Fourteenth International Conference on Domain Decomposition Methods
Editors: Ismael Herrera , David E. Keyes, Olof B. Widlund, Robert Yates

46. A Dirichlet/Robin Iteration-by-Subdomain Domain Decomposition Method Applied to Advection-Diffusion Problems for Overlapping Subdomains

G. Houzeaux[1], R. Codina[2]

1. Introduction. We present a domain decomposition (DD) method to solve scalar advection-diffusion-reaction (ADR) equations which falls into the category of *iteration-by-subdomain* DD methods.

Domain decomposition methods are usually divided into two families, namely overlapping and non-overlapping methods. The former are based on the Schwarz method. At the differential level, they use alternatively the solution on one subdomain to update the Dirichlet data of the other. Contrary, non-overlapping DD methods use necessarily two different transmission conditions on the interface, in such a way that both the continuity of the unknown and its first derivatives are achieved on the interface (for ADR equations). Let us mention the Dirichlet/Neumann method introduced in [4, 8, 10]; the γ-Dirichlet/Robin method [2]; the Robin/Robin method [6, 9, 7]; the coercive γ-Robin/Robin method [2]; the Neumann/Neumann method [5, 3, 1], etc.

In the literature, all the mixed DD methods mentioned previously have been mainly studied in the context of disjoint partitioning. However, there exists no particular reason for restricting their application only to non-overlapping subdomains. This paper gives a possible line of study for the generalization of the mixed method to overlapping subdomains. We expect that the overlapping mixed DD methods will enjoy some properties of their disjoint brothers as well as some properties of the classical Schwarz method, as for example the dependence on the overlapping length.

Our motivation to study these types of methods has been to maintain the implementation advantages of the Schwarz method when used together with a numerical approximation of the problem. The possibility to have some overlapping simplifies enormously the discretization of the subdomains. However, very often this overlapping needs to be very small in practice, and thus the convergence rate of the Schwarz method becomes very small. Contrary to the Schwarz method, the limit case of zero overlapping will be possible using the formulation proposed herein. We have chosen to study an overlapping Dirichlet/Robin method, using the coercive bilinear form presented in [2] in the context of the γ-D/R and γ-R/R methods. This simplifies the analysis of the DD method as no assumption has to be made on the direction of the flow and its amplitude on the interfaces of the overlapping subdomains.

We would like to stress that our approach *is not* to view domain decomposition as a preconditioner for solving the linear systems of equations arising after the space discretization of the differential equations. In our case, the domain is decomposed at *the continuous level*. We are not concerned with the scaling properties with respect to the number of subdomains of the iteration-by-subdomain strategy we propose. For our purposes, it is enough to analyze *two* subdomains. More precisely, our final goal is to devise a Chimera type strategy taking Dirichlet/Robin(Neumann) transmission conditions rather than the classical Dirichlet/Dirichlet (Schwarz) approach. This paper must be understood as a theoretical basis for such a formulation. We recall briefly the Chimera method, of which we give an example in Figure 1.1. Firstly, independent meshes are generated for the background mesh and the mesh around the cylinder. Secondly, the mesh around the cylinder is placed on the background mesh. Then, according to some criteria (order of interpolation, geometrical overlap prescribed, etc.), we can impose in a simple way a Dirichlet condition on some nodes of the

[1]Universitat Politècnica de Catalunya - CIMNE, houzeaux@cimne.upc.es
[2]Universitat Politècnica de Catalunya - CIMNE, ramon.codina@upc.es

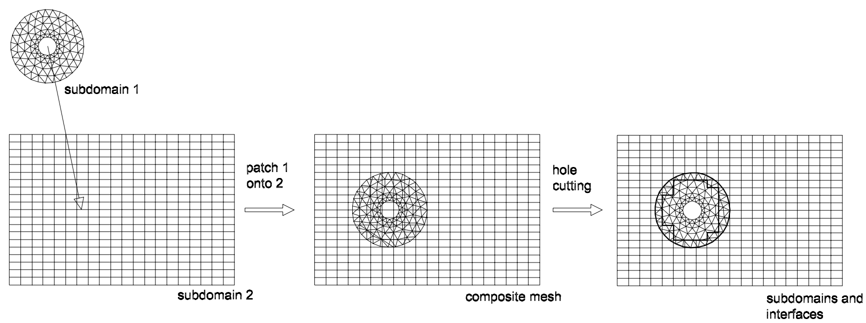

Figure 1.1: Chimera method.

background located inside the cylinder subdomain (this task is called hole cutting). Doing so, we form an apparent interface on the background subdomain to set up an iteration-by-subdomain method. Note that a natural condition of Neumann or Robin type is in general not possible as the apparent interface is irregular. Finally, by imposing a Dirichlet, Neumann or Robin condition on the outer boundary of the cylinder subdomain we can define completely an iteration-by-subdomain method to couple both subdomains. The Chimera method was first thought as a tool to simplify the meshing of complicated geometry. It is also a powerfull tool to treat subdomains in relative motion.

2. Problem statement. Let us consider the advection-diffusion-reaction problem of finding u such that:

$$\left\{ \begin{array}{l} Lu := -\varepsilon\Delta u + \nabla\cdot(\boldsymbol{a}u) + \sigma u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{array} \right. \tag{2.1}$$

where Ω is a d-dimensional domain ($d = 1, 2, 3$) with boundary $\partial\Omega$, ε is the diffusion constant of the medium, f is the force term, $\boldsymbol{a}$ is the advection field (not necessarily solenoidal) and σ is a source (reaction) term.

We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$, and by $V := H_0^1(\Omega)$ the space where u will be sought. Likewise, we use the notation

$$\langle\cdot,\cdot\rangle_\omega := \langle\cdot,\cdot\rangle_{H^s(\omega)\times H^{-s}(\omega)}, \tag{2.2}$$

for the duality pairing between the space $H^s(\omega)$ and its topological dual $H^{-s}(\omega)$, with $s = 1$ when ω is d-dimensional and with $s = 1/2$ when ω is $(d-1)$-dimensional.

Let us consider our differential problem 2.1. We restrict ourselves to solutions in V. To guarantee existence, we take $f \in H^{-1}(\Omega)$ and $\boldsymbol{a}, \sigma, \nabla\cdot\boldsymbol{a} \in L^\infty(\Omega)$. Since

$$\int_\Omega v\boldsymbol{a}\cdot\nabla u\, d\Omega = -\int_\Omega u\boldsymbol{a}\cdot\nabla v\, d\Omega - \int_\Omega uv\nabla\cdot\boldsymbol{a}\, d\Omega \quad \forall\, u, v \in V, \tag{2.3}$$

we transform the convective term into a skew symmetric operator, and we can enunciate our problem as follows: find $u \in V$ such that

$$a(u, v) = \langle f, v\rangle \quad \forall\, v \in V, \tag{2.4}$$

where the bilinear form is

$$a(w, v) \;:=\; \varepsilon(\nabla w, \nabla v) + \frac{1}{2}(\boldsymbol{a}\cdot\nabla w, v) - \frac{1}{2}(w, \boldsymbol{a}\cdot\nabla v) + (\sigma_0 w, v), \tag{2.5}$$

with $\sigma_0 = \sigma + \frac{1}{2}\nabla\cdot\boldsymbol{a}$.

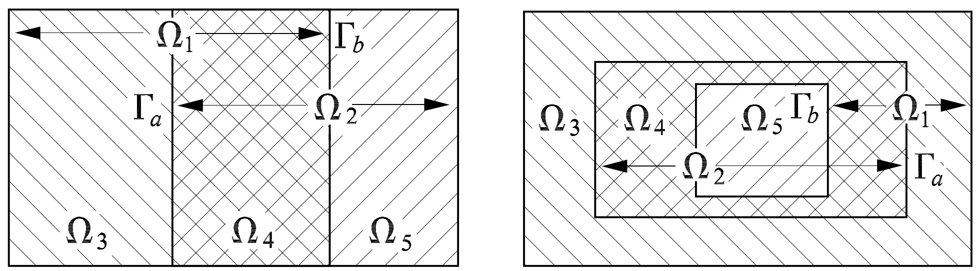

Figure 3.1: Examples of decomposition of a domain Ω into two overlapping subdomains Ω_1 and Ω_2.

3. Overlapping Dirichlet/Robin method.

3.1. Domain partitioning and definitions. We perform a geometrical decomposition of the original domain Ω into three disjoint and connected subdomains Ω_3, Ω_4 and Ω_5 such that

$$\Omega = \text{int}\left(\overline{\Omega_3 \cup \Omega_4 \cup \Omega_5}\right). \tag{3.1}$$

From this partition, we define Ω_1 and Ω_2 , as two overlapping subdomains:

$$\Omega_1 := \text{int}\left(\overline{\Omega_3 \cup \Omega_4}\right), \quad \Omega_2 := \text{int}\left(\overline{\Omega_5 \cup \Omega_4}\right). \tag{3.2}$$

Finally, we define Γ_a as the part of $\partial\Omega_2$ lying in Ω_1, and Γ_b as the part of $\partial\Omega_1$ lying in Ω_2. The geometrical nomenclature is shown in Figure 3.1. Γ_b and Γ_a are the *interfaces* of the domain decomposition method we now present. Ω_4 is the overlap zone. In the following, index i or j refers to a subdomain or an interface.

Let us introduce the following definitions:

$$\begin{aligned}
(w,v)_{\Omega_i} &:= \int_{\Omega_i} wv\, d\Omega, && (3.3)\\
a_i(w,v) &:= \varepsilon(\nabla w, \nabla v)_{\Omega_i} + \frac{1}{2}(\boldsymbol{a}\cdot\nabla w, v)_{\Omega_i} - \frac{1}{2}(w, \boldsymbol{a}\cdot\nabla v)_{\Omega_i} + (\sigma_0 w, v)_{\Omega_i} && (3.4)\\
V_i &:= \{v \in H^1(\Omega_i) \mid v_{|\partial\Omega\cap\partial\Omega_i} = 0\}, && (3.5)\\
V_i^0 &:= H_0^1(\Omega_i), && (3.6)
\end{aligned}$$

where i can be any of the five subdomains introduced previously, i.e. $i = 1, 2, 3, 4$ or 5. Let us define the linear and continuous trace operators T_a and T_b on Γ_a and Γ_b, respectively. We explicitly define the trace space on Γ_a and Γ_b as $\Lambda_a := \{\mu_a \in H^{1/2}(\Gamma_a)\}$ and $\Lambda_b := \{\mu_b \in H^{1/2}(\Gamma_b)\}$, respectively.

3.2. Variational formulation. We propose to solve the following problem: find $u_1 \in V_1$ and $u_2 \in V_2$ such that

$$\left\{\begin{array}{ll}
a_1(u_1, v_1) = \langle f, v_1\rangle_{\Omega_1} & \forall\, v_1 \in V_1^0,\\
u_1 = u_2 & \text{on } \Gamma_b,\\
a_2(u_2, v_2) = \langle f, v_2\rangle_{\Omega_2} & \forall\, v_2 \in V_2^0,\\
a_3(u_1, E_3\mu_a) + a_2(u_2, E_2\mu_a) = \langle f, E_3\mu_a\rangle_{\Omega_3} + \langle f, E_2\mu_a\rangle_{\Omega_2} & \forall\, \mu_a \in \Lambda_a,
\end{array}\right. \tag{3.7}$$

where E_i denotes any possible extension operator from Λ_a to $H^1(\Omega_i)$, that is to say,

$$E_i:\ \Lambda_a \longrightarrow H^1(\Omega_i), \quad T_a E_i \mu_a = \mu_a \quad \forall\, \mu_a \in \Lambda_a. \tag{3.8}$$

Equations 3.7_1 and 3.7_3 are the equations for the unknown in subdomains Ω_1 and Ω_2 respectively. Equation 3.7_2 is the condition that ensures continuity of the primary variable across Γ_b, and levels the solution in both subdomains. Finally, Eq. 3.7_4 is the equation for the primary variable on the interface Γ_a.

Theorem 3.1 *Problems 3.7 and 2.4 are equivalent.*

The proof can be obtained as in the case of the Dirichlet/Neumann method applied to disjoint subdomains. See for example [10].

3.3. Alternative formulation. We develop an alternative formulation for the domain decomposition method given by Eqs. 3.7_{1-4}.

Lemma 3.1 *The solution of the domain decomposition problem satisfies*

$$\frac{\partial u_1}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1 = \frac{\partial u_2}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2 \quad \text{on } \Gamma_a, \tag{3.9}$$

where $\partial(\cdot)/\partial n_2 = \boldsymbol{n}_2\cdot\nabla(\cdot)$, $\boldsymbol{n}_2$ *being the exterior normal to* Ω_2 *on* Γ_a.

In addition, we have the following result.

Theorem 3.2 *System of Eqs.* 3.7_{1-4} *can be reformulated as follows: find* $u_1 \in V_1$ *and* $u_2 \in V_2$ *such that*

$$\left\{\begin{array}{ll} a_1(u_1,v_1) = \langle f, v_1\rangle_{\Omega_1} & \forall\, v_1 \in V_1^0, \\ u_1 = u_2 & \text{on } \Gamma_b, \\ a_2(u_2,v_2') = \langle f, v_2'\rangle_{\Omega_2} + \langle \varepsilon\frac{\partial u_1}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1, v_2'\rangle_{\Gamma_a} & \forall\, v_2' \in V_2. \end{array}\right. \tag{3.10}$$

The interpretation of the domain decomposition method now appears clearly. A Dirichlet problem is solved in Ω_1 using as Dirichlet data on the interface Γ_b the solution in Ω_2, whereas a mixed Dirichlet/Robin problem is solved in Ω_2 using as Robin data on Γ_a the solution in Ω_1. This formulation justifies the name *overlapping Dirichlet/Robin method* to designate this domain decomposition method.

Remark 3.1 *The system of Eqs.* 3.10_{1-3} *could have been derived directly from the following DD problem applied at the differential level:*

$$\left\{\begin{array}{ll} Lu_1 = f & \text{in } \Omega_1, \\ u_1 = 0 & \text{on } \partial\Omega_1\cap\partial\Omega, \\ u_1 = u_2 & \text{on } \Gamma_b, \\ Lu_2 = f & \text{in } \Omega_2, \\ u_2 = 0 & \text{on } \partial\Omega_2\cap\partial\Omega, \\ \varepsilon\frac{\partial u_2}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2 = & \varepsilon\frac{\partial u_1}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1 \quad \text{on } \Gamma_a. \end{array}\right. \tag{3.11}$$

3.4. Interface equations. A convenient way to study DD methods is to derive equations for the interface unknown(s). To do so, the problem is first rewritten into two purely Dirichlet problems for which the Dirichlet data are the unknowns on the interfaces. Starting form Eqs. 3.11_{1-6}, the problems to consider are:

$$\left\{\begin{array}{lll} Lw_1 = f & \text{in } \Omega_1, \\ w_1 = 0 & \text{on } \partial\Omega_1\cap\partial\Omega, \\ w_1 = \lambda_b & \text{on } \Gamma_b, \end{array}\right. \qquad \left\{\begin{array}{ll} Lw_2 = f & \text{in } \Omega_2, \\ w_2 = 0 & \text{on } \partial\Omega_2\cap\partial\Omega, \\ w_2 = \lambda_a & \text{on } \Gamma_a. \end{array}\right. \tag{3.12}$$

Now let us decompose w_1 and w_2 into L-homogeneous and Dirichlet-homogeneous parts,

$$w_1 = u_1^0 + u_1^*, \quad w_2 = u_2^0 + u_2^*, \tag{3.13}$$

where the L-homogeneous parts u_1^0 and u_2^0 are the solutions of

$$\begin{cases} Lu_1^0 = 0 & \text{in } \Omega_1, \\ u_1^0 = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ u_1^0 = \lambda_b & \text{on } \Gamma_b, \end{cases} \qquad \begin{cases} Lu_2^0 = 0 & \text{in } \Omega_2, \\ u_2^0 = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega, \\ u_2^0 = \lambda_a & \text{on } \Gamma_a, \end{cases} \tag{3.14}$$

and the Dirichlet-homogeneous parts u_1^* and u_2^* are the solutions of

$$\begin{cases} Lu_i^* = f & \text{in } \Omega_i, \\ u_i^* = 0 & \text{on } \partial\Omega_i, \end{cases} \tag{3.15}$$

for $i = 1, 2$. We refer to u_1^0 as the L-homogeneous extension of λ_b into Ω_1, and we denote it by $\mathcal{L}_1\lambda_b$. Similarly, we call u_2^0 the L-homogeneous extension of λ_a into Ω_2, and we denote it by $\mathcal{L}_2\lambda_a$. In the case when $L = -\Delta$, $\mathcal{L}$ is the harmonic extension and is usually denoted by H. The Dirichlet-homogeneous parts u_1^* and u_2^* are rewritten as $\mathcal{G}_1 f$ and $\mathcal{G}_2 f$, respectively.

Comparing systems 3.12 with system 3.11, we have that $w_i = u_i$ for $i = 1, 2$ if and only if the following two conditions are satisfied:

$$\begin{cases} \varepsilon\dfrac{\partial w_2}{\partial n_2} - \dfrac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)w_2 = \varepsilon\dfrac{\partial w_1}{\partial n_2} - \dfrac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)w_1 \quad \text{on } \Gamma_a, \\ w_1 = w_2 \quad \text{on } \Gamma_b. \end{cases} \tag{3.16}$$

Using the previous definitions, conditions 3.16 can be rewritten as

$$\begin{cases} \varepsilon\dfrac{\partial \mathcal{L}_2\lambda_a}{\partial n_2} - \dfrac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)\mathcal{L}_2\lambda_a = \varepsilon\dfrac{\partial \mathcal{L}_1\lambda_b}{\partial n_2} - \dfrac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)\mathcal{L}_1\lambda_b \\ \qquad +\varepsilon\dfrac{\partial \mathcal{G}_1 f}{\partial n_2} - \dfrac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)\mathcal{G}_1 f - \varepsilon\dfrac{\partial \mathcal{G}_2 f}{\partial n_2} + \dfrac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)\mathcal{G}_2 f \quad \text{on } \Gamma_a, \\ \lambda_b = T_b\mathcal{L}_2\lambda_a + T_b\mathcal{G}_2 f \quad \text{on } \Gamma_b. \end{cases} \tag{3.17}$$

Let us clean up this system by introducing some definitions. In the first equation, we recognize the Steklov-Poincaré operator S_2 associated to subdomain Ω_2, defined as

$$S_2 : \; H^{1/2}(\Gamma_a) \longrightarrow H^{-1/2}(\Gamma_a), \tag{3.18}$$

$$S_2\lambda_a := \varepsilon\frac{\partial \mathcal{L}_2\lambda_a}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)\mathcal{L}_2\lambda_a \quad (\text{evaluated on } \Gamma_a). \tag{3.19}$$

Note that $\mathcal{L}_2\lambda_a = \lambda_a$ on Γ_a. We define $\tilde{S}_b$, a Steklov-Poincaré-like operator acting on Γ_b, as

$$\tilde{S}_b : \; H^{1/2}(\Gamma_b) \longrightarrow H^{-1/2}(\Gamma_a), \tag{3.20}$$

$$\tilde{S}_b\lambda_b := -\varepsilon\frac{\partial \mathcal{L}_1\lambda_b}{\partial n_2} + \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)\mathcal{L}_1\lambda_b \quad (\text{evaluated on } \Gamma_a). \tag{3.21}$$

We also define $\tilde{T}_b$, the trace on Γ_b of the L-extension of λ_a into Ω_2:

$$\tilde{T}_b : \; H^{1/2}(\Gamma_a) \longrightarrow H^{1/2}(\Gamma_b), \tag{3.22}$$

$$\tilde{T}_b\lambda_a := T_b\mathcal{L}_2\lambda_a. \tag{3.23}$$

Finally, χ and χ' are defined as follows

$$\chi = \varepsilon\frac{\partial \mathcal{G}_1 f}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)\mathcal{G}_1 f - \varepsilon\frac{\partial \mathcal{G}_2 f}{\partial n_2} + \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)\mathcal{G}_2 f, \tag{3.24}$$

$$\chi' = T_b\mathcal{G}_2 f, \tag{3.25}$$

where we have $\chi \in H^{-1/2}(\Gamma_a)$ and $\chi' \in H^{1/2}(\Gamma_b)$. Owing to the previous definitions, the system of two equations for the interface unknowns reads

$$\left\{\begin{array}{rcll} S_2\lambda_a & = & -\tilde{S}_b\lambda_b + \chi & \text{in } H^{-1/2}(\Gamma_a), \\ \lambda_b & = & \tilde{T}_b\lambda_a + \chi' & \text{in } H^{1/2}(\Gamma_b). \end{array}\right. \tag{3.26}$$

Let us introduce now the operator

$$\tilde{S}_1 : \; H^{1/2}(\Gamma_a) \longrightarrow H^{-1/2}(\Gamma_a), \tag{3.27}$$

$$\tilde{S}_1\lambda_a := \tilde{S}_b\tilde{T}_b\lambda_a, \tag{3.28}$$

and define S as

$$S = \tilde{S}_1 + S_2. \tag{3.29}$$

After substituting λ_b given by Eq. 3.26_2 into Eq. 3.26_1, we finally obtain the following system of equations for the interface unknowns

$$\left\{\begin{array}{rcll} S\lambda_a & = & \chi - \tilde{S}_b\chi' & \text{in } H^{-1/2}(\Gamma_a), \\ \lambda_b & = & \tilde{T}_b\lambda_a + \chi' & \text{in } H^{1/2}(\Gamma_b). \end{array}\right. \tag{3.30}$$

Once λ_a and λ_b are obtained, we can solve the two Dirichlet problems 3.14 to obtain the L-homogeneous parts u_1^0 and u_2^0. The Dirichlet-homogeneous parts u_1^* and u_2^* are obtained by solving Eqs. 3.15 for $i = 1, 2$. Hence, the solutions u_1 and u_2 are calculated by adding up their respective L and Dirichlet-homogeneous contributions.

Let us go back to system 3.30. We can show that S_2 is both continuous (with constant M_{S_2}) and coercive (with constant N_{S_2}) and $\tilde{S}_1$ is continuous (with constant M_{S_1}) and non-negative. As a result we have the following theorem:

Theorem 3.3 *The operator S defined in 3.29 is invertible and system 3.30 has a unique solution $\{\lambda_a, \lambda_b\}$.*

The solutions of our interface problem can be written as

$$\left\{\begin{array}{rcll} \lambda_a & = & S^{-1}(\chi - \tilde{S}_b\chi') & \text{in } H^{1/2}(\Gamma_a), \\ \lambda_b & = & \tilde{T}_bS^{-1}(\chi - \tilde{S}_b\chi') + \chi' & \text{in } H^{1/2}(\Gamma_b), \end{array}\right. \tag{3.31}$$

4. Iterative scheme.

4.1. Relaxed sequential algorithm. In this section, we derive an iterative procedure to solve the domain decomposition problem 3.7. The sequential version of the iterative overlapping D/R algorithm is defined solving first the Dirichlet problem, and then the Robin problem. Now we investigate the interface iterates produced by this relaxed iterative procedure. We enable relaxation of relaxation parameter $\theta > 0$ of one of the transmission condition at the same time. The Dirichlet-relaxed iterative scheme, denoted D_θ/R, is given for any $k \geq 0$ by

$$\left\{\begin{array}{rcl} S_2\lambda_a^{k+1} & = & -\tilde{S}_b\lambda_b^k + \chi, \\ \lambda_b^{k+1} & = & \theta(\tilde{T}_b\lambda_a^{k+1} + \chi') + (1-\theta)\lambda_b^k. \end{array}\right. \tag{4.1}$$

In terms of the interface unknowns, the Robin-relaxed iterative scheme, denoted D/R_θ, produces the following iterates for any $k \geq 0$:

$$\left\{\begin{array}{rcl} S_2\lambda_a^{k+1} & = & \theta(-\tilde{S}_b\lambda_b^k + \chi) + (1-\theta)S_2\lambda_a^k, \\ \lambda_b^{k+1} & = & \tilde{T}_b\lambda_a^{k+1} + \chi'. \end{array}\right. \tag{4.2}$$

Let us rewrite the Dirichlet and Robin-relaxed schemes as Richardson procedures. It can be shown that S_2 is invertible. We can therefore reformulate the system for the interface unknowns 3.26 as follows:

$$\left\{ \begin{array}{l} Q_a\lambda_a = \chi_a, \\ Q_b\lambda_b = \chi_b, \end{array} \right. \tag{4.3}$$

where we have defined Q_a, Q_b, χ_a and χ_b by

$$Q_a = I_a + S_2^{-1}\tilde{S}_b\tilde{T}_b, \quad Q_b = I_b + \tilde{T}_b S_2^{-1}\tilde{S}_b, \tag{4.4}$$

$$\chi_a = S_2^{-1}\chi - S_2^{-1}\tilde{S}_b\chi', \quad \chi_b = \tilde{T}_b S_2^{-1}\chi + \chi'. \tag{4.5}$$

and where I_a is the identity on $H^{1/2}(\Gamma_a)$ and I_b is the identity on $H^{1/2}(\Gamma_b)$. By solving the Dirichlet-relaxed and Robin-relaxed systems for λ_a^{k+1} and λ_b^{k+1}, we can show that both schemes lead to the same following iterates for any $k \geq 1$:

$$\left\{ \begin{array}{lll} \lambda_a^{k+1} & = & \theta(\chi_a - Q_a\lambda_a^k) + \lambda_a^k, \\ \lambda_b^{k+1} & = & \theta(\chi_b - Q_b\lambda_b^k) + \lambda_b^k. \end{array} \right. \tag{4.6}$$

We recognize here two *stationary Richardson procedures* for solving Eqs. 4.3_1 and 4.3_2. We note that the Richardson procedure for solving λ_a is similar to that produced by the classical Dirichlet/Neumann method.

4.2. Convergence. This section studies the convergence of the DD algorithm, given by Eqs. 4.1_{1-2} for the D_θ/R method and Eqs. 4.2_{1-2} for the D/R_θ method. The result we can prove is:

Theorem 4.1 *Assume that ε is large enough so that*

$$\kappa^* := 2N_{S_2} - 2\|\boldsymbol{a}\|_{\infty,\Gamma_a} C_2^2 \, \frac{M_{\tilde{S}_1} + M_{S_2}}{N_{S_2}} > 0, \tag{4.7}$$

where N_{S_2}, $M_{\tilde{S}_1}$ and M_{S_2} are the coercivity constant of S_2, and the continuity constants of $\tilde{S}_1$ and S_2, respectively. Then, there exists $\theta_{\max}$ such that for any given $\lambda_a^0 \in \Lambda_a$ and $\lambda_b^0 \in \Lambda_b$ and for all $\theta \in (0, \theta_{\max})$, the sequences $\{\lambda_a^k\}$ and $\{\lambda_b^k\}$ given by 4.6 converge in Λ_a and Λ_b, respectively. The upper bound of the relaxation parameter $\theta_{\max}$ can be estimated by

$$\theta_{\max} = \frac{\kappa^* N_{S_2}^2}{M_{S_2}(M_{\tilde{S}_1} + M_{S_2})^2} \tag{4.8}$$

More precisely, convergence is linear.

Remark 4.1 *This result carries over to the discrete variational problems provided the stability and continuity properties of the continuous case are inherited. In particular, the rate of convergence will be independent of the number of degrees of freedom.*

REFERENCES

[1] Y. Achdou, P. L. Tallec, F. Nataf, and M. Vidrascu. A domain decoposition preconditioner for an advection-diffusion problem. *Comp. Meth. Appl. Mech. Engrg*, 184:145–170, 2000.

[2] A. Alonso, R. L. Trotta, and A. Valli. Coercive domain decomposition algorithms for advection-diffusion equations and systems. *J. Comput. Appl. Math.*, 96:51–76, 1998.

[3] L. C. Berselli and F. Saleri. New substructuring domain decomposition methods for advection-diffusion equations. *J. Comput. Appl. Math.*, 116:201–220, 2000.

[4] P. E. Bjørstad and O. B. Widlund. Iterative methods for the solution of elliptic problems on regions partitioned into substructures. *SIAM J. Numer. Anal.*, 23(6):1093–1120, 1986.

[5] P. Le Tallec. Domain decomposition methods in computational mechanics. In J. T. Oden, editor, *Computational Mechanics Advances*, volume 1 (2), pages 121–220. North-Holland, 1994.

[6] P.-L. Lions. On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. In T. F. Chan, R. Glowinski, J. Périaux, and O. Widlund, editors, *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations , held in Houston, Texas, March 20-22, 1989*, Philadelphia, PA, 1990. SIAM.

[7] G. Lube, L. Müller, and F.-C. Otto. A non-overlapping domain decomposition method for the advection-diffusion problem. *Computing*, 64:49–68, 2000.

[8] L. D. Marini and A. Quarteroni. A relaxation procedure for domain decomposition methods using finite elements. *Numer. Math*, (5):575–598, 1989.

[9] F. Nataf and F. Rogier. Factorization of the convection-diffusion operator and the Schwarz algorithm. M^3AS, 5(1):67–93, 1995.

[10] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations.* Oxford Science Publications, 1999.