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25. The Direct Approach to Domain Decomposition Methods

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1. Introduction. Recently, Herrera presented a general and unifying theory of domain decomposition methods (DDM), and this paper is part of a group of articles [5], included in these Proceedings, devoted to present an overview of this theory and some of its applications. According to it, DDM are classified into direct and indirect methods. This paper is devoted to briefly present direct methods from the point of view of the unified theory. A related and more detailed discussion may be found in [6]. It must be mentioned that Direct Methods subsume Schwartz and Steklov-Poincaré Methods among others [6], [7].

2. Notations. The notations will be as in [6].In what follows, unless otherwise explicitly stated, Ω will be an open, bounded region. The closure of any set Ω will be denoted by $\overline{\Omega}$. The (outer) boundary of Ω will be denoted by $\partial\Omega$.

As usual, a collection $\Pi = \{\Omega_1, ..., \Omega_E\}$ of open subregiones Ω_i $(i = 1, ..., E)$ of Ω, is said to be a <u>*partition*</u> of Ω, <u>*iff*</u>

i. $\Omega_i \cap \Omega_j = \phi, for every i \neq j$ and

ii. $\bar{\Omega} = \bigcup\limits_{i=1}^{i=E} \bar{\Omega}_i$

In addition, the partitions considered throughout this paper are assumed to be such that the subregiones Ω_i are *manifolds with corners* [6]. The manifold $\bigcup\limits_{i=1}^{E} \partial\Omega_i$ will be referred to as the *'generalized boundary'*, while the *'internal boundary'* of Ω -to be denoted by Σ- is defined as the closed complement of $\partial\Omega$, considered as a subset of the generalized boundary. Observe that the internal boundary -and the generalized boundary as well- are concepts whose definition is relative to both the region Ω and the partition Π. Thus, when deemed necessary, the notation $\Sigma(\Omega, \Pi)$, which is more precise, will be used.

A partition $\Pi' = \left\{\Omega_1^{'}, ..., \Omega_{E'}^{'}\right\}$ of Ω, is said to be a <u>*sub-partition*</u> of Π, when for each given any $i = 1, .., E'$, there is a subset of natural numbers $\mathrm{N}(i) \subset \{1, .., E\}$, such that

$$\bar{\Omega}_i^{'} = \bigcup_{j \in \mathcal{N}(i)} \bar{\Omega}_j \tag{2.1}$$

Given a sub-partition $\Pi' = \left\{\Omega_1^{'}, ..., \Omega_{E'}^{'}\right\}$, the function $\mu' : \{1, ..., E\} \to \{1, ..., E'\}$ is defined, for every $j = 1, ..., E$, by the equation $\mu'(j) = i$, whenever $j \in \mathrm{N}(i)$. Two partitions: $\Pi' = \left\{\Omega_1^{'}, ..., \Omega_{E'}^{'}\right\}$ and $\Pi'' = \left\{\Omega_1^{''}, ..., \Omega_{E''}^{''}\right\}$, respectively, are said to be <u>*conjugate*</u> with respect to a partition Π, when:

i. They are both sub-partitions of Π ;

ii. In the measure of the generalized boundary, the sets

$$\Sigma^{'} - \left\{\Sigma^{'} \cap \left(\bigcup_{i=1}^{i=E^{'}} \Omega_i^{''}\right)\right\} \quad and \quad \Sigma^{''} - \left\{\Sigma^{''} \cap \left(\bigcup_{i=1}^{i=E^{''}} \Omega_i^{'}\right)\right\} \tag{2.2}$$

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have measure zero;
iii. And

$$\Sigma' \cup \Sigma'' = \Sigma \tag{2.3}$$

Here, $\Sigma' = \Sigma(\Omega, \Pi')$ and $\Sigma'' = \Sigma(\Omega, \Pi'')$.

When $\Pi' = \left\{\Omega_1^{'}, ..., \Omega_{E'}^{'}\right\}$ and $\Pi'' = \left\{\Omega_1^{''}, ..., \Omega_{E''}^{''}\right\}$ are conjugate partitions, in addition to the mapping $\mu^{'}$ introduced above, it will be necessary to consider a second mapping $\mu^{''}$, associated with $\Pi^{''}$, which is defined correspondingly.

The formulation and treatment of boundary problems with prescribed jumps requires the introduction of a special class of Sobolev spaces in which some of their functions are fully discontinuous [6]. The *jump* of u across Σ_{ij}, is defined by

$$[v] \equiv v_+ - v_- \tag{2.4}$$

and the *average* by

$$\dot{v} \equiv \frac{1}{2}(v_+ + v_-) \tag{2.5}$$

3. The General Problem with Prescribed Jumps (BVPJ). The direct approach to Domain Decomposition Methods, here presented, as well as Herrera's unified theory, can be applied to a very general class of boundary value problems for which jumps are prescribed in the internal boundaries. Given Ω, the region of definition of the problem, and a partition of Ω (or domain-decomposition) $\Pi \equiv \{\Omega_1, ..., \Omega_E\}$, let $\Sigma \equiv \Sigma(\Omega, \Pi)$ be the internal boundary. Then, using a notation similar to that presented in [8], the general form of such boundary value problem with prescribed jumps (BVPJ) is

$$\mathcal{L}u = \mathcal{L}u_\Omega \equiv f_\Omega; \quad in \quad \Omega_i, i = 1, ..., E \tag{3.1}$$

$$B_j u = B_j u_\partial \equiv g_j; \quad in \quad \partial\Omega \tag{3.2}$$

$$[J_k u] = [J_k u_\Sigma] \equiv j_k; \quad in \quad \Sigma \tag{3.3}$$

where the B_j's and J_j's are certain differential operators (the j's and k's run over suitable finite ranges of natural numbers) and $u_\Omega \equiv \left(u_\Omega^1, ..., u_\Omega^E\right)$, together with u_∂ and u_Σ are given functions of the space of trial functions. In addition, f_Ω, g_j and j_j may be defined by Eq. (3.1).

It must be emphasized that the scope of the methodology presented in this and the other papers of this series is quite wide, since in principle it is applicable to any partial differential equation or system of such equations that is linear, independently of its type and including equations with discontinuous coefficients. But, of course, every kind of equation has its own peculiarities, which require special developments that have to be treated separately.

4. The Elliptic Equation of Second Order. In this Section we describe the overlapping direct method under investigation, for the second-order differential equation of elliptic type, when the problem is defined in a space of arbitrary dimension. For definiteness, only boundary conditions of Dirichlet type will be presented, but the procedure is applicable to any kind of boundary conditions for which the problem is well posed, as was done in [4]. With the notation introduced in Section 2, a region Ω and a partition $\Pi \equiv \{\Omega_1, ..., \Omega_E\}$ of Ω, will be considered. The solution to the boundary value problem with prescribed jumps in this case, will be sought in a Sobolev space of the kind introduced in that Section. More precisely, a function $u \in \hat{H}^2(\Omega) \equiv H^2(\Omega_1) \oplus ... \oplus H^2(\Omega_E)$ is sought, such that

$$\mathcal{L}u \equiv -\nabla \bullet (\underline{\underline{a}} \bullet \nabla u) + \nabla \bullet (\underline{b}u) + cu = f_\Omega; \text{in}\Omega_i, i = 1, ..., E \tag{4.1}$$

subjected to the boundary conditions

$$u = u_\partial; \quad in \quad \partial\Omega \tag{4.2}$$

and jump conditions

$$[u] = j^0 = [u_\Sigma]; \quad on \quad \Sigma \tag{4.3}$$

$$[\underline{\underline{a}} \bullet \nabla u] \bullet \underline{n} = j^1 = [\underline{\underline{a}} \bullet \nabla u_\Sigma] \bullet \underline{n}; \quad on \quad \Sigma \tag{4.4}$$

The above formulation and the methodology that follows applies even if the coefficients of the differential operator are discontinuous. In the particular case when the coefficients are continuous, the jump condition of Eq. (4.4), in the presence of Eq. (4.3), is equivalent to

$$\left[\frac{\partial u}{\partial n}\right] = \left[\frac{\partial u_\Sigma}{\partial n}\right]; \quad on \quad \Sigma \tag{4.5}$$

In what follows, it will be assumed that this problem possesses one and only one solution. Conditions under which this assumption is fulfilled, are well-known.

According to the unified theory one has to choose an information-target, that is referred as *'the sought information'*, as a suitable part of the complementary information defined on Σ. In the procedure that is explained next, the *sought information* is taken to be the average, across Σ, of the solution of the BVPJ. This choice is suitable, because the boundary value problem defined by the system of equations (4.1) to (4.4), when this latter equation is replaced by

$$\dot{\hat{u}} = \dot{u}_I \tag{4.6}$$

is local and well-posed. Here, $u_I \in \hat{D}_1$ is a given function, This can be verified using the relation

$$u_+ = \dot{u} + \frac{1}{2}[u] \,\text{and}\, u_- = \dot{u} - \frac{1}{2}[u] \tag{4.7}$$

It permits evaluating the values of the function, on both sides of the internal boundary Σ, when the average is known. When this information is complemented with the boundary data on the external boundary, a Dirichlet problem can be formulated in each one of the subdomains of the partition.

In the Theorem that follows, two conjugate partitions $\Pi' = \left\{\Omega_1^{'}, ..., \Omega_{E'}^{'}\right\}$ and $\Pi'' = \left\{\Omega_1^{''}, ..., \Omega_{E''}^{''}\right\}$, as well as the mappings $\mu^{'}$ and $\mu^{''}$ associated to them in the manner explained in Section 2, will be considered. Also, the notations $\Sigma' \equiv \Sigma(\Omega, \Pi')$ and $\Sigma'' \equiv \Sigma(\Omega, \Pi'')$ will be adopted.

Theorem 4.1 *.- Let* $\Pi' = \left\{\Omega_1^{'}, ..., \Omega_{E'}^{'}\right\}$ *and* $\Pi'' = \left\{\Omega_1^{''}, ..., \Omega_{E''}^{''}\right\}$ *be two partitions of* Ω *which are conjugate with respect to* Π*, and let* $\left\{\widehat{u}^1, ..., \widehat{u}^{E'}\right\}$ *and* $\left\{\widehat{u}^1, ..., \widehat{u}^{E'}\right\}$ *be two families of functions, such that*

1) For every $i = 1, ..., E'$*, the function* $\widehat{u}^i \in \hat{H}^2(\Omega_i^{'}, \Pi^{'})$ *fulfills Eqs.(4.1) to (4.3) and satisfies Eq.(4.4) in* $\Sigma^{'}$

2) For every $j = 1, ..., E''$*, the function* $\breve{u}^j \in \hat{H}^2(\Omega_j^{''}, \Pi^{''})$ *fulfills Eqs.(4.1) to (4.3) and satisfies Eq.(4.4) in* $\Sigma^{''}$.

Then, define $u' = \left(u'^1, ..., u'^E\right) \in \hat{H}^2\left(\Omega, \Pi\right)$ *and* $u'' = \left(u''^1, ..., u''^E\right) \in \hat{H}^2\left(\Omega, \Pi\right)$*, by*

$$u'^i = \widehat{u}^{\mu'(i)}\Big|_{\Omega_i} ; i = 1, ..., E \tag{4.8}$$

$$u''^j = \widehat{u}^{\mu''(j)}\Big|_{\Omega j} ; i = 1, ..., E \tag{4.9}$$

Under these assumptions the following statements are equivalent:
i. $u^{'}$ *and* $u^{''}$ *are solutions of the BVPJ in* Ω*;*
ii.

$$u' \equiv u'' \tag{4.10}$$

iii.

$$\dot{u}'\left(\underline{x}\right) = \dot{u}''\left(\underline{x}\right), \quad a.e. \quad on \quad \Sigma = \Sigma^{'} \cup \Sigma^{''} \tag{4.11}$$

Proof..- That *i)* implies *ii)* is immediate, because of the assumption of uniqueness of solution for the BVPJ. That *ii)* implies *iii)* follows from the jump condition of Eq.(4.3) and the definition of the average across Σ. Eq.(4.11) in the presence of Eq.(4.3), in turn imply

$$u'\left(\underline{x}+\right) = \dot{u}'\left(\underline{x}\right) + \frac{1}{2}\left[u'\right] = \dot{u}'\left(\underline{x}\right) + \frac{1}{2}j^0 = \dot{u}''\left(\underline{x}\right) + \frac{1}{2}j^0 = \dot{u}''\left(\underline{x}\right) + \frac{1}{2}\left[u''\right] = u''\left(\underline{x}+\right) \tag{4.12}$$

Recalling that $\Sigma = \Sigma' \cup \Sigma''$ and that $\Sigma \cup \partial\Omega = \bigcup_{i=1}^{E} \partial\Omega_i$, it is seen that the boundary values of $u^{'}$ and $u^{''}$ coincide on each side of Σ. This, together with the assumed uniqueness of solution of the boundary value problem at each one of the sub-regions of the partition, imply $u' \equiv u''$. ■

It is timely to point out the connections between the method discussed in this paper and the Schwarz alternating methods. Indeed, this latter approach can be derived from Eqs.(4.1) to (4.3) and (4.11), when an iterative procedure is adopted for fulfilling Eq.(4.11). To show this, let $u^{2n}(n = 0, 1, ...)$ and $u^{2n+1}(n = 0, 1, ...)$ satisfy Eqs.(4.1) to (4.3), together with

$$\overbrace{u^{2n+1}}^{\bullet} = \overbrace{u^{2n}}^{\bullet}, \quad \text{on} \quad \Sigma',(n = 0, 1, ...) \tag{4.13}$$

$$\overbrace{u^{2n+2}}^{\bullet} = \overbrace{u^{2n+1}}^{\bullet}, \quad \text{on} \quad \Sigma'',(n = 0, 1, ...) \tag{4.14}$$

Then, if the sequence $u^{2n}(n = 0, 1, ...)$ converges to $\widehat{u}$, while the sequence $u^{2n+1}(n = 0, 1, ...)$ converges to $\breve{u}$, one has $\widehat{u} = \breve{u} = u$, and this function fulfills Eqs. (4.1) to (4.3), together with Eq.(4.11). In the cases when a variational principle can be applied, the projection interpretation is possible and the Schwarz alternating procedure can be derived (see, for example, [2], [3], [1]).

5. The One-Dimensional Case. The one dimensional version of the problem described in Section 4 corresponds to the two-point boundary value problem of the general differential equation of second order. Let be $\Omega \equiv (0, l)$ and $\Pi \equiv \{(0, x_1), (x_1, x_2), ..., (x_{E-1}, x_E = l)\}$. Then

$$\mathcal{L}u \equiv -\frac{d}{dx}(a\frac{du}{dx}) + \frac{d}{dx}(bu) + cu = f_\Omega, \quad in \quad (x_{i-1}, x_i), i = 1, ..., E \tag{5.1}$$

Assume that the boundary and jump conditions are:

$$u(0) = g_{\partial 0}, u(l) = g_{\partial l} \tag{5.2}$$

$$[u] = j_i^0 \equiv [u_\Sigma] \quad and \quad \left[\frac{du}{dx}\right] = j_i^1 \equiv \left[\frac{du_\Sigma}{dx}\right]; i = 1, ..., E-1 \tag{5.3}$$

respectively. In addition, it will be assumed that the Dirichlet problem is well-posed in each one of the subintervals and that $u(x) \in H^2(\Omega)$ is the unique solution of this BVPJ, in Ω. As in Section 4, the sought information will be the average of the solution, across Σ.

In every subinterval $(x_{i-1}, x_{i+1}), i = 1, ..., E-1$ define the function $u^i(x)$ to be the restriction of $u(x)$ to Ω_i. Then, for every $i = 1, ..., E-1$, $u^i(x)$, is the unique solution of a boundary value problem with prescribed jumps defined in the subinterval (x_{i-1}, x_{i+1}), which is derived from the following conditions:

$$\mathcal{L}u^i = f_\Omega, \quad \text{in} \quad (x_{i-1}, x_{i+1}); i = 1, ..., E-1 \tag{5.4}$$

$$[u^i]_i = j_i^0; \left[\frac{du^i}{dx}\right]_i = j_i^1; i = 1, ..., E-1 \tag{5.5}$$

$$u^i(x_{i-1}+) = u(x_{i-1}+) = \dot{u}(x_{i-1}) + \frac{1}{2}j_{i-1}^0; i = 2, ...E-1 \tag{5.6}$$

$$u^i(x_{i+1}-) = u(x_{i+1}-) = \dot{u}(x_{i+1}) - \frac{1}{2}j_{i+1}^0; i = 1, ..., E-2 \tag{5.7}$$

$$u^1(0) = u(0) = g_{\partial 0} \tag{5.8}$$

$$u^{E-1}(l) = u(l) = g_{\partial l} \tag{5.9}$$

Let the functions $u_H^i(x)$ and $u_P^i(x)$ be defined in (x_{i-1}, x_{i+1}) by the following conditions:

$$\mathcal{L}u_H^i = 0, \quad in \quad (x_{i-1}, x_{i+1}); i = 1, ..., E \tag{5.10}$$

$$[u_H^i]_i = \left[\frac{du_H^i}{dx}\right]_i = 0; i = 1, ...E-1 \tag{5.11}$$

$$u_H^i(x_{i-1}+) = u(x_{i-1}+) = \dot{u}(x_{i-1}) + \frac{1}{2}j_{i-1}^0; i = 2, ...E-1 \tag{5.12}$$

$$u_H^i(x_{i+1}-) = u(x_{i+1}-) = \dot{u}(x_{i+1}) - \frac{1}{2}j_{i+1}^0; i = 1, ..., E-2 \tag{5.13}$$

$$u_H^1(x_0) = u(0) = g_{\partial 0}; \tag{5.14}$$

$$u_H^{E-1}(x_E) = u(l) = g_{\partial l}; \tag{5.15}$$

together with

$$\mathcal{L}u_P^i = f_\Omega, \quad in \quad (x_{i-1}, x_i) \quad and \quad (x_i, x_{i+1}), \quad separately, \quad for \quad i = 1, ..., E-1 \tag{5.16}$$

$$u_P^i(x_{i-1}+) = u_P^i(x_{i+1}-) = 0, \quad for \quad i = 1, ..., E-1 \tag{5.17}$$

$$[u_P^i]_i = j_i^0 \quad \text{and} \quad \left[\frac{du_P^i}{dx}\right]_i = j_i^1; i = 1, ..., E-1 \tag{5.18}$$

Then, it can be verified that

$$u^i(x) = u_H^i(x) + u_P^i(x); i = 1, ..., E-1 \tag{5.19}$$

Even more:

$$u_H^i(x) = u_H^i(x_{i-1}-)\phi_-^i(x) + u_H^i(x_{i+1}+)\phi_+^i(x) \tag{5.20}$$

when $\phi_-^i(x)$ and $\phi_+^i(x)$ are defined by the conditions:

$$\mathcal{L}\phi_+^i = 0; \phi_+^i(x_{i-1}) = 0, \phi_+^i(x_{i+1}) = 1 \tag{5.21}$$

$$\mathcal{L}\phi_-^i = 0; \phi_-^i(x_{i-1}) = 1, \phi_-^i(x_{i+1}) = 0 \tag{5.22}$$

together with

$$[\phi_+^i]_i = [\phi_-^i]_i = \left[\frac{d\phi_+^i}{dx}\right]_i = \left[\frac{d\phi_-^i}{dx}\right]_i = 0 \tag{5.23}$$

From Eqs. (5.6), (5.7), (5.19), and (5.20), it follows that

$$\dot{u}(x_i) - \dot{u}_P^i(x_i) = \dot{u}_H^i(x_i) = \{\dot{u}(x_{i-1}) + \frac{1}{2}j_{i-1}^0\}\phi_-^i(x_i) + \{\dot{u}(x_{i+1}) - \frac{1}{2}j_{i+1}^0\}\phi_+^i(x_i) \tag{5.24}$$

Hence

$$-\rho_-^i\dot{u}_{i-1} + \dot{u}_i - \rho_+^i\dot{u}_{i+1} = \mu_i; i = 2, ..., E-2 \tag{5.25}$$

$$\dot{u}_i - \rho_+^i\dot{u}_{i+1} = \mu_i; i = 1 \tag{5.26}$$

$$-\rho_-^i\dot{u}_{i-1} + \dot{u}_i = \mu_i; i = E-1 \tag{5.27}$$

where

$$\rho_-^i = \phi_-^i(x_i), \rho_+^i = \phi_+^i(x_i); i = 1, ..., E-1 \tag{5.28}$$

$$\mu_i = \frac{\rho_-^i}{2}j_{i-1}^0 + \dot{u}_P^i(x_i) - \frac{\rho_+^i}{2}j_{i+1}^0; i = 2, ..., E-2 \tag{5.29}$$

$$\mu_i = \rho_-^i g_{\partial 0} + \dot{u}_P^i(x_i) - \frac{\rho_+^i}{2}j_{i+1}^0; i = 1 \tag{5.30}$$

$$\mu_i = \frac{\rho_-^i}{2}j_{i-1}^0 + \dot{u}_P^i(x_i) + \rho_+^i g_{\partial l}; i = E-1 \tag{5.31}$$

Eqs. (5.25) to (5.27), constitute an $E-1$ tridiagonal system of equations, which can be solved for $\dot{u}_i(i = 1, ..., E-1)$.

Once the averages $\dot{u}_i(i = 1, ..., E-1)$ are known, it is possible to apply *'optimal interpolation'* to obtain the solution in the interior of each one of the subintervals of the partition. This kind of interpolation consists in deriving enough information for defining well-posed problems in each of those subintervals. To this end, apply the identities

$$u(x_i+) \equiv \dot{u}_i + \frac{1}{2}[u]_i = \dot{u}_i + \frac{1}{2}j_i^0 \quad \text{and} \quad u(x_i-) \equiv \dot{u}_i - \frac{1}{2}[u]_i = \dot{u}_i - \frac{1}{2}j_i^0 \tag{5.32}$$

When these values are complemented with the prescribed boundary values of Eq. (5.2), well-posed boundary value problems in each one of the subintervals of the partition can be defined. In this manner, all that is required to reconstruct the solution of the BVPJ is to solve such "local problems", in each one of the subintervals. Using the previous developments, one can apply Eqs. (5.19), and (5.20), to obtain $u(x)$ in the interior of the subintervals of the partition.

Up to now, all the developments have been exact. However, one can apply the system of equations (5.25) to (5.27), as well as Eqs. (5.19), and (5.20), only if the functions ϕ_-^i, ϕ_+^i and u_P^i, $(i = 1, ..., E-1)$, are available. In general applications it will be necessary to resort to numerical approximations for the construction of such functions and the system of equations so obtained will not be exact any longer. Instead, its precision will depend on the error introduced by the numerical procedure that is applied for solving the problems defined by Eqs. (5.10) to (5.18). A similar comment can be made with respect to the construction of the solution of the local boundary value problem whose solution is given by Eqs. (5.19) and (5.20). In reference [6] collocation was used, obtaining in this manner a non-standard method of collocation.

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