B_{\delta_{m(i)}} = B^T_{\delta_{m(i)}} > 0$. We note that $B_{\gamma_{m(j)}}$ is generally a rectangular matrix.

The matrix B is block-diagonal,

$$B = blockdiag\{D_{\delta_{m(i)}}\} \tag{3.6}$$

for i = 1,. . . ,N, and $\delta_{m(i)} \subset \partial\Omega_i$ where

$$D_{\delta_{m(i)}} \begin{pmatrix} \underline{u}_{i|\delta_{m(i)}} \\ \underline{u}_{j|\gamma_{m(j)}} \end{pmatrix} \equiv (I \ \ (-B^{-1}_{\delta_{m(i)}} B_{\gamma_{m(j)}})) \begin{pmatrix} \underline{u}_{i|\delta_{m(i)}} \\ \underline{u}_{j|\gamma_{m(j)}} \end{pmatrix}. \tag{3.7}$$

Introducing a space of Lagrange multipliers $V := Im(B)$ to enforce the constraints $Bv_r = 0$, we obtain a saddle point formulation of (3.3),

$$\begin{pmatrix} \tilde{S} & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u_r^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} \tilde{f}_r \\ 0 \end{pmatrix}, \tag{3.8}$$

where $u_r^* \in W_r(\Gamma)$ and $\lambda^* \in V$. We obtain the problem

$$F\lambda^* = d, \tag{3.9}$$

where

$$F = B\tilde{S}^{-1}B^T, \qquad d = B\tilde{S}^{-1}\tilde{f}_r.$$

We now define a preconditioner for F. Let

$$S^{(j)} = K_{bb}^{(j)} - K_{bi}^{(j)}(K_{ii}^{(j)})^{-1}K_{ib}^{(j)}, \tag{3.10}$$

be the standard Schur complement of $K^{(j)}$ where $K_{ii}^{(j)}$ and $K_{bb}^{(j)}$ are the submatrices of $K^{(j)}$ corresponding to the interior and boundary unknowns of $\bar{\Omega}_j$, respectively. Let

$$S_{rr}^{(j)} = K_{rr}^{(j)} - K_{ri}^{(j)}(K_{ii}^{(j)})^{-1}K_{ir}^{(j)} \tag{3.11}$$

denote the Schur complement of $K^{(j)}$, without the rows and columns corresponding to the vertices. It is the restriction of $S^{(j)}$ to the space of functions which vanish at the vertices. Let

$$S := diag_{i=1}^{N}(S^{(i)}), \qquad S_{rr} := diag_{i=1}^{N}(S_{rr}^{(i)}).$$

We can take a preconditioner M of F of the form

$$M = (BS_{rr}B^T)^{-1}, \qquad M^{-1} = BS_{rr}B^T. \tag{3.12}$$

This preconditioner, called the standard one, was analyzed in [4] for two cases. In the first case there is Neumann-Dirichlet (N-D) ordering of substructures Ω_i; a Neumann substructure Ω_i is one where all sides are chosen as mortars while for a Dirichlet substructure all sides are nonmortars. In the second case, we do not have such ordering. For this preconditioner a bound was established for the condition number of FETI-DP method which is proportional to $(1+\log(H/h))^2$ in the first case while we need $(1+\log(H/h))^4$ in the second case.

We will now design a preconditioner for FETI-DP method which is similar to the one used in a FETI method on matching meshes in [7]. It is analyzed in the general case and a bound is obtained for the condition number of this method that is proportional to $(1+\log(H/h))^2$ only.

Let us introduce a scaling in $D_{\delta_{m(i)}}$, cf. (3.7), given by

$$\tilde{D}_{\delta_{m(i)}}\begin{pmatrix}\underline{u}_{i|\delta_{m(i)}}\\ \underline{u}_{j|\gamma_{m(j)}}\end{pmatrix} \equiv \{I \;\; (-\alpha_{ij}^{(m)}B_{\delta_{m(i)}}^{-1}B_{\gamma_{m(j)}})\}\begin{pmatrix}\underline{u}_{i|\delta_{m(i)}}\\ \underline{u}_{j|\gamma_{m(j)}}\end{pmatrix} \tag{3.13}$$

where $\alpha_{ij}^{(m)} = (h_{\delta_{m(i)}}/h_{\gamma_{m(j)}})$ and, cf. (3.6), let

$$\tilde{B} = blockdiag(\tilde{D}_{\delta_{m(i)}}) \tag{3.14}$$

for $i = 1, \ldots, N$, and $\delta_{m(i)} \subset \partial\Omega_i$. The preconditioner $\tilde{M}$ for F is of the form

$$\tilde{M}^{-1} = (B\tilde{B}^T)^{-1}\tilde{B}S_{rr}\tilde{B}^T(\tilde{B}B^T)^{-1}. \tag{3.15}$$

Remark We could also take

$$\widehat{M}^{-1} = diag(BB^T)^{-1}BS_{rr}B^T diag(BB^T)^{-1} \tag{3.16}$$

This corresponds to the preconditioner introduced in [8] for a FETI method on matching and nonmatching triangulations. To our knowledge, there is no full analysis of that method.

4. Convergence analysis. In this section we prove that the preconditioner $\tilde{M}$ is spectrally equivalent to F, except for a $(1+\log(H/h))^2$ factor; see Theorem 1. We follow the approach of [9], [6]. We first prove two auxiliary results.

Let us introduce the operator $P = \tilde{B}^T(B\tilde{B}^T)^{-1}B$ defined on W_r. We note that P is a projection, $P^2 = P$.

Lemma 1 Let $h_{\delta_{m(i)}} \sim h_{\gamma_{m(j)}}$, $\delta_{m(i)} \subset \partial\Omega_i$, i = 1,. . . , N be satisfied. Then for $w_r \in W_r$

$$|Pw_r|^2_{S_{rr}} \le C(1+\log(H/h))^2 |w_r|^2_{\tilde{S}} \tag{4.1}$$

holds where the constant C is independent of $H = max_i H_i$ and $h = min_i h_i$.

Proof Let w be the discrete harmonic extension of w_r to the interior points and to the vertices in the sense of $<\tilde{S}u, u>$. We have

$$|w_r|^2_{\tilde{S}} = |w|^2_{S}, \quad w \in W_c. \tag{4.2}$$

Using this fact, we estimate $|Pw_r|_{S_{rr}}$ in terms of $|w|^2_S$. We construct $I^H w$ the function which is linear on the edges and which takes the values of w at the vertices. Setting $u \equiv w - I^H w$ and noting that $BI^H w = 0$, we have

$$|Pw_r|^2_{S_{rr}} = |Pu|^2_{S_{rr}} = \sum_{i=1}^{N} |Pu|^2_{S^{(i)}}. \tag{4.3}$$

We note that $Pu = 0$ at the vertices. Using that and setting $v = (B\tilde{B}^T)^{-1}Bu$, we have

$$|Pw_r|^2_{S^{(i)}} = |\tilde{B}^T v|^2_{S^{(i)}} \le C\{ \sum_{\delta_{m(i)} \subset \partial\Omega_i} |\tilde{B}^T v|^2_{S_{\delta_{m(i)}}} + \sum_{\gamma_{m(i)} \subset \partial\Omega_i} |\tilde{B}^T v|^2_{S_{\gamma_{m(i)}}} \}, \tag{4.4}$$

where $S_{\delta_{m(i)}}$ and $S_{\gamma_{m(i)}}$ are matrix representations of the $H^{1/2}_{00}$ - norm on $\delta_{m(i)}$ and $\gamma_{m(i)}$, respectively; see Lemma 2 below. From the structure of $\tilde{B}$, see (3.13) and (3.14), it follows that

$$|\tilde{B}^T v|^2_{S_{\delta_{m(i)}}} = |v_i|^2_{S_{\delta_{m(i)}}} \tag{4.5}$$

and that

$$|\tilde{B}^T v|^2_{S_{\gamma_{m(i)}}} = |\tilde{B}^T_{ji} v_j|^2_{S_{\gamma_{m(i)}}}$$

where, here and below, $\tilde{B}_{ji} = \alpha^{(m)}_{ji} B^{-1}_{\delta_{m(j)}} B_{\gamma_{m(i)}} \equiv \alpha^{(m)}_{ji} B_{ji}$, $\gamma_{m(i)} = \delta_{m(j)}$, $\delta_{m(j)} \subset \partial\Omega_j$, and v_i and v_j are restrictions of v to $\bar{\Omega}_i$ and $\bar{\Omega}_j$, respectively.

We now prove that

$$|\tilde{B}^T_{ji} v_j|^2_{S_{\gamma_{m(i)}}} \le C|v_j|^2_{S_{\delta_{m(j)}}}. \tag{4.6}$$

We note that $v = 0$ at the cross points. We have

$$|\tilde{B}^T_{ji} v_j|^2_{S_{\gamma_{m(i)}}} = \sup_{\varphi} \frac{|< S^{1/2}_{\gamma_{m(i)}} \tilde{B}^T_{ji} v_j, \varphi >_{\gamma_{m(i)}}|^2}{|\varphi|^2_{\gamma_{m(i)}}} =$$

$$= \sup_t \frac{| < v_j, \tilde{B}_{ji} t >_{\delta_{m(j)}} |^2}{|S^{-1/2}_{\gamma_{m(i)}} t|^2_{\gamma_{m(i)}}},$$

where $< \cdot, \cdot >_{\gamma_{m(i)}}$ and $< \cdot, \cdot >_{\delta_{m(j)}}$ are ℓ_2-inner products. Hence,

$$|\tilde{B}^T_{ji} v_j|^2_{S_{\gamma_{m(i)}}} \leq |S^{1/2}_{\delta_{m(j)}} v_j|^2_{\delta_{m(j)}} \sup_t \frac{|S^{-1/2}_{\delta_{m(j)}} \tilde{B}_{ji} t|^2_{\delta_{m(j)}}}{|S^{-1/2}_{\gamma_{m(i)}} t|^2_{\gamma_{m(i)}}}. \tag{4.7}$$

Let, here and below, $\pi_{\delta_{m(j)}}(t, 0)$ correspond to $B_{ji}t$ for a piecewise linear, continuous function, also denoted by t, and defined on $\gamma_{m(i)}$ by a vector t with components that vanish at the end of $\gamma_{m(i)}$. Using Lemma 2, below, and the $H^{-1/2}$-stability of $\pi_{\delta_{m(j)}}$, see [1], we get

$$|S^{-1/2}_{\delta_{m(j)}} \tilde{B}_{ji} t|^2_{\delta_{m(j)}} \leq C h^{-2}_{\gamma_{m(i)}} ||\pi_{\delta_{m(j)}}(t, 0)||^2_{H^{-1/2}(\delta_{m(j)})} \leq$$

$$\leq C h^{-2}_{\gamma_{m(i)}} \parallel t \parallel^2_{H^{-1/2}(\gamma_{m(i)})} \leq C |S^{-1/2}_{\gamma_{m(i)}} t|^2.$$

Here $H^{-1/2}$ is the dual to $H^{1/2}_{00}$. Using this bound in (4.7), we get

$$|\tilde{B}^T_{ji} v_j|^2_{S_{\gamma_{m(i)}}} \leq C |S^{1/2}_{\delta_{m(j)}} v_j|^2_{\delta_{m(j)}},$$

which proves (4.6). Using (4.5) and (4.6) in (4.4), we have

$$|\tilde{B}^T v|^2_{S^{(i)}} \leq C\{ \sum_{\delta_{m(i)} \subset \partial\Omega_i} |v_i|^2_{S_{\delta_{m(i)}}} + \sum_{\delta_{m(j)}} |v_j|^2_{S_{\delta_{m(j)}}} \}, \tag{4.8}$$

where the second sum is taken over $\delta_{m(j)} \subset \Omega_j$ such that $\gamma_{m(i)} = \delta_{m(j)}$ with $\gamma_{m(i)} \subset \partial\Omega_i$.

We now estimate the term $|S^{1/2}_{\delta_{m(i)}} v_i|^2$ of (4.8) as follows. We have

$$|v|^2_{S_{\delta_{m(i)}}} \leq 2\{ |(B\tilde{B}^T)^{-1} Bu - \frac{1}{2} Bu|^2_{S_{\delta_{m(i)}}} + \frac{1}{4} |Bu|^2_{S_{\delta_{m(i)}}} \}. \tag{4.9}$$

We first estimate the second term. Using the structure of B, see (3.7), we have

$$|Bu|^2_{S_{\delta_{m(i)}}} \leq 2\{ |u_i|^2_{S_{\delta_{m(i)}}} + |B_{ij} u_j|^2_{S_{\delta_{m(i)}}} \}, \tag{4.10}$$

where $\delta_{m(i)} = \gamma_{m(j)}$, $\gamma_{m(j)} \subset \Omega_j$. We note that

$$|B_{ij} u_j|^2_{S_{\delta_{m(i)}}} \leq C \parallel \pi_{\delta_{m(i)}}(u_j, 0) \parallel^2_{H^{1/2}_{00}(\delta_{m(i)})} \leq$$

$$\leq C |u_j|^2_{H^{1/2}_{00}(\gamma_{m(j)})} \leq C |u_j|^2_{S_{\gamma_{m(j)}}}.$$

Here we have used the $H^{1/2}_{00}$ - stability of $\pi_{\delta_{m(i)}}$, see [1]. Using this in (4.10), we have

$$|Bu|^2_{S_{\delta_{m(i)}}} \leq C\big\{ |u_i|^2_{S_{\delta_{m(i)}}} + |u_j|^2_{S_{\gamma_{m(j)}}} \big\}. \tag{4.11}$$

To estimate the first term of (4.9), we first use the fact that $|(B\tilde{B}^T)^{-1}| \leq 1$ since $B\tilde{B}^T = I_{\delta_{m(i)}} + B_{ij}\tilde{B}_{ij}^T$ on $\delta_{m(i)}$; this follows from the structure of B. Here $I_{\delta_{m(i)}}$ is the identity matrix of a dimension equal to the number of nodal points of $\delta_{m(i)}$. Using that and $S_{\delta_{m(i)}} \leq CI_{\delta_{m(i)}}$, we have

$$|(B\tilde{B}^T)^{-1}Bu - \frac{1}{2}Bu|^2_{S_{\delta_{m(i)}}} \leq C|(B\tilde{B}^T)^{-1}(Bu - \frac{1}{2}(B\tilde{B}^T)Bu)|^2_{\ell_2} \leq C|Bu - \frac{1}{2}B\tilde{B}^T Bu|^2_{\ell_2}. \tag{4.12}$$

Setting $z = Bu$ and noting that on $\delta_{m(i)}$

$$(z - \frac{1}{2}B\tilde{B}^T z)_{|\delta_{m(i)}} = \frac{1}{2}(z_i - B_{ij}\tilde{B}_{ij}^T z_i),$$

we have

$$|z - \frac{1}{2}B\tilde{B}^T z|^2_{\ell_2} = \frac{1}{4}|z_i - B_{ij}\tilde{B}_{ij}^T z_i|^2_{\ell_2}.$$

Let $g \equiv \tilde{B}_{ij}^T z_i$. We note that $z_i = \pi(z_i, 0)$ on $\delta_{m(i)}$. Using that

$$\begin{aligned} |z_i - B_{ij}g|^2_{\ell_2} &\leq \frac{C}{h_{\delta_{m(i)}}} \parallel z_i - \pi_{\delta_{m(i)}}(g,0) \parallel^2_{L^2(\delta_{m(i)})} = \\ &= \frac{C}{h_{\delta_{m(i)}}} \parallel \pi_{\delta_{m(i)}}(z_i - g, 0) \parallel^2_{L^2(\delta_{m(i)})} \leq \\ &\leq \frac{C}{h_{\delta_{m(i)}}} \parallel z_i - g \parallel^2_{L^2(\delta_{m(i)})}, \end{aligned} \tag{4.13}$$

in view of the L_2 - stability of $\pi_{\delta_{m(i)}}$); see [1].

The question is now how to estimate the right hand side of (4.13). We do that as follows. Let $\bar{z}_i$ be a piecewise constant function on $\delta_{m(i)}$ with respect to the triangulation on $\delta_{m(i)}$ and with values $z_i(x_k)$ at $x_k \in \delta_{m(i)h}$, the set of nodal points on $\delta_{m(i)}$. Using this, we get

$$\frac{1}{h_{\delta_{m(i)}}} \parallel z_i - g \parallel^2_{L^2(\delta_{m(i)})} \leq \frac{2}{h_{\delta_{m(i)}}} \parallel \bar{z}_i - g \parallel^2_{L^2(\delta_{m(i)})} + C|z_i|^2_{S_{\delta_{m(i)}}}, \tag{4.14}$$

since

$$\parallel z_i - \bar{z}_i \parallel^2_{L^2(\delta_{m(i)})} \leq Ch_{\delta_{m(i)}} \parallel z_i \parallel^2_{H^{1/2}_{00}(\delta_{m(i)})} \leq Ch_{\delta_{m(i)}} |z_i|^2_{S_{\delta_{m(i)}}}, \tag{4.15}$$

in view of a known estimate and Lemma 2.

There remains to prove that

$$\frac{1}{h_{\delta_{m(i)}}} \parallel \bar{z}_i - g \parallel^2_{L^2(\delta_{m(i)})} \leq C|z|^2_{S_{\delta_{m(i)}}}. \tag{4.16}$$

We do this as follows. Let $\bar{g}_\gamma$ be a piecewise constant function on $\gamma_{m(j)}$ with respect to the triangulation on $\gamma_{m(j)}$ and with values $g(x_k) = (\tilde{B}_{ij}^T z_i)_k$ at $x_k \in \gamma_{m(j)h}$, the set of nodal points on $\gamma_{m(j)}$. We have,

$$\frac{1}{h_{\delta_{m(i)}}} \parallel \bar{z}_i - g \parallel^2_{L^2} \leq \frac{2}{h_{\delta_{m(i)}}} \{\parallel \bar{z}_i - \bar{g}_\gamma \parallel^2_{L^2} + \parallel g - \bar{g}_\gamma \parallel^2_{L^2}\}. \tag{4.17}$$

It is known that

$$\| g - \bar{g}_\gamma \|^2_{L^2(\gamma_{m(j)})} \leq C h_{\gamma_{m(j)}} \| g \|^2_{H^{1/2}_{00}(\gamma_{m(j)})} .$$

On the other hand,

$$\| g \|^2_{H^{1/2}_{00}(\gamma_{m(j)})} \leq C |z_i|^2_{S_{\delta_{m(i)}}},$$

in view of (4.6). Hence,

$$\frac{1}{h_{\delta_{m(i)}}} \| g - \bar{g}_\gamma \|^2_{L(\delta_{m(i)})} \leq C \frac{h_{\gamma_{m(j)}}}{h_{\delta_{m(i)}}} |z_i|^2_{S_{\delta_{m(i)}}} \leq C |z_i|^2_{S_{\delta_{m(i)}}}. \tag{4.18}$$

We now estimate $h^{-1}_{\delta_{m(i)}} \| \bar{z}_i - \bar{g}_\gamma \|^2_{L^2}$ of (4.17) as follows. We have

$$\| \bar{z}_i - \bar{g}_\gamma \|^2_{L^2(\delta_{m(i)})} = \sup_\varphi \frac{|(\bar{z}_i - \bar{g}_\gamma, \varphi)_{L^2}|^2}{\| \varphi \|^2_{L^2}}. \tag{4.19}$$

Let $Q_\delta \varphi$ and $Q_\gamma \varphi$ be the L_2 - projections on the spaces of piecewise constant functions on the triangulations of $\delta_{m(i)}$ and $\gamma_{m(j)}$, respectively. It is known that,

$$\| z_i - Q_\delta z_i \|^2_{L^2(\delta_{m(i)})} \leq C h_{\delta_{m(i)}} |z_i|^2_{H^{1/2}_{00}(\delta_{m(i)})}$$

and

$$\| z_i - Q_\gamma z_i \|^2_{L^2(\gamma_{m(j)})} \leq C h_{\gamma_{m(j)}} |z_i|^2_{H^{1/2}_{00}(\gamma_{m(j)})}.$$

Using the projections, we have

$$(\bar{z}_i - \bar{g}_\gamma, \varphi)_{L^2(\delta_{m(i)})} = (\bar{z}_i, Q_\delta \varphi)_{L^2(\delta_{m(i)})} - (\bar{g}_\gamma, Q_\gamma \varphi)_{L^2(\gamma_{m(j)})}. \tag{4.20}$$

We note that

$$(\bar{g}_\gamma, Q_\gamma \varphi)_{L^2(\gamma_{m(j)})} = h_{\gamma_{m(j)}} \sum_{x_k \in \gamma_{m(j)h}} g_\gamma(x_k)(Q_\gamma \varphi)(x_k) =$$

$$= \alpha^{(m)}_{ij} h_{\gamma_{m(j)}} (B^T_{ij} z_i, Q_\gamma \varphi)_{\ell_2} =$$
$$= h_{\delta_{m(i)}} (z_i, B_{ij} Q_\gamma \varphi)_{\ell_2} = (\bar{z}_i, \overline{B_{ij} Q_\gamma \varphi})_{L^2(\delta_{m(i)})},$$

where $\overline{B_{ij} Q_\gamma \varphi}$ is a piecewise constant function with respect to the $\delta_{m(i)}$ triangulation. Using this in (4.20), we have

$$(\bar{z}_i - \bar{g}_\gamma, \varphi)_{L^2(\delta_{m(i)})} = (\bar{z}_i, Q_\delta \varphi - \overline{B_{ij} Q_\gamma \varphi})_{L^2(\delta_{m(i)})}.$$

Hence,

$$\begin{aligned} (\bar{z}_i - \bar{g}_\gamma, \varphi)_{L^2(\delta_{m(i)})} &\leq \| z_i - \bar{z}_i \|_{L^2} \| Q_\delta \varphi - \overline{B_{ij} Q_\gamma \varphi} \|_{L^2} + \\ &+ \| z_i \|_{H^{1/2}_{00}(\delta_{m(i)})} \| Q_\delta \varphi - \overline{B_{ij} Q_\gamma \varphi} \|_{H^{-1/2}(\delta_{m(i)})} . \end{aligned} \tag{4.21}$$

We note that $\overline{B_{ij} Q_\gamma \varphi} = \overline{\pi_{\delta_{m(i)}}(Q_\gamma \varphi, 0)}$. Using that, we have

$$\| Q_\delta \varphi - \overline{B_{ij} Q_\gamma \varphi} \|_{H^{-1/2}(\delta_{m(i)})} \leq \| Q_\delta \varphi - \varphi \|_{H^{-1/2}(\delta_{m(i)})} +$$

$$\| \varphi - \pi_{\delta_{m(i)}}(Q_\gamma\varphi, 0) \|_{H^{-1/2}(\delta_{m(i)})} + \| \pi_{\delta_{m(i)}}(Q_\gamma\varphi, 0) - \overline{\pi_{\delta_{m(i)}}(Q_\gamma\varphi, 0)} \|_{H^{-1/2}(\delta_{m(i)})} .$$

Using known estimates for these terms, we get

$$\| Q_\delta\varphi - \overline{B_{ij}Q_\gamma\varphi} \|^2_{H^{-1/2}(\delta_{m(i)})} \leq C(h_{\delta_{m(i)}} + h_{\gamma_{m(j)}}) \| \varphi \|^2_{L^2(\delta_{m(i)})} . \tag{4.22}$$

It is easy to see that

$$\| Q_\delta\varphi - \overline{B_{ij}Q_\gamma\varphi} \|^2_{L^2(\delta_{m(i)})} \leq C \| \varphi \|^2_{L^2(\delta_{m(i)})} . \tag{4.23}$$

Using the estimates (4.22), (4.23), and (4.15) in (4.21), we get

$$(\bar{z}_i - \bar{g}_\gamma, \varphi)_{L^2(\delta_{m(i)})} \leq Ch_{\delta_{m(i)}} \| z_i \|_{H^{1/2}_{00}(\delta_{m(i)})} \| \varphi \|_{L^2(\delta_{m(i)})} .$$

In turn, substituting this into (4.19), we have

$$\| \bar{z}_i - \bar{g}_\gamma \|^2_{L^2(\delta_{m(i)})} \leq Ch_{\delta_{m(i)}} \| z_i \|^2_{H^{1/2}_{00}(\delta_{m(i)})} \leq Ch_{\delta_{m(i)}} |z_i|^2_{S_{\delta_{m(i)}}} .$$

Using this and (4.18) in (4.17) and the resulting inequality in (4.14), we get

$$\frac{1}{h_{\delta_{m(i)}}} \| z_i - g \|^2_{L^2(\delta_{m(i)})} \leq C|z_i|^2_{S_{\delta_{m(i)}}} .$$

In turn, using this estimate in (4.13) and the resulting inequality in (4.12), we have

$$|(B\tilde{B}^T)^{-1}Bu - 1/2Bu|^2_{S_{\delta_{m(i)}}} \leq C|Bu|^2_{S_{\delta_{m(i)}}} \leq C\{|u_i|^2_{S_{\delta_{m(i)}}} + |u_j|^2_{S_{\gamma_{m(j)}}}\};$$

we have also used (4.11). Using this and again (4.11) in (4.9) and the resulting inequality in (4.8), we get, cf. (4.4),

$$|Pw_r|^2_{S^{(i)}} \leq C\{ \sum_{\delta_{m(i)} \subset \partial\Omega_i} |u_i|^2_{S_{\delta_{m(i)}}} + \sum_{\gamma_{m(i)} = \delta_{m(j)}} |u_j|^2_{S_{\gamma_{m(j)}}} \}, \tag{4.24}$$

where the second sum is taken over $\gamma_{m(i)} \subset \Omega_i$. It is known that for $u = w - I^H w$ we have

$$|u_i|^2_{S_{\delta_{m(i)}}} \leq C(1 + \log(H/h))^2 |w_i|^2_{S_i}$$

Using this in (4.24) and summing the resulting inequality with respect i, we get (4.1), in view of (4.2). The proof is complete.

Lemma 2 Let $h_{\delta_{m(i)}} \sim h_{\gamma_{m(j)}}$. Then for $u \in W(\delta_{m(i)})$, which vanishes at the ends of $\delta_{m(i)}$ the following hold:

$$C_0 < S_{\delta_{m(i)}} u, u >_{\ell_2} \leq \| u \|^2_{H^{1/2}(\delta_{m(i)})} \leq C_1 < S_{\delta_{m(i)}} u, u >_{\ell_2} . \tag{4.25}$$

and

$$C_2 h^2_{\delta_{m(i)}} < S^{-1}_{\delta_{m(i)}} u, u > \leq \| u \|^2_{H^{-1/2}(\delta_{m(i)})} \leq C_3 h^2_{\delta_{m(i)}} < S^{-1}_{\delta_{m(i)}} u, u > \tag{4.26}$$

where C_i are positive constants independent of $h_{\delta_{m(i)}}$.

Proof The proof of (4.25) can be found for example in [3]. The proof of (4.26) follows from Proposition 7.5 in [10].
Cororally (see the proof of Lemma 1 in [4])

$$|B_{ij}t|^2_{S^{-1}_{\delta_{m(i)}}} \leq C|t|^2_{S^{-1}_{\gamma_{m(j)}}}. \tag{4.27}$$

Proof Let $\pi_{\delta_{m(i)}}(t,0)$ correspond to $B_{ij}t$ on $\delta_{m(i)}$ where t is a piecewise linear continuous function, also denoted by t, and defined by the vector t. Using (4.26), we have

$$h^2_{\delta_{m(i)}}|B_{ij}t|^2_{S^{-1}_{\delta_{m(i)}}} \leq C \parallel \pi_{\delta_{m(i)}}(t,0) \parallel^2_{H^{-1/2}(\delta_{m(i)})} . \tag{4.28}$$

We show below that

$$\parallel \pi_{\delta_{m(i)}}(t,0) \parallel^2_{H^{-1/2}(\delta_{m(i)})} \leq C(1+\frac{h_{\delta_{m(i)}}}{h_{\gamma_{m(j)}}}) \parallel t \parallel^2_{H^{-1/2}(\gamma_{m(j)})} . \tag{4.29}$$

Using this in (4.28), that $h_{\delta_{m(i)}} \sim h_{\gamma_{m(j)}}$, and (4.26), we get (4.27).

There remains to prove (4.29). We have

$$\begin{aligned}\parallel \pi_{\delta_{m(i)}}(t,0) \parallel_{H^{-1/2}(\delta_{m(i)})} &\leq \parallel t \parallel_{H^{-1/2}(\delta_{m(i)})} \\ &+ \parallel \pi_{\delta_{m(i)}}(t,0) - t \parallel_{H^{-1/2}(\delta_{m(i)})}\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}&\parallel \pi_{\delta_{m(i)}}(t,0) - t \parallel_{H^{-1/2}(\delta_{m(i)})} = \\ &= \max_g \frac{(\pi_{\delta_{m(i)}}(t,0) - t, g - Q_{\delta_{m(i)}}g)_{L^2(\delta_{m(i)})}}{\parallel g \parallel_{H^{1/2}_{00}(\delta_{m(i)})}}.\end{aligned} \tag{4.31}$$

Here $Q_{\delta_{m(i)}}$ is the L_2 orthogonal projection onto the mortar space $M(\delta_{m(i)})$. Using a known estimate for $g - Q_{\delta_{m(i)}}g$, the L_2 - stability of $\pi_{\delta_{m(i)}}$, and an inverse inequality, we get

$$\parallel \pi_{\delta_{m(i)}}(t,0) - t \parallel_{H^{-1/2}(\delta_{m(i)})} \leq C(\frac{h_{\delta_{m(i)}}}{h_{\gamma_{m(i)}}})^{1/2} \parallel t \parallel_{H^{-1/2}(\delta_{m(i)})} .$$

Using this bound in (4.30), we get (4.29). The proof is complete.
We now in the position to formulate and prove the main result.

Theorem 1 Let the assumptions of Lemma 1 be satisfied. Then for $\lambda \in V = Im(B)$

$$< \tilde{M}\lambda, \lambda > \leq < F\lambda, \lambda > \leq C(1+\log(H/h))^2 < \tilde{M}\lambda, \lambda > \tag{4.32}$$

holds, where C is independent of h and H.

Proof The right hand side of (4.32): We have, cf. [9],

$$< F\lambda, \lambda > = \max_{w_r \in W_r} \frac{| < \lambda, Bw_r > |^2}{|w_r|^2_{\tilde{S}}}.$$

Using Lemma 1, we get

$$< F\lambda, \lambda > \leq C(1+\log(H/h))^2 \max_{w_r} \frac{| < \lambda, Bw_r > |^2}{|Pw_r|^2_{S_{rr}}},$$

where $P = \tilde{B}^T(B\tilde{B}^T)^{-1}B$. In turn, by straightforward manipulations, see also (3.15), we have

$$< F\lambda, \lambda > \leq C(1+\log(H/h))^2 \max_{w_r} \frac{|< \lambda, Bw_r >|^2}{< (B\tilde{B}^T)^{-1}\tilde{B}S_{rr}\tilde{B}^T(B\tilde{B}^T)^{-1}Bw_r, Bw_r >} =$$

$$= C(1+\log(H/h))^2 \max_{w_r} \frac{|< \tilde{M}^{1/2}\lambda, \tilde{M}^{-1/2}Bw_r >|^2}{< \tilde{M}^{-1/2}Bw_r, \tilde{M}^{-1/2}Bw_r >} =$$

$$= C(1+\log(H/h))^2 < \tilde{M}\lambda, \lambda >$$

This proves the right hand side of (4.32).

The left hand side of (4.32): We have, cf. [9],

$$< F\lambda, \lambda > = \| \tilde{S}^{-1/2}B^T\lambda \|^2 = \max_{v} \frac{|< \lambda, Bv >|^2}{\| \tilde{S}^{1/2}v \|^2}.$$

Taking $v \in$ range (P) and using that $v = Pv$, and (4.2), we get

$$< F\lambda, \lambda > \geq \max_{v} \frac{< \lambda, Bv >}{< Pv, Pv >_{S_{rr}}}.$$

Setting $\mu = Bv$ and using the definition of P, we have

$$< F\lambda, \lambda > \geq \max_{\mu} \frac{|< \lambda, \mu >|^2}{< \tilde{M}^{-1}\mu, \mu >} =$$

$$= \max_{\mu} \frac{|< \tilde{M}^{1/2}\lambda, \tilde{M}^{-1/2}\mu >|^2}{< \tilde{M}^{-1/2}\mu, \tilde{M}^{-1/2}\mu} = < \tilde{M}\lambda, \lambda > .$$

This proves the left-hand side of (4.32). The proof of Theorem 1 is complete.

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