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31. Successive Subspace Correction method for Singular System of Equations

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1. Introduction. The method of successive subspace corrections, an abstraction of general iterative methods such as multigrid and Multiplicative Schwarz methods, is an algorithm for finding the solution of a linear system of equations. In this paper, we shall study in particular, Multiplicative Schwarz methods in a Hilbert space framework and present a sharp result on the convergence of the methods for singular system of equations.

For the symmetric positive definite (SPD) problems, a variety of literatures on the convergence analysis are available. Among others, we would like to refer to the upcoming paper by Xu and Zikatanov (Refer to [3]). In [3], the convergence rate of the method of subspace corrections has been beautifully established by introducing a new identity for the product of nonexpansive operators.

The main result in this paper is in that we obtained an appropriate identity for the non-SPD problems, which is suitably applied to devise or improve algorithms for singular and especially nearly singular system of equations. The related results and the corresponding estimate of the convergence rate of multigrid methods for singular system of equations shall be reported in the forthcoming paper.

The rest of this paper is organized as follows. In section 2, we set up a problem and review a successive subspace correction method in a Hilbert space setting. In section 3, we estabilsh the convergence factor of the algorithm and present an identity for the convergence rate of the method of successive subspace correction for singular system of equations. In section 4, we adapt our identity for Multiplicative Schwarz method and present various identities for the special algorithm such as Gauss-Seidel and Block Gauss-Seidel method. In the final section 5, we give some concluding remarks and future works.

2. MSC: The Method of Subspace Corrections. Let V be a Hilbert space with an inner product $(\cdot,\cdot)_V = (\cdot,\cdot)$ and an induced norm $\|\cdot\|_V = \|\cdot\|$. Let V^* denote the dual space of V. We consider the following variational problem: Find $u \in V$ for any given $f \in V^*$ such that

$$a(u,v) = \langle f, v\rangle \quad \forall v \in V \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ is a dual paring and $a(\cdot,\cdot)$ is a *symmetric* and *nonnegative* definite bilinear form satisfying $a(u,v) \leq \|a\|\|u\|\|v\|$ where $\|a\| > 0$ is a constant. We shall define $\mathcal{N}$ and $\mathcal{N}^\circ$ by $\mathcal{N} = \{v \in V : a(v,w) = 0 \quad \forall w \in V\}$ and $\mathcal{N}^\circ = \{f \in V^* : \langle f, v\rangle = 0 \quad \forall v \in \mathcal{N}\}$ respectively. The latter is often called the polar set of $\mathcal{N}$. By usual convention, for any set $W \subset V$, $W^\perp$ shall denote the orthogonal complement of W with respect to the inner product, $(\cdot,\cdot)_V$. Throughout this paper, we shall assume that $f \in \mathcal{N}^\circ$ and the continuous bilinear form $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ satisfies the following coercivity conditions on $\mathcal{N}^\perp$, namely: There exists a constant $\alpha > 0$ such that $a(v,v) \geq \alpha\|v\|^2$. This assumption implies that the problem (2.1) is well-posed on $\mathcal{N}^\perp$. We would like to remark that the problem (2.1) is not well-posed on V in a sense that it has infinitely many solutions, namely if u is a solution to (2.1), then $u + c$ will be again a solution to the problem for any $c \in \mathcal{N}$.

Now we shall discuss the method of successive subspace correction for solving 2.1. The idea of the method of successive subspace correction is to solve the residual equation on some

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properly chosen subspaces. A decomposition of V consists of a number of closed subspaces $V_i \subset V, (1 \le i \le J)$ satisfying $V = \sum_{i=1}^{J} V_i$.

Associated with each subspaces V_i, we introduce a continuous bilinear form $a_i(\cdot,\cdot)$ which can be viewed as an approximation of $a(\cdot,\cdot)$ restricted on V_i. We shall assume that the following inf-sup conditions are satisfied for all $i = 1, 2, ..., J$,

$$\inf_{v_i \in V_i} \sup_{w \in V_i} \frac{a_i(v_i, w_i)}{\|v_i\|\|w_i\|} = \inf_{w_i \in V_i} \sup_{v \in V_i} \frac{a_i(v_i, w_i)}{\|v_i\|\|w_i\|} = \alpha_i > 0 \tag{2.2}$$

and for all $i = 1, 2, ..., J$, there exists $\beta_i > 0$ such that

$$a(v_i, v_i) \ge \beta_i \|v_i\|^2 \quad \forall v_i \in V_i. \tag{2.3}$$

These inf-sup conditions are often known as Babuska-Brezzi conditions or B-B conditions. (See e.g. [4]) This is equivalent to say that the approximate subspace problems and subspace problems are uniquely solvable. While we can not in general impose the inf-sup condition for $a(\cdot,\cdot)$ on V_i due to the fact that V_i may contain a non trivial subspace of $\mathcal{N}$. In this paper, we shall assume that $a(\cdot,\cdot)$ satisfies the B-B conditions since we are mainly concerned with Multiplicative Schwarzs methods.

2.1. SSC: Successive Subspace Corrections.. The method of successive subspace corrections (MSSC) is an iterative algorithm that corrects residual equation successively on each subspace.

Algorithm[MSSC] Let $u^0 \in V$ be given.
for $l = 1, 2, ...$
  $u_0^{l-1} = u^{l-1}$
  for $i = 1 : J$
    Let $e_i \in V_i$ solve
      $a_i(e_i, v_i) = f(v_i) - a(u_{i-1}^{l-1}, v_i) \quad \forall v_i \in V_i$
    $u_i^{l-1} = u_{i-1}^{l-1} + e_i$
  endfor
  $u^l = u_J^{l-1}$
endfor

We note that the above algorithm is well-defined, thanks to the inf-sup conditions for (2.2). For the analysis of this algorithm, let us introduce another class of linear operators $T_i : V \mapsto V_i$ defined by $a_i(T_i v, v_i) = a(v, v_i) \quad \forall v_i \in V_i$. Again, thanks to inf-sup condition (2.2), each T_i is well-defined and $\mathcal{R}(T_i) = V_i$. In the special case when the subspace equation is solved exactly, we shall use the notation P_i for T_i, namely $T_i = P_i$ if $a_i(\cdot,\cdot) = a(\cdot,\cdot)$.

It is easy to see that for given $u \in V$ a solution to (2.1),

$$u - u_i^{l-1} = (I - T_i)(u - u_{i-1}^{l-1}).$$

By a simple recursive application of the above identity, we obtain that

$$u - u^l = E_J(u - u^{l-1}) = ... = E_J^l(u - u^0) \tag{2.4}$$

where

$$E_J = (I - T_J)(I - T_{J-1}) \cdots (I - T_1). \tag{2.5}$$

which is often called an error transfer operator. Because of this special form of E_J, the successive subspace correction method is also known as the Multiplicative or Product (Schwarz) method. The general notion of subspace corrections by means of space decomposition was described in Xu[2].

3. An identity for the convergence factor of MSSC. In view of (2.4), the convergence of the method of subspace correction is equivalent to $\lim_{l\to\infty} E_J^l = 0$. As was discussed before in this paper, for the case when $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ is a symmetric positive definite bilinear form, the uniform convergence result under some natural conditions on the subspace solvers T_i was established as an identity for the convergence factor $\|E_J\|_a = \sup_{\|v\|_a=1} \|E_J v\|_a$, namely the norm of the product of nonexpansive operators. (Refer to [3].) In our case when $a(\cdot,\cdot)$ is *nonnegative* definite, two types of convergences can be considered, namely the classical convergence (or norm convergence in the space V):

$$\|u^l - u\|_V \to 0 \text{ as } l \to \infty$$

and quotient norm or energy norm convergence (Refer to [1]):

$$\|u^l - u\|_{V/\mathcal{N}} \to 0 \text{ as } l \to \infty,$$

where $V/\mathcal{N}$ is the quotient space. We shall present that the following quantity is both the norm and the quotient norm convergence factor for the MSSC under some suitable conditions.
Definition[Convergence Factor]

$$\|E_J\|_{\mathcal{L}(\mathcal{N}^\perp, V)_a} = \sup_{v\in\mathcal{N}^\perp} \frac{|E_J v|_a}{\|v\|_a}$$

In the sequel of this paper, we shall establish an identity for the convergence factor $\|E_J\|_{\mathcal{L}(\mathcal{N}^\perp, V)_a}$ under certain assumptions.

3.1. Assumptions on subspace solvers. We shall now try to derive conditions on the subspaces and subspace solvers for the convergence of the MSSC.

First of all, we shall assume that

Assumption[$A0$] A decomposition of V consists of closed subspaces $V_i \subset V, \quad i = 1, 2, ..., J$ satisfying

$$V = \sum_{i=1}^{J} V_i.$$

This assumption is necessary for any quantitative convergence even for SPD problems. (See [3] page 15.)

Assumption[$A1$] There exists $\alpha_i > 0$ such that

$$a(v_i, v_i) \geq \alpha_i \|v_i\|^2 \quad \forall v_i \in V_i$$

$$\inf_{v_i\in V_i} \sup_{w\in V_i} \frac{a_i(v_i, w_i)}{\|v_i\|\|w_i\|} = \inf_{w_i\in V_i} \sup_{v\in V_i} \frac{a_i(v_i, w_i)}{\|v_i\|\|w_i\|} = \beta_i > 0$$

This assumption implies that the subspace problems are well-posed and that $T_i : V_i \mapsto V_i$ is isomorphic for each $i = 1, 2, 3, ..., J$.

Assumption[$A2$] For each $1 \leq i \leq J$, there exists $\omega \in (0, 2)$ such that

$$a(T_i v, T_i v) \leq \omega a(T_i v, v) \quad \forall v \in V.$$

Let us discuss the assumption ($A1$) briefly for the finite dimensional case. For the notational simplicity and invoking Riesz Representation theorem (See e.g. [5] (e.g. $a(\cdot,\cdot) \Leftrightarrow A$ and $a_i(\cdot,\cdot) \Leftrightarrow R_i$), let us put the discretization of the system of equation (2.1) as following operator equation: Find $u \in \mathbb{R}^n$ such that

$$Au = b$$

$\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range of A and kernel of A respectively. The iterative method is based on the classical matrix splitting as follows:

$$A = D - L - L^T$$

where D is the diagonal and L is the strictly lower triangular matrix. In this situation, one can easily show that the sufficient condition that $(A1)$ holds true is that A has a positive diagonal and the symmetric part of the approximate subspace operators, say R_i to A is positive definite.

Remark 3.1 *For the case multigrid method with nested subspaces, with the assumption that $a(\cdot,\cdot)$ satisfies the inf-sup condtion on $R(T_i)$, an appropriate identity can be deriven.*

3.2. On the Convergence factor of the MSSC. In this subsection, we shall see that the convergence factor of the MSSC is given by $\|E_J\|_{\mathcal{L}(\mathcal{N}^\perp,V)_a}$ as mentioned before. Let us begin with simple but important lemma.

Lemma 3.1 *Let $E_J = (I - T_J)\cdots(I - T_1)$. Then*

$$a(E_J v, E_J v) \leq \|E_J\|_{\mathcal{L}(\mathcal{N}^\perp,V)_a} a(v,v) \quad \forall v \in V$$

The following lemma and the theorem shall reveal that $\|E_J\|_{\mathcal{L}(\mathcal{N}^\perp,V)_a}$ is indeed both the norm and quotient norm convergence rate of the MSSC.

Theorem 3.1 *Assume $(A1)$ and $(A2)$. Then for any initial guess $u^0 \in V$, the followings hold true:*

$$\|u - u^k\| \leq C\|E_J\|^{k-1}_{\mathcal{L}(\mathcal{R}A,V)_a}\|u - u^{k-1}\|$$

and

$$\|u - u^k\|_{V/\mathcal{N}} \leq C\|E_J\|^{k}_{\mathcal{L}(\mathcal{R}A,V)_a}\|u - u^{k-1}\|_{V/\mathcal{N}},$$

where u is a solution to (2.1).

3.3. An identity for the convergence factor for the MSSC. We are in a position to present the identity for the convergence factor for the MSSC. The theorem presented below is based on the aforementioned assumptions $(A0)$, $(A1)$ and $(A2)$. Let us first introduce an operator $Q_A : V \mapsto \mathcal{N}^\perp$ defined by $\forall v \in V$ and $\forall w \in \mathcal{N}^\perp$, $(Q_A v, w) = (v, w)$ and define $Q_{i,A}$ by the restriction of Q_A on $\mathcal{R}(T_i) = V_i$. We also denote a space $Q_A W$ for any set $W \subset V$ by $Q_A W = \{Q_A w \in V : w \in W\}$. We shall also introduce linear operators $T_{i,A} : V \mapsto V$ defined by $T_{i,A} = Q_{i,A} T_i$.

Lemma 3.2 *Let us define $E_{J,A}$ by $E_{J,A} = (I - T_{J,A})\cdots(I - T_{1,A})$. Then,*

$$\|E_J\|_{\mathcal{L}(\mathcal{N}^\perp,V)_a} = \|E_{J,A}\|_a = \sup_{v \in \mathcal{N}^\perp} \frac{\|E_{J,A} v\|_a}{\|v\|_a}$$

Proof. The proof is completed by the simple observation that

$$|E_J v|_a^2 = \|Q_A E_J v\|_a^2 = \|E_{J,A} v\|_a^2.$$

■

We would like to remark that we use the notation $\|\cdot\|_a$ rather than $|\cdot|_a$. This is because $E_{J,A}$ is invariant operator on $\mathcal{N}^\perp$ and $a(\cdot,\cdot)$ is SPD on $\mathcal{N}^\perp$. We shall use this rule in the sequel of this paper if no confusion arises.

In view of the lemma (3.2), the convergence factor for the MSSC is transformed into the norm of a product of nonexpansive operators on $\mathcal{N}^\perp$. Now, by this observation, the acquisition of an identity for the convergence factor of the MSSC is in showing the three assumptions on $T_{i,A}$'s (Refers to [3]) under which we can apply the known theory in Xu and Zikatanov [3] and obtain the desired result.

Lemma 3.3 *Assume* $(A0), (A1)$ *and* $(A2)$. *Then the followings hold true.*

- *Each* $\mathcal{R}(T_{i,A}) = Q_A\mathcal{R}(T_i)$ *is closed and* $Q_{i,A} : \mathcal{R}(T_i) \mapsto \mathcal{R}(T_{i,A})$ *is an isomorphism.*
- *Each* $T_{i,A} : \mathcal{R}(T_{i,A}) \mapsto \mathcal{R}(T_{i,A})$ *is an isomorphism.*
- *The following holds true: for each* $1 \le i \le J$, *there exists* $\omega \in (0, 2)$ *such that*

$$a(T_{i,A}v, T_{i,A}) \le \omega a(T_{i,A}v, v) \quad \forall v \in V.$$

- $\mathcal{N}^\perp = \sum_{i=1}^J \mathcal{R}(T_{i,A})$.

Theorem 3.2 *Under the assumptions* $(A0), (A1)$ *and* $(A2)$, *we obtain the following identity:*

$$\|E_J\|_{\mathcal{L}(\mathcal{N}^\perp, V)_a} = \|E_{J,A}\|_a = \frac{c_0}{1 + c_0}$$

where

$$c_0 = \sup_{v \in \mathcal{N}^\perp} \inf_{\sum_{i=1}^J T_{i,A}v_i = v} \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* u_i, T_{i,A}^* u_i)_a}{(v, v)_a}$$

, $u_i = \sum_{j=i}^J T_{j,A}v_j - v_i$ *and* $\tilde{v}^T = (v_1, ..., v_J) \in \mathcal{R}(T_{1,A}) \times \cdots \times \mathcal{R}(T_{J,A})$.

Proof. From the lemma (3.3) and by applying the main result theorem 4.2 (page 10) in [3], the proof is completed. ■

We would like to point out that c_0 is a bit different from that given in [3]. One can obtain this by the following simple change of variable: $T_{i,A}v_i \leftrightarrow v_i$.

4. Multiplicative Schwarz Method.

We shall devote this section to write the expression c_0 in terms of the real subspace solvers T_i instead of $T_{i,A}$. We shall first discuss an adjoint operator of T_i.

4.1. On the adjoint operator T_i^*.

It is easy to see that it is not possible to define a unique adjoint of T_i with respect to $a(\cdot, \cdot)$ in a classical sense due to the fact that $a(\cdot, \cdot)$ is semi definite. While this is the fact, we shall see that we need to define the adjoint of T_i in some sense so that we can write c_0 in terms of the real subspace solvers T_i. In doing so, let us introduce another class of operators as follows: For each $1 \le i \le J$, we define $R_i : V_i \mapsto V_i$ and $Q_i : V \mapsto V_i$ by $(R_i v_i, w_i) = a_i(v_i, w_i)$ and $(Q_i v, w_i) = (v, w_i)$ $\forall v \in V, w_i \in V_i$ respectively. We would like to remark that by inf-sup condition (2.3), R_i is an isomorphism. We can then introduce the adjoint T_i^* and symmetrization $\bar{T}_i$ of T_i as follows:

$$T_i^* = R_i^T Q_i A \text{ and } \bar{T}_i = T_i + T_i^* - T_i^* T_i.$$

where R_i^T is the adjoint of R_i with respect to $(\cdot, \cdot)_V$. We here point out that T_i^* satisfies

$$a(T_i v, w) = a(v, T_i^* w) \quad \forall v, w \in V.$$

Correspondingly, we also define $T_{i,A}^*$ and $\bar{T}_{i,A}$ by

$$T_{i,A}^* = Q_{i,A}^* T_i^* \text{ and } \bar{T}_{i,A} = T_{i,A} + T_{i,A}^* - T_{i,A}^* T_{i,A}.$$

where $Q_{i,A}^*$ is the restriction of Q_A on $\mathcal{R}(T_i^*)$. Note that $Q_{i,A} = Q_{i.A}^*$ if $\mathcal{R}(T_i) = \mathcal{R}(T_i^*)$.

Lemma 4.1 *Assume that* $(A1), (A2)$. *Then the followings hold true:*

- $\mathcal{R}(T_i) = \mathcal{R}(T_i^*) = \mathcal{R}(\bar{T}_i) = V_i$
- T_i, T_i^* *and* $\bar{T}_i$ *are all isomorphic from* V_i *to itself.*

- *$\bar{T}_i$ is nonnegative on V and symmetric positive definite on V_i.*

Here we provide the main theorem in the paper.

Theorem 4.1 *Assume that $(A0), (A1)$ and $(A2)$. Then the convergence rate of subspace correction method above is given by the following identity.*

$$\|E_J\|_{\mathcal{L}(\mathcal{N}^\perp, V)_a} = \frac{c_0}{1+c_0}$$

where

$$c_0 = \sup_{v\in\mathcal{N}^\perp} \inf_{\sum_i v_i = v \in \mathcal{N}^\perp} \inf_{\sum_i c_i = c \in \mathcal{N}} \frac{\sum_{i=1}^J (T_i \bar{T}_i^{-1} T_i^* w_i, w_i)_a}{\|v\|_a}$$

$w_i = \sum_{j=i}^J (v_j + c_j) - T_i^{-1}(v_i + c_i)$. *and* $v_i, c_i \in V_i$.

Proof. By theorem (3.2) and simple change of variable, it is easy to see that we can write an identity for the convergence rate as follows:

$$\|E_J\|_{\mathcal{L}(\mathcal{N}^\perp, V)_a} = \frac{c_0}{c_0+1}$$

where

$$c_0 = \sup_{v\in\mathcal{N}^\perp} \inf_{\sum_{i=1}^J T_i w_i = v + c} \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* u_i, T_{i,A}^* u_i)_a}{(v,v)_a}$$

with $u_i = (\sum_{j=i}^J T_j w_j - w_i)$. Let us denote $\tilde{V}$ by $V_1 \times \cdots \times V_J$ and $\tilde{v}$ by $(v_1, \cdot, \cdot, \cdot, v_J) \in \tilde{V}$. We note that since $\tilde{T} : \tilde{V} \mapsto V$ is onto, c may vary arbitrarily in $\mathcal{N}$. Let us decompose $\tilde{w} \in \tilde{V}$ as followings: $\tilde{w} = \tilde{v} + \tilde{c}$ with $\tilde{v}, \tilde{c} \in \tilde{V}$ and $\tilde{T}\tilde{v} = v$ and $\tilde{T}\tilde{c} = c$. Thanks to this decomposition, we see that

$$c_0 = \sup_{v\in\mathcal{N}^\perp} \inf_{\tilde{T}\tilde{v} + \tilde{T}\tilde{c}} = \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* u_i, T_{i,A}^* u_i)_a}{(v,v)_a}$$

with $u_i = \sum_{j=i}^J T_j^* (v_j + c_j) - (v_i + c_i)$.

Now let us set

$$c_1 = \inf_{\tilde{T}(\tilde{v}+\tilde{c})} \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* u_i, T_{i,A}^* u_i)_a}{(v,v)_a}$$

$$c_2 = \inf_{\tilde{T}\tilde{v} = v} \inf_{\tilde{T}\tilde{c} = c} \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* u_i, T_{i,A}^* u_i)_a}{(v,v)_a}$$

and we shall show that $c_1 = c_2$. It is clear that $c_1 \geq c_2$, since if

$$\tilde{w} = \arg(\inf_{\tilde{T}\tilde{w} = v + c} \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* u_i, T_{i,A}^* u_i)_a}{(v,v)_a}) \in \tilde{V}$$

with $u_i = \sum_{j=i} T_j w_j - w_i$. We can choose any decomposition of $\tilde{w} = \tilde{v} + \tilde{c}$ such that $\tilde{T}\tilde{v} = v$ and $\tilde{T}\tilde{c} = c$ with $\tilde{v}, \tilde{c} \in \tilde{V}$. Let us show the reverse inequality. Now for any given $\tilde{v} \in \tilde{V}$, let

$$\tilde{c}(\tilde{v}) = \arg\{\inf_{\tilde{T}\tilde{c} = c} \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* u_i, T_{i,A}^* u_i)_a}{(v,v)_a})\}$$

with $u_i = \sum_{i=1}^J T_j (v_j + c_j) - (v_i + c_i)$ and now again set

$$\tilde{v} = \arg(\inf_{\tilde{T}\tilde{v} = v} \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* u_i, T_{i,A}^* u_i)_a}{(v,v)_a})\}$$

with $u_i = \sum_{i=1}^J T_j(v_j + c_j(\tilde{v})) - (v_i + c_i(\tilde{v}))$. Then it is easy to see that

$$\tilde{v} + \tilde{c} = \arg\{\inf_{\tilde{T}\tilde{v}=v} (\inf_{\tilde{T}\tilde{c}=c} \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* u_i, T_{i,A}^* u_i)_a}{(v,v)_a})\}$$

with $u_i = (\sum_{j=i}^J T_j(v_j + c_j) - (v_i + c_i))$ and $\tilde{T}(\tilde{v} + \tilde{c}) = v + c$, which implies that $c_1 \leq c_2$.

Hence it has been shown that

$$c_0 \quad = \quad \sup_{v \in \mathcal{N}^\perp} \inf_{\tilde{T}\tilde{v}=v\in\mathcal{N}^\perp} \inf_{\tilde{T}\tilde{c}=c\in\mathcal{N}} \frac{\sum_{i=1}^J (\bar{T}_{i,A}^{-1} T_{i,A}^* (v_i + c_i), (v_i + c_i))_a}{(v,v)_a}$$

Finally, we insert an explicit expression for $\bar{T}_{i,A}^{-1}$ as follows and obtain:

$$a(\bar{T}_i^{-1} Q_{i,A}^{-1} Q_{i,A} T_i^* v, w) = a(\bar{T}_i^{-1} T_i^* v, w) \quad \forall v \in \mathcal{N}^\perp \text{ and } \forall w_i \in V_i$$

This completes the proof. ■

Let us consider some special cases : in the case we use exact solvers $T_i \Leftrightarrow P_i$, c_0 in the theorem (4.1) is given by

$$c_0 = \sup_{v\in\mathcal{N}^\perp} \inf_{\sum_i v_i = v \in \mathcal{N}^\perp} \inf_{\sum_i c_i = c \in \mathcal{N}} \frac{\sum_{i=1}^J |P_i(\sum_{j=i+1}^J (v_j + c_j)|_a^2}{\|v\|_a} \tag{4.1}$$

where $v_i, c_i \in V_i$. and in particular, for Gauss-Seidel method, c_0 is given by

$$c_0 = \sup_{v\in\mathcal{N}^\perp} \inf_{c\in\mathcal{N}} \frac{(S(v-c), v-c)}{(v,v)_a} \tag{4.2}$$

where $A = D - L - L^T$ and $S = L^T D^{-1} L$.

5. Conclusion and extensions. We would like to remark that we can also consider the sharp result on the convergence rate of Multigrid methods with a nested subspace decomposition by modifying the assumption $(A1)$ slightly, in which case, the subspace problems are not well-posed. The theory presented in this paper can be applied to devise algorithms for Singular system of equations and especially Nearly singular system of equations. We shall report such related and further results in the forthcoming paper.

REFERENCES

[1] J. Xu. *Theory of Multilevel Methods.* PhD thesis, Cornell University, May 1989.

[2] J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34(4):581–613, December 1992.

[3] J. Xu and L. Zikatanov. The method of alternating projections and the method of subspace corrections in hilbert space. To Appear in J. Amer. Math. Soc., 2001.

[4] J. Xu and L. Zikatanov. Some observation on babuska brezzi conditions. To Appear in Numer. Math., 2001.

[5] K. Yosida. *Functional Analysis.* Springer-Verlag, sixth edition, 1980.