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18. Balancing Neumann-Neumann for (In)Compressible Linear Elasticity and (Generalized) Stokes — Parallel Implementation

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1. Introduction. In this paper, an extension of the Balancing Neumann-Neumann method for a class of symmetric, indefinite problems is presented, with an emphasis on implementational and algorithmic aspects; theoretical results on a bound for the condition number of the relevant operator are also stated, without proof, and results of large-scale numerical experiments are reported. For a development of the theory see [7], [4].

The Balancing Neumann-Neumann domain decomposition technique (see, e.g., Mandel [5] or Mandel and Brezina [6]) has recently been extended to a class of saddle-point problems, including the Stokes Equation (see Pavarino and Widlund [7]) and the mixed formulation of linear elasticity (see Goldfeld, Pavarino and Widlund [3], [4]).

In this algorithm, after decomposing the original domain into nonoverlapping subdomains, the interior velocity/displacement and all but the subdomain-wise constant pressure unknowns are eliminated. A preconditioner for the resulting saddle-point Schur complement problem is constructed based on the solution of a coarse problem, with one pressure and a few velocity/displacement unknowns per subdomain, and on the solution of local problems with mixed or natural boundary conditions. Local Dirichlet problems must also be solved in order to compute the action of the Schur complement operator. The quality of this preconditioner can be shown to be independent of the number of subdomains and to depend only polylogarithmically on the size of the local problems, when the coefficients are constant. Numerical experiments indicate that this is still the case when there are arbitrary jumps on the coefficients.

This paper is organized as follows. In Section 2, we briefly describe the class of problems considered and their mixed finite element discretizations. The substructuring process is explained in Section 3, where we also include some remarks on the practical implementation of the Schur complement operator. In Section 4, the Balancing Neumann-Neumann preconditioner is introduced. In Section 5, the theoretical results on the quality of the preconditioner are stated and, finally, numerical experiments are reported in Section 6.

2. Problems and Discretizations. We consider the problems of linear elasticity with a mixed formulation (compressible, incompressible or almost incompressible cases), Stokes' equations and generalized Stokes' equations (with compressibility). All of them have a variational formulation of the following form: *For* $\Omega \subset \mathbb{R}^d$*, a polyhedral domain, given* $\mathbf{f} \in \left(H^{-1}(\Omega)\right)^d$, $\mathbf{g} \in \left(H^{1/2}(\partial\Omega)\right)^d$ *and* $h \in L^2(\Omega)$*, find* $(\mathbf{u}, p) \in \left(\tilde{\mathbf{g}} + \left(H_0^1(\Omega)\right)^d\right) \times L^2(\Omega)$ *satisfying*

$$\left\{\begin{array}{rcrcll} a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & \langle \mathbf{f},\mathbf{v}\rangle & \forall \mathbf{v} \in \left(H_0^1(\Omega)\right)^d \\ b(\mathbf{u},q) & - & c(p,q) & = & \langle h,q\rangle & \forall q \in L^2(\Omega) \\ & & \mathbf{u}|_{\partial\Omega} & = & \mathbf{g} & \end{array}\right. . \tag{2.1}$$

Here $\tilde{\mathbf{g}}$ is any function in $\left(H^1(\Omega)\right)^d$ such that $\tilde{\mathbf{g}}|_{\partial\Omega} = \mathbf{g}$. The choice of the bilinear forms a, b and c depends upon the problem we are solving:

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	$a(\mathbf{u},\mathbf{v})$	$b(\mathbf{v},q)$	$c(p,q)$
compressible elasticity	$2\mu\int_\Omega \varepsilon(\mathbf{u}):\varepsilon(\mathbf{v})$	$-\int_\Omega q\nabla\cdot\mathbf{v}$	$\frac{1}{\lambda}\int_\Omega pq$
incompressible elasticity	$2\mu\int_\Omega \varepsilon(\mathbf{u}):\varepsilon(\mathbf{v})$	$-\int_\Omega q\nabla\cdot\mathbf{v}$	0
Stokes	$\nu\int_\Omega \nabla\mathbf{u}:\nabla\mathbf{v}$	$-\int_\Omega q\nabla\cdot\mathbf{v}$	0
generalized Stokes	$\nu\int_\Omega \nabla\mathbf{u}:\nabla\mathbf{v}$	$-\int_\Omega q\nabla\cdot\mathbf{v}$	$\frac{1}{\lambda}\int_\Omega pq$

Here $\varepsilon(\mathbf{u}):\varepsilon(\mathbf{v}) = \frac{1}{4}\sum_{i,j=1}^{d}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)$ and $\nabla\mathbf{u}:\nabla\mathbf{v} = \sum_{i,j=1}^{d}\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}$.

To fix ideas, we will focus, in the remainder of this paper, on the elasticity problem. Therefore, $\mathbf{u}$ will be the displacement vector and the relevant coefficients will be the Lamé parameters μ and λ.

A conforming mixed finite-element discretization of (2.1) yields a linear system of the form

$$K\underline{\mathbf{u}} = K\begin{bmatrix}\mathbf{u}\\ p\end{bmatrix} = \begin{bmatrix}A & B^T\\ B & -C\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ p\end{bmatrix} = \underline{\mathbf{f}} = \begin{bmatrix}\mathbf{f}\\ h\end{bmatrix}.$$

We select an inf-sup stable pair of finite element spaces for pressure and displacement. As will become evident in the next subsection, our method requires the pressure space to be discontinuous, at least across the interface.

Note that although this paper is written in the language of finite elements only, the method here presented is equally suitable for spectral element discretizations (see [7], [3], [4]).

3. Substructuring. The domain Ω is decomposed into N nonoverlapping subdomains, $\{\Omega_i\}_{i=1,2,\ldots,N}$, the boundaries of which do not cut through any element. Denote by Γ_h the set of nodes on the interface between subdomains, i.e., the nodes belonging to more than one subdomain. As usual, K and $\underline{\mathbf{f}}$ can be generated by subassembly:

$$\begin{aligned} K &= \sum_{i=1}^{N} {R^{(i)}}^T K^{(i)} R^{(i)} = \sum_{i=1}^{N} {R^{(i)}}^T \begin{bmatrix} A^{(i)} & {B^{(i)}}^T \\ B^{(i)} & -C^{(i)}\end{bmatrix} R^{(i)}, \qquad (3.1)\\ \underline{\mathbf{f}} &= \sum_{i=1}^{N} {R^{(i)}}^T \begin{bmatrix}\mathbf{f}^{(i)}\\ h^{(i)}\end{bmatrix}, \end{aligned}$$

where the restriction matrix $R^{(i)}$ is a matrix of zeros and ones which translates global indices of the nodes into local numbering.

Assume that the basis for the pressure space can be split as follows:

- there are N *coarse* pressures, $\{\psi_{0,i}\}_{i=1,2,\ldots,N}$, defined by $\psi_{0,i} = \chi_{\Omega_i}$, where χ_{Ω_i} is the characteristic function of the set Ω_i. We also refer to these functions as the *constant* or *interface* pressures;

- the remaining, *interior* pressures, $\{\psi_{I,j_i}\}_{j_i\in J_i}$, have zero average, $\int_\Omega \psi_{I,j_i} = 0$, and are local, in the sense that $\mathrm{supp}\,(\psi_{I,j_i}) \subset \Omega_i$.

After reordering unknowns and equations, the vectors $\underline{\mathbf{u}}$ and $\underline{\mathbf{f}}$, and the stiffness matrix K are expressed as

$$\begin{aligned}
\underline{\mathbf{u}} &= \left[\begin{array}{c} \underline{\mathbf{u}}_I \\ \hline \underline{\mathbf{u}}_\Gamma \end{array}\right] = \left[\begin{array}{c} \mathbf{u}_I \\ p_I \\ \hline \mathbf{u}_\Gamma \\ p_0 \end{array}\right], \quad \underline{\mathbf{f}} = \left[\begin{array}{c} \underline{\mathbf{f}}_I \\ \hline \underline{\mathbf{f}}_\Gamma \end{array}\right] = \left[\begin{array}{c} \mathbf{f}_I \\ h_I \\ \hline \mathbf{f}_\Gamma \\ h_0 \end{array}\right], \\
K &= \left[\begin{array}{c|c} K_{II} & K_{I\Gamma} \\ \hline K_{\Gamma I} & K_{\Gamma\Gamma} \end{array}\right] = \left[\begin{array}{cc|cc} A_{II} & B_{II}^T & A_{I\Gamma} & 0 \\ B_{II} & -C_{II} & B_{I\Gamma} & 0 \\ \hline A_{\Gamma I} & B_{I\Gamma}^T & A_{\Gamma\Gamma} & B_{0\Gamma}^T \\ 0 & 0 & B_{0\Gamma} & -C_{00} \end{array}\right].
\end{aligned} \tag{3.2}$$

The (1,1)-block K_{II} is invertible, even when K is not (i.e., when the problem is incompressible and therefore C is zero and the solution is only defined up to a constant pressure.) We can eliminate the interior variables and define a Schur complement problem,

$$S\underline{\mathbf{u}}_\Gamma = \tilde{\underline{\mathbf{f}}}_\Gamma, \tag{3.3}$$

where

$$\begin{aligned}
S &= K_{\Gamma\Gamma} - K_{\Gamma I}K_{II}^{-1}K_{I\Gamma} = \left[\begin{array}{cc} S_\Gamma & B_{0\Gamma}^T \\ B_{0\Gamma} & -C_{00} \end{array}\right] \quad \text{and} \\
\tilde{\underline{\mathbf{f}}}_\Gamma &= \underline{\mathbf{f}}_\Gamma - K_{\Gamma I}K_{II}^{-1}\underline{\mathbf{f}}_I,
\end{aligned} \tag{3.4}$$

with

$$S_\Gamma = A_{\Gamma\Gamma} - \left[\begin{array}{cc} A_{\Gamma I} & B_{I\Gamma}^T \end{array}\right] \left[\begin{array}{cc} A_{II} & B_{II}^T \\ B_{II} & -C_{II} \end{array}\right]^{-1} \left[\begin{array}{c} A_{I\Gamma} \\ B_{I\Gamma} \end{array}\right]. \tag{3.5}$$

We note that K_{II} is block-diagonal, which allows us to generate S by subassembly, by means of restriction matrices $R_\Gamma^{(i)}$:

$$S = \sum_{i=1}^{N} {R_\Gamma^{(i)}}^T S^{(i)} R_\Gamma^{(i)} = \sum_{i=1}^{N} {R_\Gamma^{(i)}}^T \left(K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)} {K_{II}^{(i)}}^{-1} K_{I\Gamma}^{(i)}\right) R_\Gamma^{(i)}. \tag{3.6}$$

We present a preconditioner for the operator S. Once the system $S\underline{\mathbf{u}}_\Gamma = \tilde{\underline{\mathbf{f}}}_\Gamma$ is solved, the computations required to obtain $\underline{\mathbf{u}}_I$ are completely local.

3.1. Implementing S. Before we describe the Neumann-Neumann preconditioner, we discuss how to compute the action of the operator S on a given vector.

We have assumed that the basis functions for the pressure degrees of freedom can be divided into two groups: zero-average functions and constant functions. We now show how S can be implemented using a standard basis for the pressure, as long as the pressure space *admits* a basis of that special form.

In our actual implementation we generate, instead of the stiffness matrix in (3.2), a stiffness matrix $\tilde{K}$ using a standard nodal basis and introduce a Lagrange multiplier to enforce the zero average of the pressure. Furthermore, we never assemble the entire matrix $\tilde{K}$, but rather work with the local stiffness matrices $\tilde{K}^{(i)}$:

$$\tilde{K} = \sum_{i=1}^{N} {\tilde{R}^{(i)}}^T \tilde{K}^{(i)} \tilde{R}^{(i)}, \quad \text{where} \quad \tilde{K}^{(i)} = \left[\begin{array}{ccc} A^{(i)} & {\tilde{B}^{(i)}}^T & 0 \\ \tilde{B}^{(i)} & -\tilde{C}^{(i)} & w^{(i)} \\ 0 & {w^{(i)}}^T & 0 \end{array}\right]. \tag{3.7}$$

Note that, since a different basis has been used, $\tilde{K}^{(i)}$, $\tilde{B}^{(i)}$, and $\tilde{C}^{(i)}$ are different from $K^{(i)}$, $B^{(i)}$, and $C^{(i)}$ in equation (3.1). The entries of the vector $w^{(i)}$ are the integrals of the pressure basis functions over Ω.

In each of the local matrices $\tilde{K}^{(i)}$, we eliminate the interior velocities, *all* the pressures and the Lagrange multiplier. This corresponds to taking the Schur complement with respect to the (2,2)-block in the following matrix, which is a reordering of (3.7):

$$\left[\begin{array}{ccc|c} A_{II}^{(i)} & \tilde{B}_{I}^{(i)^T} & 0 & A_{I\Gamma}^{(i)} \\ \tilde{B}_{I}^{(i)} & -\tilde{C}^{(i)} & w^{(i)} & \tilde{B}_{\Gamma}^{(i)} \\ 0 & w^{(i)^T} & 0 & 0 \\ \hline A_{\Gamma I}^{(i)} & \tilde{B}_{\Gamma}^{(i)^T} & 0 & A_{\Gamma\Gamma}^{(i)} \end{array}\right].$$

We can show that the result of this static condensation is precisely $S_{\Gamma}^{(i)}$, the $(1,1)$-block of $S^{(i)}$ (see (3.4), (3.5), (3.6)). The remainder of the matrix $S^{(i)}$, namely the vector $B_{0\Gamma}^{(i)^T}$ and the scalar $C_{00}^{(i)}$, can be computed by means of the formula:

$$\left[\begin{array}{cc} A_{\Gamma\Gamma}^{(i)} & B_{0\Gamma}^{(i)^T} \\ B_{0\Gamma}^{(i)} & C_{00}^{(i)} \end{array}\right] = \left[\begin{array}{c} I \\ e^{(i)^T} \end{array}\right] \left[\begin{array}{cc} A_{\Gamma\Gamma}^{(i)} & \tilde{B}_{\Gamma}^{(i)^T} \\ \tilde{B}_{\Gamma}^{(i)} & -\tilde{C}^{(i)} \end{array}\right] \left[\begin{array}{cc} I & e^{(i)} \end{array}\right].$$

Here the matrix at the right side of the equation is a submatrix of (3.7) and the entries of the vector $e^{(i)}$ are the coefficients that express the constant pressure on subdomain Ω_i in terms of the regular basis functions:

$$\sum_{k=1}^{\tilde{n}_p} \left(e^{(i)}\right)_k \tilde{\psi}_k = \chi_{\Omega_i},$$

where $\left\{\tilde{\psi}_k\right\}_{k=1,\ldots,\tilde{n}_p}$ is the basis for the pressure space.

4. Preconditioner. The Balancing Neumann-Neumann preconditioner is of the form:

$$Q = Q_0 + (I - Q_0 S)\, Q_{loc}\, (I - S Q_0)\,,$$

where Q_0 is the coarse-level part of the preconditioner and Q_{loc} the local-level part.

4.1. Local Level. The local part of the preconditioner basically involves the solution of local problems with natural or mixed boundary conditions (for floating and non-floating subdomains, respectively). Q_{loc} is defined by

$$Q_{loc} = \sum_{i=1}^{N} R_{\Gamma}^{(i)^T} \left[\begin{array}{cc} D^{(i)^{-1}} & 0 \\ 0 & 0 \end{array}\right] S^{(i)^{\dagger}} \left[\begin{array}{cc} D^{(i)^{-1}} & 0 \\ 0 & 0 \end{array}\right] R_{\Gamma}^{(i)}.$$

The dagger (†) above indicates a pseudo-inverse, since $S^{(i)}$ is singular on a floating subdomain (the nullspace being constant velocities for Stokes' equation and rigid-body displacements for elasticity). The matrices $D^{(i)^{-1}}$ are diagonal and determine a partition of unity on Γ. A proper choice of this partition is necessary for the method to be insensitive to jumps in the coefficients:

$$\left(D^{(i)^{-1}}\right)_{jj} = \frac{\mu_i^{\gamma}}{\displaystyle\sum_{x_j \in \partial\Omega_k} \mu_k^{\gamma}}\,, \quad \gamma \geq \frac{1}{2}.$$

In computing the action of Q_{loc}, it is useful to remember that

$$\left[\begin{array}{ccc} I & 0 \end{array}\right] S^{(i)\dagger} \left[\begin{array}{c} I \\ 0 \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & I \end{array}\right] \left[\begin{array}{ccc} A_{II}^{(i)} & \tilde{B}_I^{(i)^T} & A_{I\Gamma}^{(i)} \\ \tilde{B}_I^{(i)} & -\tilde{C}^{(i)} & \tilde{B}_\Gamma^{(i)^T} \\ A_{\Gamma I}^{(i)} & \tilde{B}_\Gamma^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{array}\right]^\dagger \left[\begin{array}{c} 0 \\ 0 \\ I \end{array}\right].$$

Note that the matrix in the right side of the equation is a submatrix of (3.7). The coarse step preceding the local step ensures that the right-hand sides are consistent and in this case a good approximation for the pseudo-inverse can be obtained by perturbing the original system, replacing the original $A^{(i)}$ by $A^{(i)} + \varepsilon I$ or $A^{(i)} + \varepsilon M^{(i)}$, where $M^{(i)}$ is the local mass matrix for the displacement variables and ε is a small positive constant.

4.2. Coarse Level. The application of the coarse term Q_0 amounts to the solution of a coarse, global problem:

$$Q_0 \quad = \quad R_0^T (R_0 S R_0^T)^\dagger R_0,$$

where

$$R_0 = \left[\begin{array}{cc} L^T & 0 \\ 0 & I \end{array}\right].$$

The columns of the matrix R_0^T span the coarse space: the identity block corresponds to the coarse pressures, one per subdomain; the displacement coarse space is determined by the columns of the matrix L. In order to ensure solvability of the local problems with natural boundary conditions, L must contain the traces of the elements of a basis of the nullspace of $A^{(i)}$ scaled by $D^{(i)^{-1}}$ for all i corresponding to the floating subdomains (cf. subsection 4.1). These scaled rigid-body displacements can also be added for non-floating subdomains, as long as care is taken to avoid linearly dependence; this can be accomplished by dropping the contribution of one non-floating subdomain.

In order to obtain an inf-sup stable coarse space, we need to enrich L further. Two alternatives are: adding the traces of either the coarse bi/tri-linear functions (the space Q_1^H) or the quadratic coarse edge/face bubble functions for the normal directions.

Remark We can show that QS is positive-definite on range $(I - Q_0 S)$. If an initial guess is chosen such that the initial error is in range $(I - Q_0 S)$, then the error on every step of a Krylov method will also be restricted to range $(I - Q_0 S)$, since $Q_0 S$ is a projection. The importance of this observation is that it allow us to use the preconditioned conjugate gradient method as our iterative solver, even though our original operator is indefinite.

5. Theoretical Bound. A theoretical bound for the condition number of the preconditioned operator QS restricted to the appropriate subspace to which the iterates are confined is proved in [7], [4], for the constant coefficient case:

$$\kappa \leq C \left(1 + \log\left(\frac{H}{h}\right)\right)^2.$$

We note that κ does not depend on the number of subdomains and depends only polylogarithmically on the size of the subdomain problems. The constant C depends, in the incompressible or quasi-incompressible cases, on the inf-sup constants of the original and coarse spaces. This is the reason why we enrich the displacement coarse space to achieve inf-sup stability.

Figure 6.1: material properties of a heterogeneous problem.

s	r	s	r	$\cdots$	s	r
r	a	r	a	$\cdots$	r	a
s	r	s	r	$\cdots$	s	r
r	a	r	a	$\cdots$	r	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
s	r	s	r	$\cdots$	s	r
r	a	r	a	$\cdots$	r	a

$s-$ steel-like
$a-$ aluminium-like
$r-$ rubber-like

$\mu_s = 8.20 \quad \lambda_s = 10.00 \quad \nu_s = 0.275$
$\mu_a = 2.60 \quad \lambda_a = 5.60 \quad \nu_a = 0.341$
$\mu_r = 0.01 \quad \lambda_r = 0.99 \quad \nu_r = 0.495$

6. Numerical Experiments. Our algorithm has been implemented in C, using the PETSc library (see [1], [2]). Parallel numerical experiments were run on the Linux cluster Chiba City at Argonne National Laboratory (with 256 Dual Pentium III processors with 512MB of local RAM). We report on results for compressible/almost-incompressible elasticity only, although similar results have been obtained for incompressible elasticity, Stokes and generalized Stokes equations.

We consider an elasticity problem defined on a square heterogeneous domain, which is composed of an arrangement of three different materials in the pattern depicted in figure 6.1. Note that the material r is almost incompressible, with a Poisson ratio close to 0.5. The problem is discretized with $Q_2 - Q_0$ finite elements and the domain Ω divided into $\sqrt{N} \times \sqrt{N}$ square subdomains, each of them composed of a single material. The saddle point Schur complement (3.3) is solved iteratively by PCG with our balancing Neumann-Neumann preconditioner and the coarse space $V_0 = \{\text{scaled rigid body motions}\} + Q_1^H$. The initial guess is a random vector modified so that the initial error is in the range of $(I - Q_0 S)$, the right hand side is a random, uniformly distributed vector, and the stopping criterion is $\|r_k\|_2 / \|r_0\|_2 \leq 10^{-6}$, where r_k is the residual at the $k-$th iterate.

In the lower half of Table 6.1, we show the results for increasing mesh sizes, always with 64 subdomains. The condition number and the iteration count grow weakly as we increase the size of the local problems, as can also be observed in the left part of Figure 6.2.

The last two columns of this table display CPU-times for these runs. The last column gives the total time for the code to run, while the column labeled "fact." gives the time spent on LU factorizations; there are three of them: two local, namely Dirichlet and Neumann subdomain-level problems, and one global coarse problem. We note that the cost of the factorizations grows rapidly and dominates the cost of the computation. The upper part of Table 6.1 shows results for an increasing number of subdomains of fixed size (about 58,000 degrees of freedom). The corresponding graph, on the right in Figure 6.2, shows an almost horizontal tail, indicating independence of the condition number and the iteration count on the number of subdomains. This is numerical evidence that our result in section 5 remains valid in the case of discontinuous coefficients. The fact that the factorization time remains constant for the entire range of problem sizes tested (from 16 to 169 subdomains) indicates that the cost associated with the factorization of the coarse problem is still tiny compared with that of the local problems. One can expect this scenario to change if the number of subdomains increases significantly.

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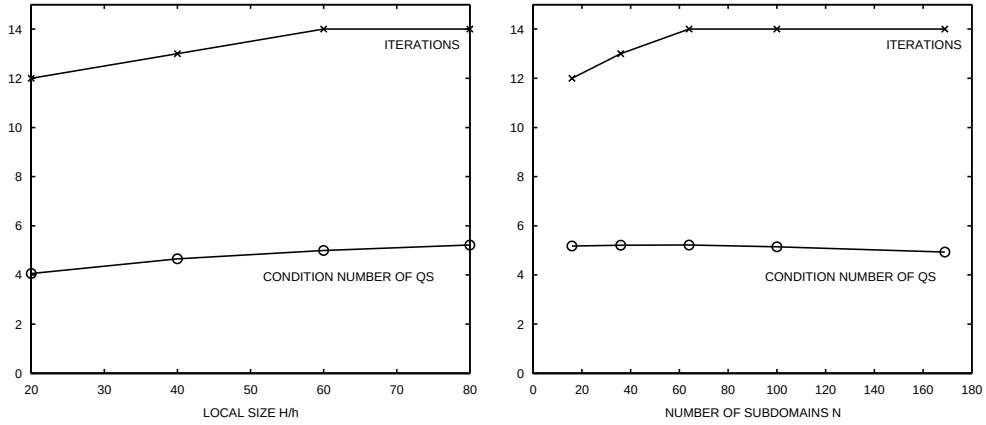
Table 6.1: results for elasticity problem in heterogeneous medium with $Q_2 - Q_0$ finite elements and $V_0 = (\text{scaled rigid-body motions}) + \left(Q_1^H\right)^2$. The iterative method is PCG and the termination criterion is $\|r_{\text{final}}\| \leq 10^{-6} \|r_0\|$. The initial guess and right-hand side are randomly generated. The ordering of the displacement variables is determined by quasi-minimal degree.

fixed H/h, local problem with 80×80 elements (58,242 dof)						
grid size	# subd.	dof	iter.	cond.	time (sec.)	
(# elem.)		($\times 10^6$)			fact.	other
320×320	4×4	0.92	12	5.14	258.0	63.4
480×480	6×6	2.08	13	5.12	253.7	63.7
640×640	8×8	3.69	14	5.13	260.8	84.5
800×800	10×10	5.77	14	5.06	262.8	93.9
1040×1040	13×13	9.74	14	4.87	261.2	102.7
fixed number of subdomains $N = 8 \times 8$						
grid size	loc. dof	dof	iter.	cond.	time (sec.)	
(# elem.)	($\times 10^3$)	($\times 10^6$)			fact.	other
160×160	3.8	0.23	12	4.00	1.4	16.7
320×320	14.7	0.92	13	4.57	18.2	22.7
480×480	32.9	2.08	14	4.91	84.2	42.1
640×640	58.2	3.69	14	5.13	260.8	84.5

Figure 6.2: results for elasticity problem in heterogeneous medium with $Q_2 - Q_0$ finite elements: PCG iteration count and condition number of QS vs. local size H/h (left) and number of subdomains N (right), from Table 6.1.



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