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43. A Robin-Robin preconditioner for strongly heterogeneous advection-diffusion problems

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1. Introduction. We consider an advection-diffusion problem with discontinuous viscosity coefficients. We apply a substructuring technique and we extend to the resulting Schur complement the Robin-Robin preconditioner used for problems with constant viscosity. In Section 2 the algorithm is analyzed theorically by means of Fourier techniques, and we show that its convergence rate is independent of the coefficients: this allows to treat large discontinuities. Section 3 is dedicated to the variational generalization to an arbitrary number of subdomains, while in Section 4 we give some numerical result in 3D.

1.1. Statement of the problem. Let Ω be bounded domain in $\mathbf{R}^2$. We consider the following general advection-diffusion problem

$$\begin{array}{rcll} -\text{div}\,(\nu(x)\nabla u) + \vec{b}\cdot\nabla(u) + au & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega_D \end{array} \tag{1.1}$$

where $\vec{b}$ is the convective field $\vec{b} = (b_x, b_y)$ while the constant a may arise from an Euler implicit time discretization for the time dependent problem, and represent the inverse of the time step, i.e. $a = 1/\Delta t$.
We assume the function $\nu(x)$ to be piecewise constant

$$\nu(x) = \begin{cases} \nu_1 & \text{if } x \in \Omega_1 \\ \nu_2 & \text{if } x \in \Omega_2 \end{cases}$$

with $\nu_1 < \nu_2$, where Ω_1 and Ω_2 are two non overlapping subsets which cover Ω $\Omega_1 \cup \Omega_2 = \Omega$. Γ denotes the interface between the two subdomains, i.e. $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, while $\mathcal{L}_j$ $(j = 1, 2)$ denotes the operator

$$\mathcal{L}_j(w) := -\nu_j \Delta w + \vec{b}\cdot\nabla w + aw$$

2. The Continuous Algorithm. We introduce, at the continuous level, the operator

$$\begin{array}{rcl} \Sigma : H^{1/2}_{00}(\Gamma) \times L^2(\Omega) & \longrightarrow & H^{-1/2}(\Gamma) \\ (u_\Gamma, f) & \longmapsto & \left(\nu_1 \frac{\partial u_1}{\partial n_1} + \nu_2 \frac{\partial u_2}{\partial n_2}\right)_\Gamma \end{array} \tag{2.1}$$

where u_j $(j = 1, 2)$ is the solution to problem

$$\begin{array}{rcll} \mathcal{L}_j(u_j) & = & f & \text{in } \Omega_j \\ u_j & = & 0 & \text{on } \partial\Omega_D \cap \partial\Omega_j \\ u_j & = & u_\Gamma & \text{on } \Gamma \end{array} \tag{2.2}$$

It is evident that u_Γ satisfies the Steklov-Poincaré equation

$$\mathcal{S}(u_\Gamma) = \chi \tag{2.3}$$

where $\mathcal{S}(.) := \Sigma(., 0)$ and $\chi := -\Sigma(0, f)$. We split the operator $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$, with

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$$\mathcal{S}_j : u_\Gamma \mapsto \left(\nu_j \frac{\partial u_j}{\partial n_j} - \frac{\vec{b}\cdot\vec{n}_j}{2} u_j\right)_\Gamma ,$$

for $j = 1,2$ (since $\vec{n}_1 = -\vec{n}_2$ and $u_1 = u_2 = u_\Gamma$, the terms $\frac{1}{2}\vec{b}\cdot\vec{n}_j\, u_j$ cancel by summation). Following ([1]), ([2]) and ([8]), we propose as a preconditioner for the Steklov-Poincaré equation at the continuous level a weighted sum of the inverses of the operators $\mathcal{S}_1$ and $\mathcal{S}_2$,

$$\mathcal{T} = N_1 \mathcal{S}_1^{-1} N_1 + N_2 \mathcal{S}_2^{-1} N_2, \tag{2.4}$$

with $N_1 = \frac{\nu_1}{\nu_1+\nu_2}$, $N_2 = \frac{\nu_2}{\nu_1+\nu_2}$, which is defined by

$$\begin{array}{rcl} \mathcal{T} : H^{-1/2}(\Gamma) & \longrightarrow & H^{1/2}_{00}(\Gamma) \\ g & \longmapsto & (N_1\, v_1 + N_2\, v_2)_\Gamma \end{array} \tag{2.5}$$

where v_j $(j = 1,2)$ is the solution to

$$\begin{array}{rcll} \mathcal{L}_j(v_j) & = & 0 & \text{in } \Omega_j \\ v_j & = & 0 & \text{on } \partial\Omega_D \cap \partial\Omega_j \\ \left(\nu_j \frac{\partial v_j}{\partial n_j} - \frac{\vec{b}\cdot\vec{n}_j}{2} v_j\right)_\Gamma & = & N_j\, g & \text{on } \Gamma. \end{array} \tag{2.6}$$

2.1. The vertical strip case - Uniform velocity. In this section we consider the case where $\Omega = \mathbf{R}^2$ is decomposed into the left ($\Omega_1 =]-\infty, 0[\times \mathbf{R}$) and right ($\Omega_2 =]0, +\infty[\times \mathbf{R}$) half-planes, we assume the convective field to be uniform $\vec{b} = (b_x, b_y)$, with the additional requirement on the solutions u_j to be bounded as $|x| \to +\infty$. We express the action of the operator $\mathcal{S}$ in terms of its Fourier transform in the y direction as

$$\mathcal{S} u_\Gamma = \mathcal{F}^{-1}\left(\hat{\mathcal{S}}(\xi)\hat{u}_\Gamma(\xi)\right), \quad u_\Gamma \in H^{1/2}_{00}(\Gamma)$$

where ξ is the Fourier variable and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. We consider, for $j = 1,2$, the problem

$$\begin{array}{rcll} \mathcal{L}_j(u_j) & = & 0 & \text{in } \Omega_j \\ u_j & = & u_\Gamma & \text{on } \Gamma, \end{array} \tag{2.7}$$

and we have to compute $\hat{\mathcal{S}}\hat{u}_\Gamma$. Performing a Fourier transform in the y direction on the operators $\mathcal{L}_j$, we get

$$\left(a + b_x\partial_x - \nu_j\partial_{xx} + i b_y \xi + \nu_j \xi^2\right)\hat{u}_j(x,\xi) = 0, \tag{2.8}$$

for $j = 1,2$, where $i^2 = -1$. For a given ξ, equation (2.8) is an ordinary differential equation in x whose solutions have the form $\alpha_j(\xi)\exp\{\lambda_j^-(\xi)x\} + \beta_j(\xi)\exp\{\lambda_j^+(\xi)x\}$, where

$$\lambda_j^\pm(\xi) = \frac{b_x \pm \sqrt{b_x^2 + 4a\nu_j + 4\nu_j^2\xi^2 + 4 i b_y \nu_j \xi}}{2\nu_j}, \tag{2.9}$$

with $\mathrm{Re}(\lambda_j^\pm) \gtreqless 0$, as $\mathrm{Re}(z)$ indicates the real part of a complex number z. The solutions u_j $(j = 1,2)$ must be bounded at infinity, so $\alpha_1(\xi) = \beta_2(\xi) = 0$, while the Dirichlet condition on the interface provides $\beta_1(\xi) = \alpha_2(\xi) = \hat{u}_\Gamma$. Hence,

$$\hat{\mathcal{S}}\hat{u}_\Gamma = \frac{1}{2}\left(\sqrt{b_x^2 + 4a\nu_1 + 4\nu_1^2\xi^2 + 4ib_y\nu_1\xi} + \sqrt{b_x^2 + 4a\nu_2 + 4\nu_2^2\xi^2 + 4ib_y\nu_2\xi}\right)\hat{u}_\Gamma \tag{2.10}$$

In a similar way we compute $\hat{\mathcal{T}}\hat{g}$ for $g \in H^{-1/2}(\mathbf{R})$, and we have $(\hat{\mathcal{T}}\circ\hat{\mathcal{S}})\hat{u}_\Gamma = \Phi(\xi)\hat{u}_\Gamma$, with

$$\Phi(\xi) = N_1^2 \cdot [1 + z(\xi)] + N_2^2 \cdot \left[1 + \frac{\bar{z}(\xi)}{|z(\xi)|^2}\right], \tag{2.11}$$

where we have set $z(\xi) := \sqrt{\frac{b_x^2+4a\nu_2+4\nu_2^2\xi^2+4ib_y\nu_2\xi}{b_x^2+4a\nu_1+4\nu_1^2\xi^2+4ib_y\nu_1\xi}}$. We have $1 < |z(\xi)| \le \nu_2/\nu_1$, with $|z(\xi)|$ decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$.

Theorem 2.1 (Main Result) *In the case where the plane* $\mathbf{R}^2$ *is decomposed into the left and right half planes, and the convective field is uniform, the reduction factor for the associated* GMRES *algorithm can be bounded from above by a constant independent of the time step* Δt*, the convective field* $\vec{b}$ *and the viscosity coefficients* ν_1 *and* ν_2.

Proof. Let $\Phi(\xi)$ be the function defined in (2.11). The GMRES reduction factor is given, for a positive real matrix A with symmetric part M, by

$$\rho_{\text{GMRES}} = 1 - \frac{(\lambda_{\min}(M))^2}{\lambda_{\max}(A^T A)}.$$

Therefore, it is enough to show that

$$\frac{\max_\xi |\Phi(\xi)|^2}{(\min_\xi \operatorname{Re}\Phi(\xi))^2} \in O(1) \tag{2.12}$$

independentely of a, b_x, b_y, ν_1 and ν_2.
If $b_y \neq 0$, since $\operatorname{Re} z(\xi) \ge 0$, we have from (2.11)

$$\operatorname{Re}\Phi(\xi) \ge N_1^2 + N_2^2 > \frac{\nu_2^2}{(\nu_1+\nu_2)^2}, \tag{2.13}$$

for all ξ, as well as, focusing on $|\Phi(\xi)|^2$,

$$|\Phi(\xi)|^2 \le \left[N_1^2 + N_2^2 + N_1^2 \cdot |z(\xi)| + \frac{N_2^2}{|z(\xi)|}\right]^2 + \left[N_1^2 \cdot |z(\xi)| - \frac{N_2^2}{|z(\xi)|}\right]^2 = \Psi(\xi) \tag{2.14}$$

which is increasing in $(-\infty, 0)$ and decreasing in $(0, +\infty)$.
i) If $b_x \neq 0$, we define $\eta := 4a/b_x^2$ and we have

$$\Psi(0) = \left[N_1^2\left(1+\sqrt{\frac{1+\eta\nu_2}{1+\eta\nu_1}}\right) + N_2^2\left(1+\sqrt{\frac{1+\eta\nu_1}{1+\eta\nu_2}}\right)\right]^2 + \left[N_1^2\sqrt{\frac{1+\eta\nu_2}{1+\eta\nu_1}} - N_2^2\sqrt{\frac{1+\eta\nu_1}{1+\eta\nu_2}}\right]^2$$

The right hand term is decreasing as a function of η. This provides

$$\max_\xi |\Phi(\xi)|^2 \le (2N_1^2 + 2N_2^2)^2 + (N_1^2 - N_2^2)^2 \tag{2.15}$$

From (2.13) and (2.15), we get

$$\frac{\max_\xi |\Phi(\xi)|^2}{(\min_\xi \operatorname{Re}\Phi(\xi))^2} \le 5 + 6\cdot\left(\frac{\nu_1}{\nu_2}\right)^2 + 5\cdot\left(\frac{\nu_1}{\nu_2}\right)^4 < 16. \tag{2.16}$$

ii) If $b_x = 0$ (flux parallel to the interface), $|z(0)| = \sqrt{\nu_2/\nu_1}$, and we have

$$\max_\xi |\Phi(\xi)|^2 \le \left[N_1^2\left(1+\sqrt{\frac{\nu_2}{\nu_1}}\right) + N_2^2\left(1+\sqrt{\frac{\nu_1}{\nu_2}}\right)\right]^2 + \left[N_1^2\sqrt{\frac{\nu_2}{\nu_1}} - N_2^2\sqrt{\frac{\nu_1}{\nu_2}}\right]^2 \tag{2.17}$$

From (2.13) and (2.17), we get

$$\frac{\max_\xi |\Phi(\xi)|^2}{(\min_\xi \operatorname{Re}\Phi(\xi))^2} \le 1 + 2\sum_{n=1}^{7}\left(\frac{\nu_1}{\nu_2}\right)^{n/2} + \left(\frac{\nu_1}{\nu_2}\right)^4 < 16. \tag{2.18}$$

If $b_y = 0$, the complex function $\Phi(\xi)$ reduces to a real one which is symmetric in ξ, decreasing in $[0, +\infty)$ and satisfies $\Phi(\xi) \geq 1$ for all ξ. Therefore

$$\frac{\max_\xi |\Phi(\xi)|^2}{(\min_\xi \mathrm{Re}\, \Phi(\xi))^2} = \left[\frac{\max_\xi \Phi(\xi)}{\min_\xi \Phi(\xi)}\right]^2 \leq \left[\max_\xi \Phi(\xi)\right]^2 = [\Phi(0)]^2 .$$

i) if $b_x \neq 0$, we define $\eta := 4a/b_x^2$, and we have

$$[\Phi(0)]^2 = \left[N_1^2 \left(1 + \sqrt{\frac{1+\eta\nu_2}{1+\eta\nu_1}}\right) + N_2^2 \left(1 + \sqrt{\frac{1+\eta\nu_1}{1+\eta\nu_2}}\right)\right]^2 \tag{2.19}$$

where the right hand side attains its maximum for $\eta = 0$. Hence

$$\left[\frac{\max_\xi \Phi(\xi)}{\min_\xi \Phi(\xi)}\right]^2 < \left[2 \cdot \frac{\nu_1^2}{(\nu_1+\nu_2)^2} + 2 \cdot \frac{\nu_2^2}{(\nu_1+\nu_2)^2}\right]^2 < 4. \tag{2.20}$$

ii) if $b_x = 0$ (purely elliptic case) we simply have

$$\left[\frac{\max_\xi \Phi(\xi)}{\min_\xi \Phi(\xi)}\right]^2 < \left[N_1^2 \left(1 + \sqrt{\frac{\nu_2}{\nu_1}}\right) + N_2^2 \left(1 + \sqrt{\frac{\nu_1}{\nu_2}}\right)\right]^2 < 4. \tag{2.21}$$

□

Remark 2.1 The argument above is based only on the assumption $\nu_1 < \nu_2$, and it can be easily seen that a symmetric argument would give the same result as long as $\nu_2 < \nu_1$. In a forthcoming paper ([6]), a more detailed proof of the main result will be given. More, it appears that the condition number of the preconditioned system improves with the growth of the ratio ν_2/ν_1.

3. Variational Generalization. We consider in $\mathbf{R}^d$ (with $d = 2, 3$) the domain $\Omega = \bigcup_{k=1}^N \Omega_k$, with $\Omega_j \cap \Omega_k = \emptyset$ for $j \neq k$, in which we solve

$$\begin{array}{rcll} -\mathrm{div}\,(\nu(x)\nabla u) + \vec{b}(x)\cdot\nabla(u) + a(x)u &=& f & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega_D \end{array} \tag{3.1}$$

with piecewise constant viscosity $\nu(x) = \nu_k$ in $\Omega_k(x)$. We restrict ourselves to well-posed problems, and we assume $\vec{b} \in W^{1,\infty}(\Omega)$ and there exists $\mu > 0$ such that $a - 1/2\mathrm{div}(\vec{b}) \geq \mu > 0$. We introduce the space $\mathbb{H}(\Omega) = \left\{v \in H^1(\Omega) : v_{|\partial\Omega_D} = 0\right\}$, and the variational form of (3.1)

$$\text{Find } u \in \mathbb{H}(\Omega): \qquad a(u,v) = L(v) \quad \forall v \in \mathbb{H}(\Omega), \tag{3.2}$$

with

$$a(u,v) = \int_\Omega \nu\nabla u\nabla v + (\vec{b}\cdot\nabla u)v + auv, \qquad L(v) = \int_\Omega fv.$$

We define the local interfaces $\Gamma_k := \partial\Omega_k \setminus \partial\Omega$ and the global interface $\Gamma = \cup_k \Gamma_k$, and we introduce the local form

$$a_k(u,v) = \int_{\Omega_k} \left\{\nu_k \nabla u\nabla v + (\vec{b}\cdot\nabla u)v + auv\right\} - \int_{\Gamma_k} \frac{1}{2}\vec{b}\cdot\vec{n}_k uv$$

where the interface terms $-\int_{\Gamma_k} 1/2\, \vec{b}\cdot\vec{n}_k uv$ added locally cancel each other by summation, but their presence guarantees nevertheless that the local bilinear form is positive on the space of restrictions $\mathbb{H}(\Omega_k) = \left\{v_k = v_{|\Omega_k},\ v \in \mathbb{H}(\Omega)\right\}$. Summing up on k, and letting $L_k(v) := \int_{\Omega_k} fv$, the variational problem (3.2) is equivalent to

$$\text{Find } u \in \mathbb{H}(\Omega): \qquad \sum_{k=1}^n \{a_k(u,v) - L_k(v)\} = 0 \quad \forall v \in \mathbb{H}(\Omega). \tag{3.3}$$

3.1. Finite Element Approximation. In order to approximate problem (3.3) with finite elements, we assume that the domain Ω is polygonal, and that the triangulations respect the geometry of subdomain decomposition: the interfaces Γ_k will coincide with interelement boundaries, and each subdomain can be obtained as the union of a given subset of elements in the original triangulation.

In several cases of practical interest, problem (3.1) is advection-dominated and must be stabilized. We will use *Galerkin Least-Squares* techniques (*GALS*), which consists in adding to the original variational formulation the element residuals

$$\int_{T_i} \delta_i(h) \Big(-\mathrm{div}\,(\nu \nabla u) + \vec{b} \cdot \nabla(u) + au - f \Big) \Big(-\mathrm{div}\,(\nu \nabla v) + \vec{b} \cdot \nabla(v) + av \Big)$$

where T_i is an element of the triangulation, with a suitable choice of the local positive stabilization parameter $\delta_i(h)$. The stabilized finite elements formulation then reads

$$\text{Find } u_h \in \mathbb{H}_h(\Omega) : \qquad \sum_{k=1}^{n} \{a_{kh}(u_h, v_h) - L_{kh}(v_h)\} = 0 \quad \forall v_h \in \mathbb{H}_h(\Omega), \tag{3.4}$$

3.2. Substructuring. The variational structure of problems (3.3) and (3.4) allows to reduce them to an interface problem by means of standard substructuring techniques. Following ([2]), we introduce the space $\mathbb{H}^0(\Omega_k) = \Big\{ v_k \in \mathbb{H}(\Omega), v_k = 0 \text{ in } \overline{\Omega \setminus \Omega_k} \Big\}$, the global and local trace spaces $\mathbb{V}$ and $\mathbb{V}_k$, the restriction operators $R_k : \mathbb{H}(\Omega) \to \mathbb{H}(\Omega_k)$ and $\bar{R}_k : \mathbb{V} \to \mathbb{V}_k$, the a_k-harmonic extension $\mathrm{Tr}_k^{-1} : \mathbb{V}_k \to \mathbb{H}(\Omega_k)$, defined as

$$a_k(\mathrm{Tr}_k^{-1} \bar{u}_k, v_k) = 0 \quad \forall v_k \in \mathbb{H}^0(\Omega_k), \qquad \mathrm{Tr}(\mathrm{Tr}_k^{-1} \bar{u}_k)_{|\Gamma_k} = \bar{u}_k, \tag{3.5}$$

with its adjoint Tr_k^{-*}. The bilinear form a_k is elliptic on $\mathbb{H}^0(\Omega_k)$ so problem (3.5) is well-posed, and we can define the local Schur complement operator $S_k : \mathbb{V}_k \to \mathbb{V}'_k$ as

$$\langle S_k \bar{u}_k, \bar{v}_k \rangle = a_k(\mathrm{Tr}_k^{-1} \bar{u}_k, \mathrm{Tr}_k^{-*} \bar{v}_k) \qquad \forall \bar{u}_k, \bar{v}_k \in \mathbb{V}_k$$

If we decompose the local degrees of freedom U_k of $u_k = R_k u$ into internal (U_k^0) and interface ($\bar{U}_k$) degrees of freedom, the matrix A_k associated to the bilinear form a_k can be represented as

$$A_k = \begin{bmatrix} A_k^0 & B_k \\ \tilde{B}_k^T & \bar{A}_k \end{bmatrix},$$

and we eliminate the local internal component U_k^0 as solution of a well-posed local problem, to get

$$S_k \bar{U}_k = \Big(\bar{A}_k - \tilde{B}_k^T (A_k^0)^{-1} B_k \Big) \bar{U}_k.$$

The global Schur complement operator

$$S = \sum_{k=1}^{N} \bar{R}_k^T S_k \bar{R}_k \tag{3.6}$$

follows and we reduce problems (3.3) and (3.4) to the interface problem $S\bar{u} = F$ in $\mathbb{V}$, with a right-hand side defined as $\langle F, \bar{v} \rangle = \sum_k L_k(Tr_k^{-*}(\bar{R}_k \bar{v}))$, where v_k is any function in $\mathbb{H}(\Omega_k)$ such that $v_k = \bar{v}$ on Γ_k.

ν_1, ν_2	ν_1/ν_2	$\vec{b} = (\pm 1, 0, 0)$		$\vec{b} = (0, 1, 1)$	$\vec{b} = (\pm 1, 3, 5)$	
$10^{-1}, 10^{-5}$	10^4	10	11	17	15	17
$10^{-2}, 10^{-6}$		12	16	13	7	8
$10^{-1}, 10^{-6}$	10^5	10	11	17	15	17
$10^{-6}, 10^{-11}$		5	5	2	7	7
$10^{-1}, 10^{-7}$	10^6	10	11	17	15	17
$10^3, 10^{-3}$		3	3	3	3	3
$1, 10^{-7}$	10^7	6	7	9	11	11

Table 4.1: Number of iterations for the two-domain problem

3.3. Construction of the preconditioner. We extend the preconditioner $\mathcal{T}$ introduced in the previous section to an arbitrary number of subdomains and we generalize the ones proposed in ([2]) and ([8]). The interface operator (3.6) is preconditioned with a weighted sum of inverses based on a partition of unity argument:

$$T = \sum_{k=1}^{N} D_k^T (S_k)^{-1} D_k, \tag{3.7}$$

with $\sum_{k=1}^{N} D_k \bar{R}_k = Id_\Gamma$. For any $F_k \in \mathbb{V}_k'$ the action of the operator $(S_k)^{-1} F_k$ is equal to the trace on Γ_k of the solution w_k of the local variational problem $a_k(w_k, v_k) = \langle F_k, Tr_k v_k \rangle$, $\forall v_k \in \mathbb{H}(\Omega_k), w_k \in \mathbb{H}(\Omega_k)$, which is associated to the operator $\mathcal{L}_k = -\text{div}(\nu_k \nabla w) + \vec{b} \cdot \nabla w + aw$ with Robin boundary condition on the interface $\nu_k \frac{\partial w}{\partial n_k} - \frac{1}{2} \vec{b} \cdot \vec{n}_k w = F_k$. In order to achieve good parallelization, the weights D_k are defined on each interface degree of freedom $\bar{u}(P)$ (with $P \in \Gamma_k$) as

$$D_k \, \bar{u}(P) \; = \; C_P \, \frac{\nu_k}{\sum_{P \in \Gamma_j} \nu_j} \, \bar{u}(P),$$

where the constant C_P is chosen to satisfy the partition of unity requirement, and depends only on the number of subdomains to which the point P belongs.

4. Numerical results in 3D. Problem (3.1) is discretized by means of GALS second order finite elements on hexaedral decomposition. The interface problem is solved by a GMRES algorithm preconditioned by the operator $\mathcal{T}$, which stops when the residual is less than 10^{-10}. We consider $\Omega = [0, 1]^3$, the unit cube, as costituted of two different materials with viscosity coefficients ν_1 and ν_2, we choose $a = 1$ and $f \equiv 0$ in the whole Ω, and we force the solution to have a boundary layer by imposing $u = 1$ on the bottom face of the cube as well as homogeneous Dirichlet conditions on the rest of the boundary $\partial\Omega$. We consider large jumps between the viscosity coefficients.

In Table 4.1 we report the number of iterations for a two-domain decomposition, where we choose different convective fields: perpendicular to the interface ($\vec{b} = \vec{e}_1$), parallel ($\vec{b} = \vec{e}_2 + \vec{e}_3$) and oblique ($\vec{b} = \vec{e}_1 + 3\,\vec{e}_2 + 5\,\vec{e}_3$). The preconditioner appears a little sensitive to the direction of the velocity but it is insensitive to the amplitude of the jumps in the viscosity coefficients.

In Table 4.2 we report the number of iterations for a eight domain decomposition. Each coefficient ν_j ($j = 1, 2$) refers to four subdomains which mutual position is varied: in Test 1 the two half cubes of the previous test are decoupled into four smaller subdomains, the configuration of Test 2 is given in Figure 4.1, while Test 3 is a black and white coloring where each subdomain of one kind is surrounded by subdomains of the other one. The convective

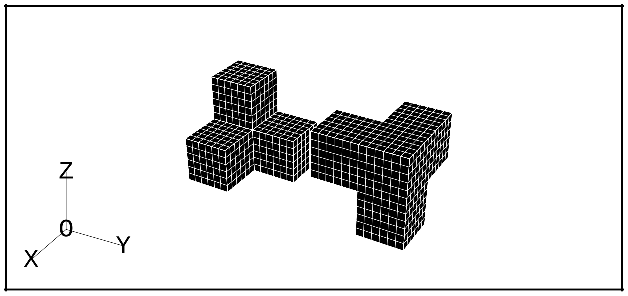

Figure 4.1: The subdomains Ω_1 (left) and Ω_2 (right) in Test 2.

ν_1, ν_2	ν_1/ν_2	Test 1	Test 2	Test 3
$10^{-1}, 10^{-5}$	10^4	33	33	34
$10^{-1}, 10^{-6}$	10^5	32	33	34
$10^{-1}, 10^{-7}$	10^6	32	33	34
$10^3, 10^{-3}$	10^6	29	28	21
$1, 10^{-7}$	10^7	29	31	29

Table 4.2: Number of iterations for the multidomain problem

field is $\vec{b} = -2\pi(y-0.5)\,\vec{e}_1 + 2\pi(x-0.5)\,\vec{e}_2 + \sin(2\pi(x-0.5))\,\vec{e}_3$. The preconditioner is again insensitive to the jumps and to the position of the subdomains.

A complete description of the tests will be given in ([6]).

5. Conclusions. The proposed preconditioner is a generalization of the Robin-Robin one to advection-diffusion problems with discontinuous coefficients. Numerical tests in 3D show, as we expected from the theoetical analysis of Section 2, that the preconditioner is fairly insensitive to the jumps in the viscosity coefficients as well as to the convective field, while it remains a little sensitive to the number of subdomains, but this seems unavoidable for advection-dominated problems. However, our knowledge of the preconditioner is not complete, and further work needs to be done: a convergence analysis in a more general setting is not yet available, the introduction of a coarse space to reduce the sensitivity to the number of subdomains should be analyzed and the algorithm should be tested on less academical situations.

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