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48. V–cycle Multigrid Convergence for Cell Centered Finite Difference Method, 3-D case.

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1. Introduction. In this paper, we study a multigrid algorithm for cell centered finite difference method for elliptic problems in three dimensions.

Cell centered finite difference methods are very popular among engineering circle working on various fluid computations such as oil reservoir simulation, underground water flow, or steady Euler equations, etc. It seems mainly due to the conservation property and simplicity of the scheme. On the other hand, as a solution process of the corresponding linear system, multigrid methods have been known fast for many class of problems[1],[7], [9],[4], [5],[2]. The performance of multigrid algorithms for two dimensional cell-centered finite difference method have been investigated in [10],[6] and W–cycle convergence has been analyzed in [3]. Recently V–cycle convergence has been shown with certain weighted prolongation operator[8]. This paper is a continuation of [8] dealing with three dimensional aspect of multigrid algorithm for cell-centered finite difference methods.

One of the main ingredient of multigrid algorithms in the nonstandard discretization is the design of prolongation operators between two consecutive levels, since for the cell centered finite difference case, the natural injection increases the energy norm even in two dimensional problems as shown in [3, 8]. Hence we consider a certain weighted prolongation and show its energy norm is bounded by one. Another natural operator is trilinear based operator. In this case, we also show the energy norm is less than equal to 1. Finally, we consider prolongation with different weight. This is motivated by the geometric configuration: when a box element is subdivided by 8 subboxes, one of the subbox shares three faces with its mother box, while it shares just one face with three neighboring box, thus the weights $\{3, 1, 1, 1\}$. In this last case, one can only show the energy norm is bounded by $\sqrt{10/9}$, but the multigrid performance is better than any other operator(either as an iterative solver or as a preconditioner). The rest of the paper is organized as follows. In section 2, we derive cell-centered FDM for a model 3-dimensional problem through the use of Raviart-Thomas-Nedelec element for the mixed formulation. In section 3, we describe the multigrid algorithm and some convergence theory. In section 4, we consider various prolongation operators together with their energy norm estimates. Finally in section 5, we present numerical experiments.

2. Derivation of Cell Centered FDM from RTN. Consider a model problem

$$\begin{aligned} -\nabla \cdot \mathcal{K}\nabla p &= f \text{ in } \Omega \\ p &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.1}$$

where Ω is a unit cube, $\mathcal{K}$ is a diagonal tensor whose entries are piecewise smooth. Let $h := h_k = 2^{-k}$ for some positive integer k. Assuming the domain has been subdivided by axis parallel planes into small cubes of equal size h with index (i, j, l), we consider the Raviart-Thomas-Nedelec (RTN) mixed finite element space. Let

$$\vec{V}_h = \{\mathbf{u}_h = (a_1 + a_2 x, b_1 + b_2 y, c_1 + c_2 z) \text{ on each element }\} \cap H(\operatorname{div} \Omega) \tag{2.2}$$

$$L_h = \{p_h : \text{ piecewise constant on each element}\}. \tag{2.3}$$

The RTN mixed method is to find $(\mathbf{u}_h, p_h) \in \vec{V}_h \times M_h$ such that

$$(\mathcal{K}^{-1}\mathbf{u}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_h) = 0, \quad \mathbf{v} \in \vec{V}_h \tag{2.4}$$

$$(\operatorname{div} \mathbf{u}_h, q) = (f, q), \quad q \in L_h. \tag{2.5}$$

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Let $\mathbf{v}$ be a test function in $\vec{V}_h$ whose only nonzero component, the x component, is one at a vertex $i+1/2$ and zero at all others. One uses the trapezoidal rule to evaluate first integral. Then we get

$$u^1_{i+1/2,j,l} Fac = h^2(p_{i,j,l} - p_{i+1,j,l}), \tag{2.6}$$

where

$$Fac = \frac{h}{2}\left[\int_{j-\frac{1}{2}}^{j+\frac{1}{2}}\int_{l-\frac{1}{2}}^{l+\frac{1}{2}} \mathcal{K}_L^{-1}(x_{i+\frac{1}{2}}, y, z)dydz + \int_{j-\frac{1}{2}}^{j+\frac{1}{2}}\int_{l-\frac{1}{2}}^{l+\frac{1}{2}} \mathcal{K}_R^{-1}(x_{i+\frac{1}{2}}, y, z)dydz\right].$$

Similarly we integrate along y-directional and z-directional volumes using y, z directional test functions to get difference equations along the y and z axes. The second equation of mixed formulations reads:

$$h^2(u^1_{i+\frac{1}{2},j,l} - u^1_{i-\frac{1}{2},j,l} + u^2_{i,j+\frac{1}{2},l} - u^2_{i,j-\frac{1}{2},l} + u^3_{i,j,l+\frac{1}{2}} - u^3_{i,j,l-\frac{1}{2}}) = h^3 f_{i,j,l}, \tag{2.7}$$

where we assumed f is piecewise constant for simplicity. By substituting the expressions for $u^1_{i+1/2,j,l}$ etc, if we denote the integral of $\mathcal{K}_L^{-1}$ simply as $\mathcal{K}_L^{-1}$, we have

$$2\left[\frac{-p_{i-1,j,l} - p_{i+1,j,l} - p_{i,j-1,l} + 6p_{i,j,l} - p_{i,j+1,l} - p_{i,j,l-1} - p_{i,j,l+1}}{\mathcal{K}_L^{-1} + \mathcal{K}_R^{-1}}\right] = f_{i,j,l}h^2. \tag{2.8}$$

When $\mathcal{K} = 1$ the stencil for interior is(without h-factor) $-1, -1, -1, 6, -1, -1, -1$ while on the boundary face $-1, -1, -1,\ 7, -1, -1, 0$ and on the boundary edge $0, -1, -1,\ 8, -1, -1, 0$ and on the corner $-1, -1, -1,\ 9, 0, 0, 0$. This is the cell-centered finite difference method. If we denote by M_k the space of functions which are piecewise constant on each cell, the problem can be viewed as seeking a solution $x \in M_k$ satisfying an algebraic equation of the form

$$A_k x = b, \tag{2.9}$$

where x is identified as the vector representation of p_h.

2.1. Multgrid Method. Now we describe a V–cycle multigrid algorithm (with one smoothing R_k, e.g, Gauss-Seidel) for solving (2.9) for $k = J$. First consider the sequence of spaces

$$M_1, \cdots, M_J.$$

One can view this sequence of space nested with obvious injection. But as we shall see other types of operator to be considered in this paper work better for multigrid.

ALGORITHM. If $k = 1$, set $B_1 b = A_1^{-1} b$. Otherwise define B_k recursively as follows:

1. Pre-smooth
$$x^1_k := R_k b.$$
2. Set
$$q = B_{k-1} P^0_{k-1}(b - A_k x^1_k).$$
3. Correct
$$x^2_k := x^1_k + I_k q.$$
4. Post-smooth
$$B_k b := x^2 + R^t_k (b - A_k x^2_k).$$

For the convergence analysis, we need two conditions to verify: One is the so-called regularity and approximation property: There exist constants $\alpha \in (0,1]$ and C_α such that, for all $k = 1, \cdots, J$,

$$A_k((I - I_k P_{k-1})u, u) \le C_\alpha^2 \left(\frac{\|A_k u\|^2}{\lambda_k} \right)^\alpha A_k(u,u)^{1-\alpha}, \quad \forall u \in M_k. \tag{2.10}$$

Here, λ_k is the largest eigenvalue of A_k, and P_{k-1} is the elliptic projection defined by

$$A_{k-1}(P_{k-1}u, v) = A_k(u, I_{k-1}^k v), \quad \forall u \in V_k, v \in M_{k-1}. \tag{2.11}$$

The next is

$$A_k(I_k v, I_k v) \le C_I A_{k-1}(v,v), \quad \forall v \in V_{k-1}. \tag{2.12}$$

With these verified one can prove the following result[5].

Theorem 2.1 *We have*
1. If $C_I \le 1$, then V–cycle multigrid algorithm satisfies

$$0 \le A_k(E_k u, u) \le \delta_k A_k(u,u), \quad \forall u \in M_k, \tag{2.13}$$

where $E_k = I - B_k A_k$ and $\delta_k = \frac{Ck}{Ck+1}$.
2. If $C_I \le 1 + Ch_k$, then B_k is a good preconditioner in the sense that

$$\eta_0 A_k(u,u) \le A_k(B_k A_k u, u) \le \eta_1 A_k(u,u), \quad \forall u \in M_k, \tag{2.14}$$

where η_1 is independent of k and $\eta_0 \le 1 - \delta_k$.

3. Energy norm estimate of various prolongations. For all the prolongation operators to be considered below, this regularity and approximation property holds(see [8] for details). Hence we concentrate (2.12) only. To make things easier we summarize 2-D result briefly first and extend it to 3-D. Referring to figure 2.1, we shall use the notation (i,j) to denote a coarse grid cell center, while we use (I_1, J_1) etc, to denote the fine grid center obtained by halving the coarse cell. We define the prolongation operator $I_k : M_{k-1} \to M_k$ as follows: With any positive number w let

$$(I_k v)_{I-1,J-1} = \frac{1}{w}((w-2)v_{i,j} + v_{i-1,j} + v_{i,j-1}) \tag{3.1}$$

$$(I_k v)_{I-1,J} = \frac{1}{w}((w-2)v_{i,j} + v_{i,j+1} + v_{i-1,j}) \tag{3.2}$$

$$(I_k v)_{I,J-1} = \frac{1}{w}((w-2)v_{i,j} + v_{i+1,j} + v_{i,j-1}) \tag{3.3}$$

$$(I_k v)_{I,J} = \frac{1}{w}((w-2)v_{i,j} + v_{i,j+1} + v_{i+1,j}) \tag{3.4}$$

One can show that in this case (2.12) holds with $C_I = (2(w-2)^2+8)/w^2$ whose minimum is obtained when $w = 4$. Thus we have weight $\{1/2, 1/4, 1/4\}$. Hence the analysis in [8] can be carried out to show that symmetric V–cycle with one smoothing yields a convergence factor $\delta < 1$. For 3D, the situation is different. The weight has to be changed to get suitable operator. We use similar notations as in 2-D. Fix a box element (i,j,l) in $k-1$ level and divided it by 8 axi-parallel subboxes, denoted by $(I,J,L), (I_1,J,L)$, $(I,J_1,L), (I_1,J_1,L)$ and $(I,J,L_1), (I_1,J,L_1)$, $(I,J_1,L_1), (I_1,J_1,L_1)$, etc. It is natural to define $I_k v$ on each subbox as a linear combination of values of v on (i,j,l) and its adjacent boxes. Referring to figure

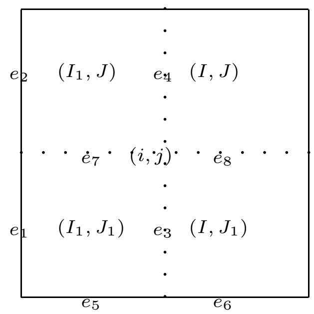

Figure 3.1: Numbering of (i, j) element and its subdivision

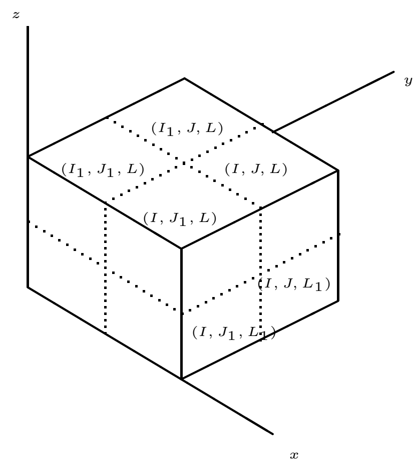

Figure 3.2: A box element and its subdivision

2.1 and 2.2, let $u^U_{I,J,L}, u^U_{I_1,J,L}$, etc., denote $I_k v$ on the upper part of the box (i, j, l), and let $u^L_{I,J,L}, u^L_{I_1,J,L}$ etc., denote its lower part. We define

$$u^U_{I_1,J_1,L} = \frac{1}{w}((w-3)v_{i,j,l} + v_{i-1,j,l} + v_{i,j-1,l} + v_{i,j,l+1}) \tag{3.5}$$

$$u^U_{I_1,J,L} = \frac{1}{w}((w-3)v_{i,j,l} + v_{i,j+1,l} + v_{i-1,j,l} + v_{i,j,l+1}) \tag{3.6}$$

$$u^U_{I,J_1,L} = \frac{1}{w}((w-3)v_{i,j,l} + v_{i+1,j,l} + v_{i,j-1,l} + v_{i,j,l+1}) \tag{3.7}$$

$$u^U_{I,J,L} = \frac{1}{w}((w-3)v_{i,j,l} + v_{i,j+1,l} + v_{i+1,j,l} + v_{i,j,l+1}) \tag{3.8}$$

and u^L are defined similarly with $l+1$ replaced by $l-1$.

This choice of weight reflects that the prolongation operator must have a certain approximation property, i.e, $\|I_k v - v\| \le Ch\|v\|_{1,h}$ for all piecewise constant functions. Here $\|\cdot\|_{1,h}$ denotes the discrete energy norm $A_k(v,v)^{1/2}$. By considering the differences between two cell centers, it is easy to see that for $v \in M_{k-1}$,

$$
\begin{aligned}
(A_{k-1}v, v)_{k-1} &= -h_{k-1}\sum_{i,j,l} v_{i,j,l}\big[(v_{i,j,l+1} - v_{i,j,l}) + (v_{i,j,l-1} - v_{i,j,l}) \\
&\quad +(v_{i,j+1,l} - v_{i,j,l}) + (v_{i,j-1,l} - v_{i,j,l}) \\
&\quad +(v_{i+1,j,l} - v_{i,j,l}) + (v_{i-1,j,l} - v_{i,j,l})\big] \\
&= h_{k-1}\sum_{i,j,l}(v_{i,j,l} - v_{i-1,j,l})^2 + (v_{i,j,l} - v_{i,j-1,l})^2 + (v_{i,j,l} - v_{i,j,l-1})^2.
\end{aligned}
$$

Let $u = I_k v$. Then

$$
(A_k u, u)_k = h_k \sum_{i,j,l}(D_i^2 + D_j^2 + D_l^2), \tag{3.9}
$$

where D_i, D_j and D_l are the differences along the x, y, z directions respectively, i.e,

$$
D_i = (u_{i,j,l} - u_{i-1,j,l}), \quad D_j = (u_{i,j,l} - u_{i,j-1,l}), \quad D_l = (u_{i,j,l} - u_{i,j,l-1}).
$$

First fix L and consider square differences along the x direction of the upper part of subdivisions. Across e_1, the square is $(u_{I_1,J_1,L} - u_{I_2,J_1,L})^2$. Similarly, across e_2, the square difference is $(u_{I_1,J_2,L} - u_{I_2,J_2,L})^2$. If we let E_i denote the contribution from edge e_i, then we see that, ignoring the $\frac{1}{w^2}$ factor,

$$
\begin{aligned}
E_1 &= \big[(w-3)v_{i,j,l} + v_{i-1,j,l} + v_{i,j-1,l} + v_{i,j,l+1} \\
&\quad -((w-3)v_{i-1,j,l} + v_{i,j,l} + v_{i-1,j-1,l} + v_{i-1,j,l+1})\big]^2 \\
&= \big[(w-4)(v_{i,j,l} - v_{i-1,j,l}) + (v_{i,j-1,l} - v_{i-1,j-1,l}) + (v_{i,j,l+1} - v_{i-1,j,l+1})\big]^2 \\
&\leq (w-2)[(w-4)(v_{i,j,l} - v_{i-1,j,l})^2 + (v_{i,j-1,l} - v_{i-1,j-1,l})^2 \\
&\quad +(v_{i,j,l+1} - v_{i-1,j,l+1})^2]
\end{aligned}
$$

where general Cauchy-Schwarz inequality

$$
(\sum w_i\alpha_i)^2 \leq (\sum w_i)(\sum w_i\alpha_i^2)
$$

has been used. Similarly, the contributions from edges $e_2, \cdots, e_8$ are estimated.

$$
\begin{aligned}
E_2 &\leq (w-2)[(w-4)(v_{i,j,l} - v_{i-1,j,l})^2 + (v_{i,j+1,l} - v_{i-1,j+1,l})^2 \\
&\quad +(v_{i,j,l+1} - v_{i-1,j,l+1})^2] \\
E_3 &= \big[(w-3)v_{i,j,l} + v_{i+1,j,l} + v_{i,j-1,l} + v_{i,j,l+1} \\
&\quad -((w-3)v_{i,j,l} + v_{i-1,j,l} + v_{i,j-1,l} + v_{i,j,l+1})\big]^2 \\
&\leq 2(v_{i+1,j,l} - v_{i,j,l})^2 + 2(v_{i,j,l} - v_{i-1,j,l})^2 \\
E_4 &\leq 2(v_{i+1,j,l} - v_{i,j,l})^2 + 2(v_{i,j,l} - v_{i-1,j,l})^2.
\end{aligned}
$$

The contribution E_5, E_6 are obtained from E_1, E_2 by interchanging the role of i, j. Thus

$$
\begin{aligned}
E_5 &\leq (w-2)[(w-4)(v_{i,j,l} - v_{i,j-1,l})^2 + (v_{i-1,j,l} - v_{i-1,j-1,l})^2 \\
&\quad +(v_{i,j,l+1} - v_{i,j-1,l+1})^2]
\end{aligned}
$$

$$
\begin{aligned}
E_6 &\leq (w-2)[(w-4)(v_{i,j,l} - v_{i,j-1,l})^2 + (v_{i+1,j,l} - v_{i+1,j-1,l})^2 \\
&\quad +(v_{i,j,l+1} - v_{i,j-1,l+1})^2]
\end{aligned}
$$

Also, E_7, E_8 are obtained from E_3, E_4 by interchanging the role of i, j. Thus

$$E_7 \leq 2(v_{i,j+1,l} - v_{i,j,l})^2 + 2(v_{i,j,l} - v_{i,j-1,l})^2 \tag{3.10}$$
$$E_8 \leq 2(v_{i,j+1,l} - v_{i,j,l})^2 + 2(v_{i,j,l} - v_{i,j-1,l})^2. \tag{3.11}$$

Now let us count the terms of the form $(v_{i,j,l} - v_{i-1,j,l})^2$. From E_1, we see that the coefficient is $(w-2)(w-4)$, while E_1 contributes $w-2$ to the neighboring boxes ($l+1$ and $j-1$) respectively. Thus the same amount come from those boxes. All together, the contribution to $(v_{i,j,l} - v_{i-1,j,l})^2$ is $(w-2)(w-4)+2(w-2) = (w-2)^2$. By the same reasoning the contributions from e_2, e_3 and e_4 are $(w-2)^2$, 4, and 4. The lower part of the subdivision has the same form except $l+1$ is replaced by $l-1$. Thus the sum of the coefficient for $(v_{i,j,l} - v_{i-1,j,l})^2$ is $2\frac{2(w-2)^2+8}{w^2}$. The same reasoning shows that the coefficients for $(v_{i,j+1,l} - v_{i,j,l})^2$ and $(v_{i,j,l+1} - v_{i,j,l})^2$ are shown to be the same. It is an elementary calculus to see $2\frac{2(w-2)^2+8}{w^2}$ has minimum 2 when $w = 4$. Considering h_k factor in A_k form, we have proved (2.12) with $C_I = 1$. Thus we obtain $\{1, 1, 1, 1\}$ as a good choice for weight.

3.1. Trilinear case. The prolongation is defined as (with $w = 64$)

$$u_{I_1,J_1,L} = \frac{3}{w}(9v_{i,j,l} + 3v_{i-1,j,l} + 3v_{i,j-1,l} + v_{i-1,j-1,l}) \tag{3.12}$$
$$+ \frac{1}{w}(9v_{i,j,l+1} + 3v_{i-1,j,l+1} + 3v_{i,j-1,l+1} + v_{i-1,j-1,l+1}) \tag{3.13}$$

where other terms are similarly defined. By the same argument as above, we can show (2.12) holds with $C_I = 1$ for trilinear prolongation also. Hence the V-cycle convergence theory follows.

3.2. Different weight. Finally consider weight $\{3, 1, 1, 1\}$. We can follow the same line of argument but we could only show $C_I \leq 10/9$. However, the numerical result shows this one performs best. This phenomenon is subject to further investigation.

4. Numerical experiment. We set $\mathcal{K} = 1$ and compare all three prolongation with natural injection whose weight can be viewed as $\{1, 0, 0, 0\}$. All three weighted operators perform well and the reduction factor seems to be independent of the number of levels. We note that the weight $\{3, 1, 1, 1\}$ works best. As a reference, we give numerical estimate on the size of prolongation operators in Table 5.

h_J	λ_{min}	λ_{max}	K	δ
1/8	0.734	1.283	1.749	0.279
1/16	0.702	1.475	2.102	0.467
1/32	0.684	1.678	2.452	0.666
1/64	0.673	1.880	2.794	0.863

Table 1. Natural injection $\{1, 0, 0, 0\}$

h_J	λ_{min}	λ_{max}	K	δ
1/8	0.615	0.999	1.626	0.378
1/16	0.581	0.999	1.721	0.410
1/32	0.556	0.999	1.797	0.432
1/64	0.536	0.999	1.864	0.450

Table 2. Weight $\{1, 1, 1, 1\}$

h_J	λ_{min}	λ_{max}	K	δ
1/8	0.684	0.999	1.460	0.308
1/16	0.661	0.999	1.514	0.326
1/32	0.644	0.999	1.553	0.333
1/64	0.634	0.999	1.578	0.336

Table 3. $\{3, 1, 1, 1\}$

h_J	λ_{min}	λ_{max}	K	δ
1/8	0.641	0.999	1.560	0.353
1/16	0.616	0.999	1.624	0.374
1/32	0.599	0.999	1.669	0.383
1/64	0.589	0.999	1.698	0.389

Table 4. Trilinear

$\{1, 0, 0, 0\}$	$\{1, 1, 1, 1\}$	$\{3, 1, 1, 1\}$	Trilinear
2	0.59	0.67	0.49
2	0.65	0.78	0.60
2	0.69	0.84	0.66
2	0.71	0.86	0.69

Table 5. Estimate of energy of I_k

Concluding remarks: We proved V-cycle multigrid convergence for the cell-centered FDM for 3-dimensional problem for two kinds of weighted prolongation operators. A third weight, $\{3, 1, 1, 1\}$, works slightly better even though the energy norm seems larger than the other two. Thus, we guess that an operator with smaller energy norm (although they guarantee convergence) does not always work better.

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