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20. A Dual-Primal FETI Method for solving Stokes/Navier-Stokes Equations

Jing Li[1]

1. Introduction. The Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) methods were first proposed by Farhat et al [3] for elliptic partial differential equations. In this method, the spatial domain is decomposed into non-overlapping subdomains, and the interior subdomain variables are eliminated to form a Schur complement problem for the interface variables. Lagrange multipliers are then introduced to enforce continuity across the interface, except at the subdomain vertices where continuity is enforced directly, i.e., the neighboring subdomains share the degrees of freedom at the subdomain vertices. A symmetric positive semi-definite linear system for the Lagrange multipliers is solved by using the preconditioned conjugate gradient (PCG) method. FETI-DP methods have been shown to be numerically scalable for second order elliptic problems. Thus, Mandel and Tezaur [6] have proved that the condition number grows at most as $C(1+log(H/h))^2$ in two dimensions, where H is the subdomain diameter and h is the element size. Klawonn et al [4] proposed new preconditioners of this type and proved that the condition numbers are bounded from above by $C(1+log(H/h))^2$ in three dimensions; these bounds are also independent of possible jumps of the coefficients of the elliptic problem. In [5], we developed a dual-primal FETI method for the two-dimensional incompressible Stokes problem and proved that the condition number is bounded from above by $C(1+log(H/h))^2$. In this paper, we will extend this algorithm to solving three-dimensional incompressible Stokes problem, give the same condition number bound and an inf-sup stability result for the coarse level saddle point problem, which appeared as an assumption in [5]. We will also extend this dual-primal FETI algorithm to solving nonlinear Navier-Stokes equations by using a Picard iteration, where in each iteration step, we will solve a non-symmetric linearized incompressible Navier-Stokes equation. Illustrative numerical results are presented by solving lid driven cavity problems.

2. FETI-DP algorithm for Stokes problem. We will consider the following Stokes problem on a three-dimensional, bounded, polyhedral domain Ω,

$$\left\{\begin{array}{rcll} -\Delta\mathbf{u}+\nabla p &=& \mathbf{f}, & \text{in } \Omega \\ -\nabla\cdot\mathbf{u} &=& 0, & \text{in } \Omega \\ \mathbf{u} &=& \mathbf{g}, & \text{on } \partial\Omega\ , \end{array}\right. \tag{1}$$

where the boundary velocity $\mathbf{g}$ satisfies the compatibility condition $\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}=0$. The domain Ω is decomposed into N non-overlapping polyhedral subdomains Ω^i of characteristic size H. The interface is defined as $\Gamma=(\cup\partial\Omega^i)\backslash\partial\Omega$ and $\Gamma^{ij}=\partial\Omega^i\cap\partial\Omega^j$ is the interface between two neighboring subdomains Ω^i and Ω^j. We will consider subdomain incompressible Stokes problems,

$$\left\{\begin{array}{rcll} -\Delta\mathbf{u}^i+\nabla p^i &=& \mathbf{f}^i, & \text{in } \Omega^i \\ -\nabla\cdot\mathbf{u}^i &=& 0, & \text{in } \Omega^i \\ \mathbf{u}^i &=& \mathbf{g}^i, & \text{on } \partial\Omega\cap\partial\Omega^i \\ \frac{\partial\mathbf{u}^i}{\partial\mathbf{n}^i}-p^i\mathbf{n}^i &=& \lambda^i, & \text{on } \Gamma^{ij}\ , \end{array}\right.$$

$$\left\{\begin{array}{rcll} -\Delta\mathbf{u}^j+\nabla p^j &=& \mathbf{f}^j, & \text{in } \Omega^j \\ -\nabla\cdot\mathbf{u}^j &=& 0, & \text{in } \Omega^j \\ \mathbf{u}^j &=& \mathbf{g}^j, & \text{on } \partial\Omega\cap\partial\Omega^j \\ \frac{\partial\mathbf{u}^j}{\partial\mathbf{n}^j}-p^j\mathbf{n}^j &=& \lambda^j, & \text{on } \Gamma^{ij}\ , \end{array}\right.$$

[1]Courant Institute of Mathematical Sciences, lijing@cims.nyu.edu. This work was supported in part by the U.S. Department of Energy under contract DE-FG02-92ER25127.

where $\lambda^i + \lambda^j = 0$. We first form subdomain discrete problems by using an inf-sup stable mixed finite element method on each subdomain. We denote the discrete finite element space for the pressures inside the subdomain Ω^i by Π_I^i, and the subdomain constant pressure space by Π_0. We denote the discrete finite element space for the velocity components on Ω^i by $\mathbf{W}^h(\Omega^i)$, which is decomposed as $\mathbf{W}^h(\Omega^i) = \mathbf{W}_I^i \oplus \mathbf{W}_\Gamma^i$, with $\mathbf{W}_I^i$ the interior velocity part and $\mathbf{W}_\Gamma^i$ the subdomain boundary velocity part. Let $\Pi_I = \prod_{i=1}^N \Pi_I^i$, $\mathbf{W}_I = \prod_{i=1}^N \mathbf{W}_I^i$, and $\mathbf{W}_\Gamma = \prod_{i=1}^N \mathbf{W}_\Gamma^i$ be the corresponding product spaces. $\widetilde{\mathbf{W}}_\Gamma$ is a subspace of $\mathbf{W}_\Gamma$ and is given by

$$\widetilde{\mathbf{W}}_\Gamma = \mathbf{W}_\Pi \oplus \mathbf{W}_\Delta,$$

where the primal subspace $\mathbf{W}_\Pi$ consists of two parts. The first is the subdomain corner velocity part, which is spanned by the nodal finite element basis function $\theta_{\mathcal{V}^{il}}$ of the subdomain corners. The other part corresponds to the integrals of the velocity over each subdomain interface, and it is spanned by the pseudoinverse $\mu_{\mathcal{F}^{ij}}^\dagger$ of the counting functions $\mu_{\mathcal{F}^{ij}}$ corresponding to each face $\mathcal{F}^{ij}$ of the subdomain Ω^i: $\mu_{\mathcal{F}^{ij}}$ is 0 at the interface nodes outside $\bar{\mathcal{F}}^{ij}$ while its value at any node on $\mathcal{F}^{ij}$ equals the number of subdomains shared by that node. Its pseudoinverse $\mu_{\mathcal{F}^{ij}}^\dagger$ is the function $1/\mu_{\mathcal{F}^{ij}}(x)$ for all interface nodes where $\mu_{\mathcal{F}^{ij}}(x) \neq 0$, and it vanishes at all other points. We also note that, we make both $\mu_{\mathcal{F}^{ij}}$ and $\mu_{\mathcal{F}^{ij}}^\dagger$ vanish at the subdomain corners. $\mathbf{W}_\Delta$ is the dual part, which is the direct sum of the local subspaces $\mathbf{W}_\Delta^i$. In the 3D case,

$$\mathbf{W}_\Delta^i := \{\mathbf{w} \in \mathbf{W}_\Gamma^i : \mathbf{w}(\mathcal{V}^{il}) = 0; \bar{\mathbf{w}}_{\mathcal{F}^{ij}} = 0,\ \forall \mathcal{V}^{il}, \mathcal{F}^{ij} \subset \partial\Omega^i\},$$

with $\bar{\mathbf{w}}_{\mathcal{F}^{ij}}$ defined by

$$\bar{\mathbf{w}}_{\mathcal{F}^{ij}} = \frac{\int_{\mathcal{F}^{ij}} \mathbf{w} d\mathbf{x}}{\int_{\mathcal{F}^{ij}} d\mathbf{x}}.$$

With these notations, we can decompose the discrete velocity and pressure space of the original problem (1) as follows

$$\mathbf{W} = \mathbf{W}_I \oplus \mathbf{W}_\Pi \oplus \mathbf{W}_\Delta,$$

and

$$\Pi = \Pi_I \bigoplus \Pi_0.$$

If we further introduce a Lagrange multiplier space Λ to enforce the continuity of the velocities across the subdomain interfaces, then we have the following discrete problem: find a vector $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, p_0, \mathbf{u}_\Delta, \lambda) \in (\mathbf{W}_I, \Pi_I, \mathbf{W}_\Pi, \Pi_0, \mathbf{W}_\Delta, \Lambda)$ such that

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T & 0 \\ B_{II} & 0 & B_{\Pi I} & 0 & B_{\Delta I} & 0 \\ A_{\Pi I} & B_{\Pi I}^T & A_{\Pi\Pi} & B_{\Pi 0}^T & A_{\Delta\Pi}^T & 0 \\ 0 & 0 & B_{\Pi 0} & 0 & 0 & 0 \\ A_{\Delta I} & B_{\Delta I}^T & A_{\Delta\Pi} & 0 & A_{\Delta\Delta} & B_\Delta^T \\ 0 & 0 & 0 & 0 & B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Pi \\ 0 \\ \mathbf{f}_\Delta \\ 0 \end{pmatrix}. \tag{2}$$

It is important to note that the B_Δ matrix here is a scaled matrix with elements given by $\{0, \pm\sqrt{\mu_{\mathcal{F}^{ij}}^\dagger}\}$ placing different weights on the face and edge nodes, unlike in the two-dimensional case where B_Δ is constructed from $\{0, \pm 1\}$. It follows immediately from the definition of B_Δ that, on each subdomain interface $\mathcal{F}^{ij}$,

$$(B_\Delta^T B_\Delta \mathbf{w})^i|_{\mathcal{F}^{ij}} = \pm(\mu_{\mathcal{F}^{ij}}^\dagger(\mathbf{w}^i - \mathbf{w}^j))|_{\mathcal{F}^{ij}}\ , \forall \mathbf{w} \in \mathbf{W}_\Gamma. \tag{3}$$

Also note that we are not requiring the pressure to be continuous across the subdomain interfaces in our algorithm. In fact, we consider only mixed methods with discontinuous pressure spaces. By defining a Schur complement operator $\tilde{S}$ as

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T \\ B_{II} & 0 & B_{\Pi I} & 0 & B_{\Delta I} \\ A_{\Pi I} & B_{\Pi I}^T & A_{\Pi\Pi} & B_{\Pi 0}^T & A_{\Delta\Pi}^T \\ 0 & 0 & B_{\Pi 0} & 0 & 0 \\ A_{\Delta I} & B_{\Delta I}^T & A_{\Delta\Pi} & 0 & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{S}\mathbf{u}_\Delta \end{pmatrix}, \tag{4}$$

solving the linear system (2) is reduced to solving the following linear system

$$\begin{pmatrix} \tilde{S} & B_\Delta^T \\ B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Delta^* \\ 0 \end{pmatrix} . \tag{5}$$

By using a further Schur complement procedure, the problem is finally reduced to solving the following linear system with the Lagrange multipliers λ as the remaining variable:

$$B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta \tilde{S}^{-1} \mathbf{f}_\Delta^*, \tag{6}$$

Our preconditioner is the standard Dirichlet preconditioner, $B_\Delta S_\Delta B_\Delta^T$, with S_Δ defined as

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Delta I}^T \\ B_{II} & 0 & B_{\Delta I} \\ A_{\Delta I} & B_{\Delta I}^T & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ S_\Delta \mathbf{u}_\Delta \end{pmatrix}. \tag{7}$$

We have now formed the preconditioned linear system

$$B_\Delta S_\Delta B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta S_\Delta B_\Delta^T B_\Delta \tilde{S}^{-1} f_\Delta^* , \tag{8}$$

which is our FETI-DP algorithm to solve the incompressible Stokes problem (1). In [5], we show that both S_Δ and $\tilde{S}^{-1}$ are symmetric, positive definite on the space $\mathbf{W}_\Delta$. Therefore a preconditioned conjugate gradient method, as well as GMRES, can be used to solve equation (8). We note that we need to apply both S_Δ and $\tilde{S}^{-1}$ to a vector in each iteration step. Multiplying S_Δ by a vector requires solving subdomain incompressible Stokes problems with Dirichlet boundary conditions, and multiplying $\tilde{S}^{-1}$ by a vector requires solving a coarse level saddle point problem, as well as subdomain problems. In [5], we made an assumption about the inf-sup stability condition of the coarse level saddle point problem. In the next section we will give an inf-sup stability estimate as well as a condition number bound of the preconditioned linear system (8).

3. Inf-sup stability of the coarse saddle point problem and a condition number estimate. We know, from the definition (4), that to find a vector $\mathbf{u}_\Delta = \tilde{S}^{-1} \cdot \mathbf{w}_\Delta \in \mathbf{W}_\Delta$, for a given $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, requires solving the following linear system

$$\begin{pmatrix} A_{II} & A_{\Delta I}^T & B_{II}^T & A_{\Pi I}^T & 0 \\ A_{\Delta I} & A_{\Delta\Delta} & B_{\Delta I}^T & A_{\Pi\Delta}^T & 0 \\ B_{II} & B_{\Delta I} & 0 & B_{\Pi I} & 0 \\ A_{\Pi I} & A_{\Pi\Delta} & B_{\Pi I}^T & A_{\Pi\Pi} & B_{\Pi 0}^T \\ 0 & 0 & 0 & B_{\Pi 0} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ p_I \\ \mathbf{u}_\Pi \\ p_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{w}_\Delta \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

In our FETI-DP algorithm, we solve this linear system by a Schur complement procedure. We first solve a coarse level problem

$$\begin{pmatrix} S_\Pi & B_{\Pi 0}^T \\ B_{\Pi 0} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Pi \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Pi^* \\ 0 \end{pmatrix} , \tag{10}$$

and then the independent subdomain problems

$$\begin{pmatrix} A_{II} & A_{\Delta I}^T & B_{II}^T \\ A_{\Delta I} & A_{\Delta\Delta} & B_{\Delta I}^T \\ B_{II} & B_{\Delta I} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ p_I \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{w}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} A_{\Pi I}^T \\ A_{\Pi\Delta}^T \\ B_{\Pi I} \end{pmatrix} \mathbf{u}_\Pi. \tag{11}$$

In (10), S_Π is defined by:

$$A_{\Pi\Pi} - \begin{pmatrix} A_{\Pi I} & A_{\Pi\Delta} & B_{\Pi I}^T \end{pmatrix} \begin{pmatrix} A_{II} & A_{\Delta I}^T & B_{II}^T \\ A_{\Delta I} & A_{\Delta\Delta} & B_{\Delta I}^T \\ B_{II} & B_{\Delta I} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^T \\ A_{\Pi\Delta}^T \\ B_{\Pi I} \end{pmatrix}, \tag{12}$$

which corresponds to a discrete Stokes harmonic extension operator $\mathcal{SH}_\Pi : \mathbf{W}_\Pi \to \prod_{i=1}^N \mathbf{W}^h(\Omega^i)$ defined as: for any given primal velocity $\mathbf{u}_\Pi \in \mathbf{W}_\Pi$, find $\mathcal{SH}_\Pi \mathbf{u}_\Pi \in \prod_{i=1}^N \mathbf{W}^h(\Omega^i)$ and $p_I \in \prod_{i=1}^N \Pi_I^i$ such that on each subdomain $\Omega^i, i = 1, ..., N$,

$$\begin{cases} a(\mathcal{SH}_\Pi \mathbf{u}_\Pi, \mathbf{v}^i) + b(\mathbf{v}^i, p_I^i) &= 0, \quad \forall \mathbf{v}^i \in \mathbf{W}^h(\Omega^i) \\ b(\mathcal{SH}_\Pi \mathbf{u}_\Pi, q_I^i) &= 0, \quad \forall q_I^i \in \Pi^i \\ \mathcal{SH}_\Pi \mathbf{u}_\Pi &= \mathbf{u}_\Pi, \quad \text{in the primal space } \mathbf{W}_\Pi. \end{cases} \tag{13}$$

If we define an inner product $s_\Pi(.,.)$, corresponding to the Schur operator S_Π, on the space $\mathbf{W}_\Pi$ as

$$s_\Pi(\mathbf{u}_\Pi, \mathbf{u}_\Pi) = \mathbf{u}_\Pi^T S_\Pi \mathbf{u}_\Pi = a(\mathcal{SH}_\Pi \mathbf{u}_\Pi, \mathcal{SH}_\Pi \mathbf{u}_\Pi)\ , \ \forall \mathbf{u}_\Pi \in \mathbf{W}_\Pi, \tag{14}$$

then the matrix form of the coarse problem (10) can be written in the following variation form: find $\mathbf{u}_\Pi \in \mathbf{W}_\Pi$ and $p_0 \in \Pi_0$ such that,

$$\begin{cases} s_\Pi(\mathbf{u}_\Pi, \mathbf{v}_\Pi) + b(\mathbf{v}_\Pi, p_0) &= <\mathbf{f}_\Pi, \mathbf{v}_\Pi>, \forall \mathbf{v}_\Pi \in \mathbf{W}_\Pi \\ b(\mathbf{u}_\Pi, q_0) &= 0, \forall q_0 \in \Pi_0. \end{cases} \tag{15}$$

We can prove the following inf-sup stability estimate for this coarse saddle point problem.

Theorem 3.1

$$\sup_{\mathbf{w}_\Pi \in \mathbf{W}_\Pi} \frac{b(\mathbf{w}_\Pi, q_0)^2}{s_\Pi(\mathbf{w}_\Pi, \mathbf{w}_\Pi)} \geq \beta_C^2 ||q_0||_{L^2}^2, \forall q_0 \in \Pi_0, \tag{16}$$

where $\beta_C = C(1 + log(H/h))^{-1/2}$. C is a constant independent of h and H, but depends on the inf-sup stability constant of subdomain Stokes problem solver.

We have given a condition number bound for the preconditioned linear system (8) for two-dimensional case in [5]. Here we use some techniques from Klawonn et al [4] to obtain the following condition number bound for the three-dimensional case:

Theorem 3.2 *The condition number of the preconditioned linear system (8) is bounded from above by $C(1 + \log(H/h))^2$, where C is independent of h and H, but depends on the inf-sup stability constant of subdomain Stokes problem solver.*

4. Extension to nonlinear Navier-Stokes equations. The nonlinear problem is:

$$\begin{cases} -\mu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega \\ -\nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega \\ \mathbf{u} &= \mathbf{g}, \quad \text{on } \partial\Omega\ , \end{cases} \tag{17}$$

where μ is the viscosity and $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$.
We solve this nonlinear problem by using a Picard iteration, where in each iteration step we solve a linearized Navier-Stokes problem:

$$\begin{cases} -\mu\Delta\mathbf{u}^{n+1} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} &= \mathbf{f}, \\ -\nabla \cdot \mathbf{u}^{n+1} &= 0, \\ \mathbf{u}^{n+1}|_{\partial\Omega} &= \mathbf{g}. \end{cases}$$

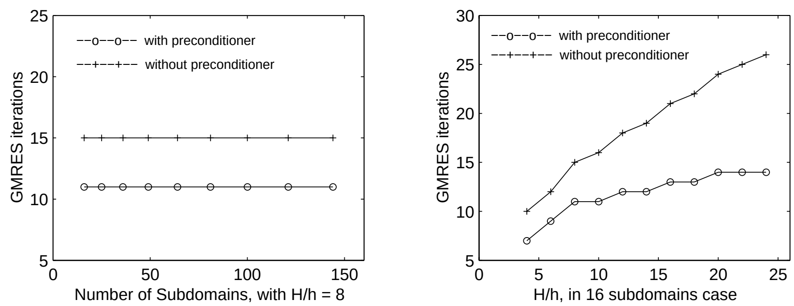

Figure 5.1: GMRES iterations counts for the Stokes solver vs. number of subdomains (left) and vs. H/h (right)

To solve this non-symmetric equation, the non-symmetric bilinear form $\int_{\Omega^i}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}\mathbf{v}$, on each subdomain Ω^i, is written as the sum of a skew-symmetric term and an interface term:

$$\left(\frac{1}{2}\int_{\Omega^i}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}\mathbf{v}-\frac{1}{2}\int_{\Omega^i}(\mathbf{u}^n\cdot\nabla)\mathbf{v}\mathbf{u}^{n+1}\right)+\frac{1}{2}\int_{\partial\Omega^i}(\mathbf{u}^n\cdot\mathbf{n})\mathbf{u}^{n+1}\mathbf{v}. \qquad (18)$$

By doing this, we are identifying the correct bilinear form describing the action of the above non-symmetric operator on any given subdomain Ω^i, and the subdomain incompressible Navier-Stokes problem appears as:

$$\left\{\begin{array}{rcll} -\Delta\mathbf{u}^{n+1}+(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}+\nabla p^{n+1} &=& \mathbf{f}, & \text{in } \Omega^i\\ -\nabla\cdot\mathbf{u}^{n+1} &=& 0, & \text{in } \Omega^i\\ \mathbf{u}^{n+1} &=& \mathbf{g}, & \text{on } \partial\Omega\cap\partial\Omega^i\\ \frac{\partial\mathbf{u}^{n+1}}{\partial\mathbf{n}}-p^{n+1}\mathbf{n}-\frac{\mathbf{u}^n\cdot\mathbf{n}}{2}\mathbf{u}^n &=& \lambda, & \text{on } \Gamma^{ij}\ . \end{array}\right. \qquad (19)$$

The idea to write the non-symmetric bilinear form $\int_{\Omega^i}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}\mathbf{v}$ as in (18) was used by Achdou et al [1] to solve advection-diffusion problems. After discretizing the subdomain problems (19), we can use the same procedure as in section 2 to design the FETI-DP algorithm. We should also note that the conjugate gradient method cannot be used here to solve the preconditioned linear system, because this problem is no longer symmetric, positive definite.

5. Numerical Experiments. We have tested our algorithm by solving a lid driven cavity problem on the domain $\Omega=[0,1]\times[0,1]$, with $\mathbf{f}=\mathbf{0}$, $g_x=1, g_y=0$ for $x\in[0,1], y=1$, and $\mathbf{g}=\mathbf{0}$ elsewhere on the boundary (cf. Elman et al [2]). We have used GMRES to solve the preconditioned linear system (8), as well as the nonpreconditioned linear system (6). The initial guess is $\lambda=0$ and the stopping criterion is $||r_k||_2/||r_0||_2\le 10^{-6}$, where r_k is the residual of the Lagrange multiplier equation at the kth iteration. Figure 5.1 gives the number of GMRES iterations for different number of subdomains with a fixed subdomain problem size $H/h=8$, and for different subdomain problem size H/h with 4×4 subdomains. We see, from the left figure, that the convergence of the augmented FETI-DP method, with or without a preconditioner, is independent of the number of subdomains, while the preconditioned version needs fewer iterations. The right figure shows that the GMRES iteration count increases, in both the preconditioned and the nonpreconditioned cases, with the increase of the size of subdomain problem, but that it is growing much slower with the Dirichlet preconditioner than without. Figure 5.2 shows that the coarse saddle point problem is inf-sup stable; cf. Theorem 3.1. We can see, from the left figure, that β_C is bounded away from zero while

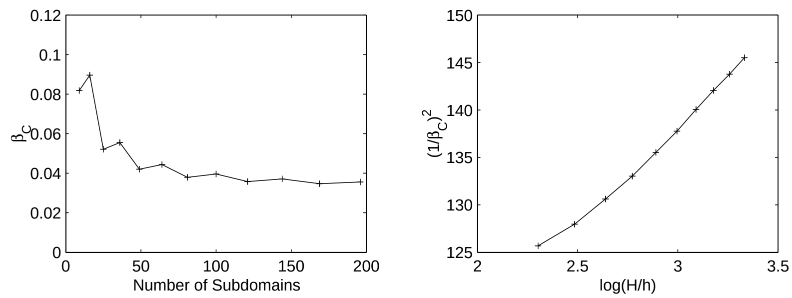

Figure 5.2: Inf-sup stability condition of the coarse level saddle point problem

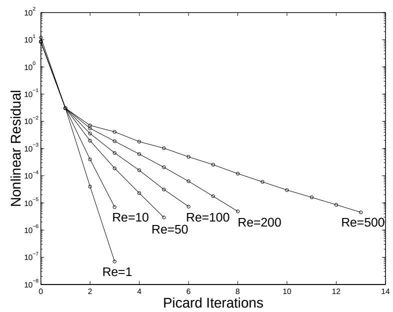

Figure 5.3: Convergence of Picard iteration for different Reynolds number

we increase the number of subdomains; from the right figure, $(1/\beta_C)^2$ appears to be a linear function of $log(H/h)$. In Figure 5.3 and Figure 5.4, we show the convergence behavior of the Picard iteration used to solve the nonlinear Navier-Stokes equation (17) for the 2D lid driven cavity problem. In our experiments, we start from a zero initial guess, and the Picard iteration is stopped when the nonlinear residual is reduced by 10^{-6}. For the GMRES solver, we reduce the linear residual by 10^{-3} in each iteration step. ¿From Figure 5.3, we see that the convergence of the Picard iteration depends on the Reynolds number: the larger is the Reynolds number, the slower is the convergence. Figure 5.4 shows that the convergence is independent of the mesh size. The left figure shows that the convergence is independent of the number of subdomains for fixed $H/h = 10$; the right figure shows that that the convergence is independent of H/h for the 64 subdomain case, except for a Reynolds number of 500. This can be explained by the fact that for high Reynolds number, the mesh has to be fine enough to achieve good convergence. **Acknowledgments.** The author is grateful to Olof Widlund for proposing this problem and giving many helpful suggestions.

REFERENCES

[1] Y. Achdou, P. L. Tallec, F. Nataf, and M. Vidrascu. A domain decomposition preconditioner for an advection-diffusion problem. *Comput. Methods Appl. Mech. Engrg.*, 184:145–170, 2000.

[2] H. C. Elman, D. J. Silvester, and A. J. Wathen. Iterative methods for problems in computational fluid dynamics. *Iterative Methods in Scientific Computing*, 1997.

[3] C. Farhat, M. Lesoinne, and K. Pierson. A scalable dual-primal domain decomposition method. *Numer. Lin. Alg. Appl.*, 7:687–714, 2000.

[4] A. Klawonn, O. Widlund, and M. Dryja. Dual-primal FETI methods for three-dimensional

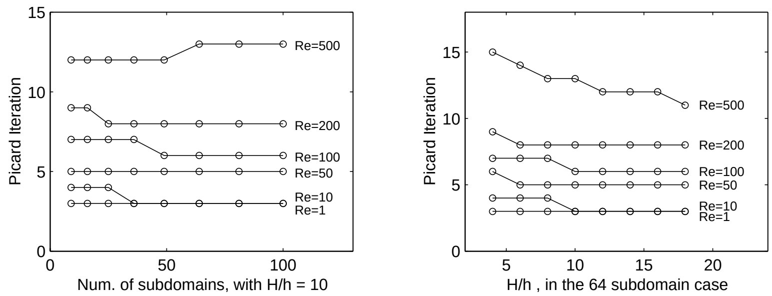

Figure 5.4: Convergence of Picard iteration for different meshes

elliptic problems with heterogeneous coefficients. Technical Report 815, Courant Institute of Mathematical Sciences, Department of Computer Science, April 2001. to appear in SIAM J. Numer. Anal.

[5] J. Li. A Dual-Primal FETI method for incompressible Stokes equations. Technical Report 816, Courant Institute of Mathematical Sciences, Department of Computer Sciences, 2001.

[6] J. Mandel and R. Tezaur. On the convergence of a dual-primal substructuring method. *Numer. Math.*, 88:543–558, 2001.