Fourteenth International Conference on Domain Decomposition Methods
Editors: Ismael Herrera , David E. Keyes, Olof B. Widlund, Robert Yates

24. Dual preconditioners for mortar discretization of elliptic problems

M. Dryja[1], W. Proskurowski[2]

1. Introduction. In this paper, we discuss a second order elliptic problem with discontinuous coefficients defined on a polygonal region Ω which is a union of two polygons, Ω_1 and Ω_2. The problem is discretized by the finite element method on non-matching triangulation across $\overline{\Gamma} = \overline{\Omega}_1 \cap \overline{\Omega}_2$. The discrete problem is described using the mortar technique in the space with constraints (the mortar condition) and in the space without constraints using Lagrange multipliers, see [2] and [1].

The goal of this paper is to compare two preconditioners, dual Neumann-Dirichlet and dual Neumann-Neumann (or FETI, see [5], [6], [7]) used for solving the discrete problem formulated in the space without constraints using Lagrange multipliers. An analysis of convergence of the discussed preconditioners is given. Such analysis to our knowledge has not yet been previously established. The theory is supported by numerical experiments.

The paper is organized as follows. In Section 2, the differential and discrete problems are formulated. In Section 3, a matrix form of discrete problems is given. The preconditioners are described and analyzed in Sections 4, while some aspects of their implementation are presented in Section 5. Finally, numerical results and comparisons of the considered preconditioners are given in Section 6.

2. Mortar discrete problem. We consider the following differential problem:
Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \qquad v \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(u, v) = (\rho(x)\nabla u, \nabla v)_{L^2(\Omega)}, \; f(v) = (f, v)_{L^2(\Omega)}.$$

We assume that Ω is a polygonal region. Let Ω be a union of two disjoint polygonal subregions $\Omega_i, i = 1, 2$, of a diameter one. We additionally assume that $\rho(x) \geq \rho_0 > 0$ is a continuous function in each Ω_i and, for simplicity of presentation, that $\rho(x) = \rho_i =$ constant on Ω_i.

In each Ω_i, a triangulation is introduced with triangular elements $e_i^{(k)}$ and a parameter $h_i = \max_k h_i^{(k)}$, where $h_i^{(k)}$ is a diameter of $e_i^{(k)}$. The resulting triangulation of Ω is non-matching across $\overline{\Gamma} = \overline{\Omega}_1 \cap \overline{\Omega}_2$. We assume that the h_i-triangulation in each Ω_i is quasi-uniform, see [3].

Let $X_i(\Omega_i)$ be the finite element space of piecewise linear continuous functions defined on the triangulation of Ω_i and vanishing on $\partial\Omega_i \cap \partial\Omega$, and let

$$X^h(\Omega) = X_1(\Omega_1) \times X_2(\Omega_2).$$

Note that $X^h \not\subset H_0^1(\Omega)$; therefore it cannot be used for discretization of (2.1). To discretize (2.1) some weak continuity on Γ for $v \in X^h$ is imposed and it is called a *mortar condition*, see [2]. To describe the mortar condition we assume that $\rho_1 \leq \rho_2$ and select a face of Ω_2, geometrically equal to Γ, as a *mortar (master)* and denote it by γ, while $\delta = \Gamma$ as a face of Ω_1 as *non-mortar (slave)*. This choice is arbitrary in the case $\rho_1 = \rho_2$, however in

[1]Department of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland, dryja@mimuw.edu.pl. This work was supported in part by the National Science Foundation under Grant NSF-CCR-9732208 and in part by the Polish Science Foundation under Grant 2P03A 02116

[2]Department of Mathematics, University of Southern California, Los Angeles, CA 90089-1113, USA, proskuro@math.usc.edu.

our case, $\rho_1 \leq \rho_2$ and it is important for the analysis of convergence to choose as the mortar side the one where the coefficient is larger. In the analysis of the FETI method we need that $\frac{h_\gamma}{h_\delta}$ be uniformly bounded, where h_δ and h_γ are the steps of triangulation on δ and γ, respectively.

Let $W_1(\delta)$ and $W_2(\gamma)$ be the restrictions of $X_1(\Omega_1)$ and $X_2(\Omega_2)$ to δ and γ, respectively. Note that they are different because they are defined on different 1-D triangulations of Γ. Let $M(\delta)$ be a space of piecewise linear continuous functions defined on the triangulation of δ with constant values on elements which intersect $\partial\delta$.

We say that $u = (u_1, u_2) \in X^h(\Omega)$ satisfies the mortar condition on $\delta(\delta = \gamma = \Gamma)$ if

$$\int_\delta (u_1 - u_2)\psi ds = 0, \quad \psi \in M(\delta). \tag{2.2}$$

Note that (2.2) for a given u_2 can be written as $u_1 = \pi(u_2, Tr\ u_1)$ where $\pi(u_2, Tr\ u_1) : L^2(\delta) \to W_1(\delta)$ is defined by

$$\begin{cases} \int_\delta \pi(u_2, Tr\ u_1)\psi ds &= \int_\delta u_2\psi ds,\ \psi \in M(\delta), \\ \\ Tr\pi(u_2, Tr\ u_1) &= Tr\ u_1. \end{cases} \tag{2.3}$$

Here $Tr\ v$ is a trace of v on $\partial\delta$. In our case $Tr\ u_1 = 0$.

Let $V^h(\Omega)$ be a subspace of $X^h(\Omega)$ of functions which satisfy the mortar condition (2.2) on δ. The discrete problem for (2.1) in V^h is of the form:

Find $u_h^* = (u_{1h}^*, u_{2h}^*) \in V^h$ such that

$$\sum_{i=1}^{2} a_i(u_{ih}^*, v_{ih}) = f(v_h), \quad v_h = (v_{1h}, v_{2h}) \in V^h, \tag{2.4}$$

where $a_i(u_i, v_i) = \rho_i(\nabla u_i, \nabla v_i)_{L^2(\Omega_i)}$. This problem has a unique solution and its error bound is known, see [2].

The discrete problem (2.4) can be rewritten as a saddle-point problem using Lagrange multipliers as follows:

Let for $u = (u_1, u_2) \in X^h(\Omega)$ and $\psi \in M(\delta)$

$$b(u, \psi) \equiv \int_\delta (u_1 - u_2)\psi dx.$$

Find $(u_h^*, \lambda_h^*) \in X^h(\Omega) \times M(\delta)$ such that

$$\begin{cases} a(u_h^*, v_h) + b(v_h, \lambda_h^*) = f(v_h),\ v_h \in X^h(\Omega), \\ \\ b(u_h^*, \psi) = 0, \qquad \psi \in M(\delta). \end{cases} \tag{2.5}$$

It is easy to see that (2.5) is equivalent to (2.4), i.e. the solution u_h^* of (2.5) is the solution of (2.4) and vice versa. Therefore the problem (2.5) has a unique solution. An analysis of (2.5) can be done straightforwardly using the inf-sup condition, including the error bound, see [1], [2].

3. Matrix form. In this section we derive the matrix form of the discrete problem (2.5).

To provide a matrix form of (2.5) we need a matrix formulation of the mortar condition, i.e. the matrix form of $b(\cdot, \cdot)$. Using the nodal basis functions, $\varphi_k^{(1)} \in W_1(\delta)$, $\varphi_k^{(2)} \in W_2(\gamma)$, and $\psi_l \in M(\delta)$, one can rewrite the equation (2.2) as

$$B_\delta u_{1\delta} - B_\gamma u_{2\gamma} = 0, \tag{3.1}$$

where $u_{1\delta}$ and $u_{2\gamma}$ are vectors that represent $u_{1|\delta} \in W_1(\delta)$ and $u_{2|\gamma} \in W_2(\gamma)$, respectively, and

$$B_\delta = \{(\psi_l, \varphi_k^{(1)})_{L^2(\delta)}\}, \qquad l,k = 1,\dots,n_\delta,$$

$$B_\gamma = \{(\psi_l, \varphi_k^{(2)})_{L^2(\gamma)}\}, \qquad l = 1,\dots,n_\delta;\ k = 1,\dots,n_\gamma.$$

Here $n_\delta = \dim(M(\delta)) = \dim(W_1(\delta)), n_\gamma = \dim(W_2(\gamma))$. Note that B_δ is a square tridiagonal matrix $n_\delta \times n_\delta$, symmetric and positive definite, and $\mathrm{cond}(B_\delta) \sim 1$, while B_γ is a rectangular matrix $n_\delta \times n_\gamma$. Hence for $(u,\lambda) \in X^h(\Omega) \times M(\delta)$

$$b(u,\lambda) = (B_\delta u_{1\delta}, \lambda)_{R^{n_\delta}} - (B_\gamma u_{2\gamma}, \lambda)_{R^{n_\delta}},$$

where here and below a vector representation of λ is also denoted by λ.

Thus (2.5) can be presented in the form

$$\begin{pmatrix} A_{II}^{(1)} & A_{I\delta}^{(1)} & 0 & 0 & 0 \\ A_{\delta I}^{(1)} & A_{\delta\delta}^{(1)} & 0 & 0 & B_\delta \\ 0 & 0 & A_{II}^{(2)} & A_{I\gamma}^{(2)} & 0 \\ 0 & 0 & A_{\gamma I}^{(2)} & A_{\gamma\gamma}^{(2)} & -B_\gamma^T \\ 0 & B_\delta & 0 & -B_\gamma & 0 \end{pmatrix} \begin{pmatrix} u_I^{(1)} \\ u_\delta^{(1)} \\ u_I^{(2)} \\ u_\gamma^{(2)} \\ \lambda_\delta \end{pmatrix} = \begin{pmatrix} F_I^{(1)} \\ F_\delta^{(1)} \\ F_I^{(2)} \\ F_\gamma^{(2)} \\ 0 \end{pmatrix} \tag{3.2}$$

Here $\left\{u_I^{(1)}, u_\delta^{(1)}\right\}^T$ and $\left\{u_I^{(2)}, u_\gamma^{(2)}\right\}^T$ correspond to the nodal values of u_1^* and u_2^* at the interior nodal points of Ω_i, δ and γ, denoted by Ω_{ih}, δ_h and γ_h, respectively, and λ_δ is a vector representation of λ^*;

$A_{II}^{(1)} = \left\{a_1(\varphi_k^{(1)}, \varphi_l^{(1)})\right\} \quad x_k, x_l \in \Omega_{1h},$

$A_{I\delta}^{(1)} = \left\{a_1(\varphi_k^{(1)}, \varphi_l^{(1)})\right\} \quad x_k \in \Omega_{1h}$ and $x_l \in \delta_h,$

$A_{\delta\delta}^{(1)} = \left\{a_1(\varphi_k^{(1)}, \varphi_l^{(1)})\right\} \quad x_k, x_l \in \delta_h;$

$A_{II}^{(2)}$, $A_{I\gamma}^{(2)}$ and $A_{\gamma\gamma}^{(2)}$ are defined in a similar way. Note that $(A_{I\delta}^{(1)}) = (A_{\delta I}^{(1)})^T$ and $(A_{I\gamma}^{(2)}) = (A_{\gamma I}^{(2)})^T$. The matrix of (3.2) is invertible.

4. Preconditioners for (2.5). In this section we define and analyze preconditioners for problem (2.5). They will be defined for the Schur complement system with respect to unknowns λ_δ, the Lagrange multipliers.

Let

$$A^{(1)} = \begin{pmatrix} A_{II}^{(1)} & A_{I\delta}^{(1)} \\ A_{\delta I}^{(1)} & A_{\delta\delta}^{(1)} \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} A_{II}^{(2)} & A_{I\gamma}^{(1)} \\ A_{\gamma I}^{(2)} & A_{\gamma\gamma}^{(2)} \end{pmatrix}.$$

Their Schur complement matrices with respect to $u_\delta^{(1)}$ and $u_\gamma^{(2)}$, respectively, are of the form

$$S_1 = A_{\delta\delta}^{(1)} - A_{\delta I}^{(1)} \left(A_{II}^{(1)}\right)^{-1} A_{I\delta}^{(1)}, \quad S_2 = A_{\gamma\gamma}^{(2)} - A_{\gamma I}^{(2)} \left(A_{II}^{(2)}\right)^{-1} A_{I\gamma}^{(2)}. \tag{4.1}$$

We consider system (3.2). We first eliminate the unknowns $u_I^{(1)}$ and $u_I^{(2)}$. Using rows 1 and 3 of (3.2) and substituting the result in rows 2 and 4 of (3.2) we obtain

$$\begin{pmatrix} S_1 & 0 & B_\delta \\ 0 & S_2 & -B_\gamma^T \\ B_\delta & -B_\gamma & 0 \end{pmatrix} \begin{pmatrix} u_\delta^{(1)} \\ u_\gamma^{(2)} \\ \lambda_\delta \end{pmatrix} = \begin{pmatrix} F_\delta^{(1)} - (A_{I\delta}^{(1)})^T (A_{II}^{(1)})^{-1} F_I^{(1)} \\ F_\gamma^{(2)} - (A_{I\gamma}^{(2)})^T (A_{II}^{(2)})^{-1} F_I^{(2)} \\ 0 \end{pmatrix}, \tag{4.2}$$

where S_1 and S_2 are given by (4.1).

Then, we eliminate the unknowns $u_\delta^{(1)}$ and $u_\gamma^{(2)}$ from this system. Using rows 1 and 2 of (4.2) and setting $\hat{\lambda}_\delta = B_\delta \lambda_\delta$, we obtain

$$S_L \hat{\lambda}_\delta = F_\lambda, \tag{4.3}$$

where

$$S_L = S_1^{-1} + B_\delta^{-1} B_\gamma S_2^{-1} B_\gamma^T B_\delta^{-1}, \tag{4.4}$$

and

$$F_\lambda = S_1^{-1}(F_\delta^{(1)} - (A_{I\delta}^{(1)})^T (A_{II}^{(1)})^{-1} F_I^{(1)}) - B_\delta^{-1} B_\gamma S_2^{-1}(F_\gamma^{(2)} - (A_{I\gamma}^{(2)})^T (A_{II}^{(2)})^{-1} F_I^{(2)}).$$

The dual Schur complement matrix S_L is symmetric and positive definite, n_δ by n_δ.

Our goal is to define preconditioners for (4.3) dual to the Neumann-Dirichlet one and dual to the Neumann-Neumann one. The latter, for the matching triangulation, is called FETI (the Finite Element Tearing and Interconnecting), see [5], [6], [7].

4.1. Neumann-Dirichlet (N-D) preconditioner. The Neumann-Dirichlet dual preconditioner for S_L is defined by S_1^{-1}.

Theorem 4.1 *For any $\lambda \in R^{n_\delta}$ and $\rho_1 \leq \rho_2$ the following holds*

$$\left(S_1^{-1}\lambda, \lambda\right)_{R^{n_\delta}} \leq (S_L \lambda, \lambda)_{R^{n_\delta}} \leq C \left(S_1^{-1}\lambda, \lambda\right)_{R^{n_\delta}} \tag{4.5}$$

where C is a positive constant independent of h_i and ρ_i, $i = 1, 2$.

For the proof see [4].

4.2. FETI (N-N) preconditioner. We now discuss FETI method for solving (4.3). This preconditioner is of the form

$$G = \left(\frac{\rho_2}{\rho_1 + \rho_2} S_1 + \frac{\rho_1}{\rho_1 + \rho_2} B_\delta^{-1} B_\gamma S_2 B_\gamma^T B_\delta^{-1} \right)^{-1}. \tag{4.6}$$

Theorem 4.2 *Let $\frac{h_\gamma}{h_\delta}$ be uniformly bounded. For any $\lambda \in R^{n_\delta}$ and $\rho_1 \leq \rho_2$ holds*

$$\frac{1}{2}(G\lambda, \lambda)_{R^{n_\delta}} \leq (S_L \lambda, \lambda)_{R^{n_\delta}} \leq C (G\lambda, \lambda)_{R^{n_\delta}} \tag{4.7}$$

where C is a positive constant independent of h_i and ρ_i, $i = 1, 2$.

For the proof see [4].

5. Implementation aspects. In this section we discuss some implementation aspects of solving the Schur complement systems.

To solve the dual Schur complement equation (4.3) we use the preconditioned conjugate gradient (PCG) iterations. Here, we only need to describe the implementation of **1.** the multiplication of a vector by the dual Schur complement matrix $S_L \in R^{n_\delta \times n_\delta}$ (defined by (4.4)), and **2.** solving a system with **a.** the Neumann-Dirichlet dual preconditioner S_1 (defined by (4.1)), and with **b.** the Neumann-Neumann dual preconditioner G (defined by (4.6)).

Let us recall that the iterations are carried out on the non-mortar side δ of the interface Γ with the number of grid equal to n_δ. The mortar condition (2.3) ensures the proper transfer of information across the interface.

1. Compute $r^k = S_L \lambda^k$ for any given $\lambda^k \in R^{n_\delta}$. The multiplication by S_L reduces to solving two independent problems:

i) Compute $r_1^k = S_1^{-1}\lambda^k$, i.e. solve

$$S_1 r_1^k = \lambda^k. \tag{5.1}$$

This reduces to solving the Neumann problem on Ω_1, and more precisely, the problem with non-homogeneous Neumann boundary conditions on δ and homogeneous Dirichlet ones on $\partial\Omega_1\backslash\delta$, see (4.1)

$$\begin{pmatrix} A_{II}^{(1)} & A_{I\delta}^{(1)} \\ A_{\delta I}^{(1)} & A_{\delta\delta}^{(1)} \end{pmatrix} \begin{pmatrix} r_I^{(1)} \\ r_1^k \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda^k \end{pmatrix}, \tag{5.2}$$

ii) Compute $r_2^k = B_\delta^{-1} B_\gamma S_2^{-1} B_\gamma^T B_\delta^{-1} \lambda^k$. This step is similar to (5.2). The only difference is that before solving the Neumann problem on Ω_2 we need first to solve $B_\delta z_\delta^{(1)} = \lambda^k$, then to compute $B_\gamma^T z_\delta^{(1)}$; and after solving the Neumann problem on Ω_2 we need to perform these operations in reversed order.

Finally, $r^k = r_1^k + r_2^k$.

2a. Compute $r_1^k = S_1\lambda^k$ for any given $\lambda^k \in R^{n_\delta}$.

We first compute, see (4.1),

$$S_1\lambda^k = A_{\delta\delta}^{(1)}\lambda^k - A_{\delta I}^{(1)}\left(A_{II}^{(1)}\right)^{-1} A_{I\delta}^{(1)}\lambda^k.$$

This reduces to solving the Dirichlet problem in Ω_1 as follows

$$A_{II}^{(1)} v_I^{(1)} = A_{I\delta}^{(1)}\lambda^k \tag{5.3}$$

and to computing $r_1^k = A_{\delta\delta}^{(1)}\lambda^k - A_{\delta I}^{(1)} v_I^{(1)}$.

2b. Compute $r^k = G^{-1}\lambda^k$ for any given $\lambda^k \in R^{n_\delta}$.

This step consists of solving two Dirichlet problems, one in Ω_1, the other in Ω_2 (with the pre- and post- multiplications by $B_\delta^{-1}B_\gamma$ and its transpose, respectively, as in **1.ii)**), see (4.6) .

6. Numerical experiments. The test example for all our experiments is the weak formulation, see (2.1), of

$$-div(\rho(x)\nabla u) = f(x_1, x_2) \text{ in } \Omega, \tag{6.1}$$

with the Dirichlet boundary conditions on $\partial\Omega$, where Ω is a union of two disjoint rectangular subregions Ω_i, $i = 1, 2$, of a diameter one, and $\rho(x) = \rho_i$ is a positive constant in each Ω_i.

The problem (6.1) is discretized by the finite element method on non–matching triangulation across the interface Γ. The grids used in our experiments are: 1. *double grids*, where the grid on one side of the interface Γ is twice the one on the other side of Γ, with every other position of the nodes coinciding, 2. *staggered grids*, where the grid size, h on both sides of Γ is the same but the nodes are staggered, with the distance of $\frac{h}{2}$ between the nearest two nodes on the opposite sides of Γ, and 3. *mixed grids*, where the grid on one side of Γ is coarse with the grid size $2h$, while the grid on the other side of Γ is fine with the grid size h and staggered by $\frac{h}{2}$. The mixed grids may better represent general non-matching grids.

We select a face of Ω_2 which coincides with the interface Γ as the mortar side, while the face of Ω_1 is the non–mortar one. We choose the following combinations of the diffusion coefficients: 1. $\rho_1 = \rho_2$, 2. $1 = \rho_1 < \rho_2 = 1000$, and 3. $1 = \rho_2 < \rho_1 = 1000$ (the case not covered by the theory).

To create a discrete driving function $f(x_1, x_2)$ we generate a random discrete solution $u(x_1, x_2)$ and multiply it by the matrix (3.2).

We solve the problems using the preconditioned conjugate gradient (PCG) iterations (see Section 5 for the implementation aspects). The iterations are terminated when the

Table 6.1: Performance of the dual Neumann-Dirichlet (dual N-D, $Q = S_1 S_L$) and dual Neumann-Neumann (dual N-N, or FETI, $Q = G^{-1} S_L$) preconditioners for the finest meshes on different grids. The number of iterations and the estimate of the condition number are displayed.

grids	precon-ditioner	n_δ	n_γ	continuous		$\rho_2 < \rho_1$		$\rho_1 < \rho_2$	
				no. iter.	$\kappa(Q)$	no. iter.	$\kappa(Q)$	no. iter.	$\kappa(Q)$
double	dual N-D	255	127	5	2.00	10	$*$	2	1.001
		127	255	4	1.34	6	1.85	2	1.001
	dual N-N	255	127	11	9.97	23	$*$	7	5.00
		127	255	6	1.73	7	2.26	5	1.28
staggered	dual N-D	256	255	8	1.93	115	997.	3	1.30
		255	256	9	3.08	114	1176.	2	1.002
	dual N-N	256	255	13	4.27	144	1003.	9	2.85
		255	256	12	5.07	146	2957.	8	1.91
mixed	dual N-D	256	127	7	2.28	16	$*$	3	1.31
		127	256	10	10.98	13	91.0	3	1.01
	dual N-N	256	127	14	19.23	35	$*$	12	9.98
		127	256	15	22.21	18	181.7	8	2.96

norm of the residual has decreased 10^6 times in the norm generated by the inverse of the preconditioner matrix.

To estimate the condition number of the PCG iteration matrix we compute the tridiagonal matrix representing the restriction of the preconditioned Schur complement matrix to the space spanned by the conjugate gradient residuals.

The preconditioners considered behave as predicted by the theory: for $\rho_1 \leq \rho_2$ the convergence is independent of the grid size, see Table 6.2. Table 6.1 presents performance of the preconditioners for the finest meshes on different grids. The N-D and dual N-D preconditioners converge somewhat faster than the N-N and dual N-N (FETI) preconditioners. All four preconditioners are robust for cases with the discontinuity ratio of 1000 across the interface, see Table 6.1.

The differences in performance on different grids are qualitatively insignificant, thus in Table 6.2 we present only one set of experiments. Comparison of the convergence rate for the preconditioned and non-preconditioned iterations (on a chosen set of problems, see Table 6.2) shows that the first remain constant independently of the grid size, while the latter depend roughly proportional to the square root of the size of the iteration matrix[3]. From this one can infer that $cond(S) = O(\frac{1}{h})$ and $cond(S_L) = O(\frac{1}{h})$ even for problems with jump discontinuity at the interface.

Additionally, we performed experiments with the grid ratio across the interface varying in the range $\frac{h_\gamma}{h_\delta} = 2^k, k = -5(1)5$, i.e. from $\frac{1}{32}$ to $\frac{32}{1}$ (and different diffusion coefficients ρ, as before). Performance for the dual N-D preconditioner was virtually independent of the grid ratio, as was for the FETI preconditioner and $\frac{h_\gamma}{h_\delta} < 1$. For $\frac{h_\gamma}{h_\delta} > 1$ the condition number of the FETI iteration matrix grows almost quadratically with the grid ratio while the number of iterations increases only very slowly (and depends also on the grid size).

7. Conclusions. The preconditioners considered behave as predicted by the theory: for $\rho_1 \leq \rho_2$ the convergence is independent of the grid size and the jump of the discontinuity. All preconditioners considered are very robust for cases with the discontinuity ratio of 1000

[3]the number of iterations for the non-preconditioned problems in the range $n_\delta = 16$ to 256 is proportional to n_δ^p, where $p = 0.5 \pm 0.03$ as computed using *polyfit* in the loglog scale.

Table 6.2: Examples of the PCG iterations convergence for (2.5) with and without preconditioners (for $\rho_1 < \rho_2$) as a function of grid sizes on the mixed grids. The number of iterations and estimate of the condition number are displayed.

		no precond.		dual N-D precond.		dual N-N precond.	
n_δ	n_γ	no. iter.	$\kappa(Q)$	no. iter.	$\kappa(Q)$	no. iter.	$\kappa(Q)$
16	7	12	14.35	4	1.30	9	9.88
32	15	18	27.06	4	1.30	12	9.96
64	31	24	52.39	4	1.31	12	9.97
128	63	33	102.9	3	1.31	12	9.98
256	127	44	203.8	3	1.31	12	9.98
7	16	7	6.29	3	1.01	7	2.81
15	32	11	12.95	3	1.01	8	2.96
31	64	17	25.68	3	1.01	8	2.96
63	128	23	51.29	3	1.01	8	2.96
127	256	33	102.7	3	1.01	8	2.96

across the interface. One should be cautious not to generalize conclusions drawn on such limited two subdomain case. Nevertheless, the results are illuminating, and we intend to extend the experimental evidence to more complex subdivisions.

REFERENCES

[1] F. Ben Belgacem. The mortar finite element method with Lagrange multipliers. *Numer. Math.*, 84(2):173–197, 1999.

[2] C. Bernardi, Y. Maday, and A. T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In H. Brezis and J.-L. Lions, editors, *Collège de France Seminar*. Pitman, 1994. This paper appeared as a technical report about five years earlier.

[3] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1994.

[4] M. Dryja and W. Proskurowski. On preconditioners for mortar discretization of elliptic problems. *Numerical Linear Algebra with Applications*, 2002. to appear.

[5] C. Farhat and F.-X. Roux. A Method of Finite Element Tearing and Interconnecting and its Parallel Solution Algorithm. *Int. J. Numer. Meth. Engrg.*, 32:1205–1227, 1991.

[6] A. Klawonn and O. B. Widlund. FETI and Neumann–Neumann Iterative Substructuring Methods: Connections and New Results. *Comm. Pure Appl. Math.*, 54:57–90, January 2001.

[7] J. Mandel and R. Tezaur. Convergence of a Substructuring Method with Lagrange Multipliers. *Numer. Math.*, 73:473–487, 1996.