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29. Optimization of Interface Operator Based on Algebraic Approach

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1. Introduction. This paper is dedicated to recent developments of an optimized two-Lagrange multiplier domain decomposition method [5], [8]). Most methods for optimizing the augmented interface operator are based on the discretization of approximations of the continuous transparent operator [4], [1], [2], [7]. At the discrete level, the optimal operator can be proved to be equal to the Schur complement of the outer domain. This Schur complement can be directly approximated using purely algebraic techniques like sparse approximate inverse methods or incomplete factorization. The main advantage of such algebraic approach is that it is much more easy to implement in existing code without any information on the geometry of the interface and the finite element formulation used. Convergence results and parallel efficiency of several algebraic optimization techniques of interface operator for acoustic analysis applications will be presented.

2. Algebraic Formulation of Domain Decomposition Methods.

2.1. General Presentation. Consider a splitting of the domain Ω as in Figure 2.1 and note by subscripts i and p the degrees of freedom located inside subdomain $\Omega^{(s)}$, $s = 1, 2$, and on the interface Γ_p. Then, the contribution of subdomain $\Omega^{(s)}$, $s = 1, 2$ to the matrix and the right-hand side of a finite element discretization of a linear PDE on Ω can be written as follows:

$$K^{(s)} = \begin{bmatrix} K_{ii}^{(s)} & K_{ip}^{(s)} \\ K_{pi}^{(s)} & K_{pp}^{(s)} \end{bmatrix} \quad , \quad b^{(s)} = \begin{bmatrix} b_i^{(s)} \\ b_p^{(s)} \end{bmatrix} \tag{2.1}$$

where $K_{pp}^{(1)}$ and $K_{pp}^{(2)}$ represent the interaction matrices between the nodes on the interface obtained by integration on $\Omega^{(1)}$ and on $\Omega^{(2)}$. The global problem is a block system obtained by assembling local contribution of each subdomain:

$$\begin{bmatrix} K_{ii}^{(1)} & 0 & K_{ip}^{(1)} \\ 0 & K_{ii}^{(2)} & K_{ip}^{(2)} \\ K_{pi}^{(1)} & K_{pi}^{(2)} & K_{pp} \end{bmatrix} \begin{bmatrix} x_i^{(1)} \\ x_i^{(2)} \\ x_p \end{bmatrix} = \begin{bmatrix} b_i^{(1)} \\ b_i^{(2)} \\ b_p \end{bmatrix}. \tag{2.2}$$

The block K_{pp} is the sum of the two blocks $K_{pp}^{(1)}$ and $K_{pp}^{(2)}$. In the same way, $b_p = b_p^{(1)} + b_p^{(2)}$ is obtained by local integration in each subdomain and sum on the interface.

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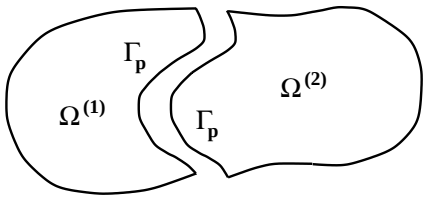

Figure 2.1: Non-overlapping domain splitting

Theorem 2.1 (existence and uniqueness) *Given any splitting of* $K_{pp} = K_{pp}^{(1)} + K_{pp}^{(2)}$ *and* $b_p = b_p^{(1)} + b_p^{(2)}$, *and any matrices* $A^{(1)}$, $A^{(2)}$ *there is only one pair of Lagrange multipliers* $\lambda^{(1)}$, $\lambda^{(2)}$ *such as the following coupled problem:*

$$\begin{bmatrix} K_{ii}^{(1)} & K_{ip}^{(1)} \\ K_{pi}^{(1)} & K_{pp}^{(1)} + A^{(1)} \end{bmatrix} \begin{bmatrix} x_i^{(1)} \\ x_p^{(1)} \end{bmatrix} = \begin{bmatrix} b_i^{(1)} \\ b_p^{(1)} + \lambda^{(1)} \end{bmatrix} \tag{2.3}$$

$$\begin{bmatrix} K_{ii}^{(2)} & K_{ip}^{(2)} \\ K_{pi}^{(2)} & K_{pp}^{(2)} + A^{(2)} \end{bmatrix} \begin{bmatrix} x_i^{(2)} \\ x_p^{(2)} \end{bmatrix} = \begin{bmatrix} b_i^{(2)} \\ b_p^{(2)} + \lambda^{(2)} \end{bmatrix} \tag{2.4}$$

$$x_p^{(1)} - x_p^{(2)} = 0 \tag{2.5}$$

$$\lambda^{(1)} + \lambda^{(2)} - A^{(1)} x_p^{(1)} - A^{(2)} x_p^{(2)} = 0 \tag{2.6}$$

is equivalent to the problem (2.2).

Proof. The admissibility condition (2.5) derives from the relation $x_p^{(1)} = x_p^{(2)} = x_p$.
If $x_p^{(1)} = x_p^{(2)} = x_p$, the first rows of local systems (2.3) and (2.4) are the same as the two first rows of global system (2.2), and adding the last rows of local systems (2.3) and (2.4) gives:

$$K_{pi}^{(1)} x_i^{(1)} + K_{pi}^{(2)} x_i^{(2)} + K_{pp}\ x_p - b_p = \lambda^{(1)} + \lambda^{(2)} - A^{(1)} x_p^{(1)} - A^{(2)} x_p^{(2)} \tag{2.7}$$

So, the last equation of global system (2.2) is satisfied only if:

$$\lambda^{(1)} + \lambda^{(2)} - A^{(1)} x_p^{(1)} - A^{(2)} x_p^{(2)} = 0 \tag{2.8}$$

Reversely, if $x_p^{(1)}$, $x_p^{(2)}$ and x_p are derived from global system (2.2), then local systems (2.3) and (2.4) define $\lambda^{(1)}$ and $\lambda^{(2)}$ in a unique way. ∎

2.2. Two-Lagrange Multiplier Domain Decomposition Method . If the local inner matrix $K_{ii}^{(s)}$ is non singular, a direct relation between $x_p^{(s)}$ and $\lambda^{(s)}$ can be obtained from (2.3) and (2.4):

$$x_p^{(s)} = [S^{(s)} + A^{(s)}]^{-1} (c_p^{(s)} + \lambda^{(s)}) \tag{2.9}$$

where $S^{(s)} = K_{pp}^{(s)} - K_{pi}^{(s)} [K_{ii}^{(s)}]^{-1} K_{ip}^{(s)}$ is the Schur complement and $c_p^{(s)} = b_p^{(s)} - K_{pi}^{(s)} [K_{ii}^{(s)}]^{-1} b_i^{(s)}$ is the condensed right hand side in subdomain $\Omega^{(s)}$.
After substitution of $x_p^{(1)}$ and $x_p^{(2)}$ in the interface continuity conditions (2.5) and (2.6) the following linear system is obtained:

$$\begin{bmatrix} [S^{(1)} + A^{(1)}]^{-1} & -[S^{(2)} + A^{(2)}]^{-1} \\ I - A^{(1)} [S^{(1)} + A^{(1)}]^{-1} & I - A^{(2)} [S^{(2)} + A^{(2)}]^{-1} \end{bmatrix} \begin{bmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{bmatrix} = \begin{bmatrix} -[S^{(1)} + A^{(1)}]^{-1} c_p^{(1)} + [S^{(2)} + A^{(2)}]^{-1} c_p^{(2)} \\ A^{(1)} [S^{(1)} + A^{(1)}]^{-1} c_p^{(1)} + A^{(2)} [S^{(2)} + A^{(2)}]^{-1} c_p^{(2)} \end{bmatrix} \tag{2.10}$$

The solution of this system by a Krylov method defines a non overlapping domain decomposition method.

2.3. Discrete Transmission Conditions as Local Preconditioner. Instead of relations (2.5) and (2.6) on the interface, it may be more interesting to consider another set of conditions:

$$C^{(1)}\ (2.5) + (2.6) \quad = \quad 0 \tag{2.11}$$
$$-C^{(2)}\ (2.5) + (2.6) \quad = \quad 0 \tag{2.12}$$

which are equivalent to the initial relations, as soon as the two matrices $C^{(1)}$ and $C^{(2)}$ are such that $C^{(1)} + C^{(2)}$ is invertible. Following the same steps than in section 2.2, the matrix and the right hand side of the linear system takes now the block form:

$$\begin{bmatrix} I - (A^{(1)} - C^{(1)})[S^{(1)} + A^{(1)}]^{-1} & I - (A^{(2)} + C^{(1)})[S^{(2)} + A^{(2)}]^{-1} \\ I - (A^{(1)} + C^{(2)})[S^{(1)} + A^{(1)}]^{-1} & I - (A^{(2)} - C^{(2)})[S^{(2)} + A^{(2)}]^{-1} \end{bmatrix} \tag{2.13}$$
$$\begin{bmatrix} (A^{(1)} - C^{(1)})[S^{(1)} + A^{(1)}]^{-1} c_p^{(1)} + (A^{(2)} + C^{(1)})[S^{(2)} + A^{(2)}]^{-1} c_p^{(2)} \\ (A^{(1)} + C^{(2)})[S^{(1)} + A^{(1)}]^{-1} c_p^{(1)} + (A^{(2)} - C^{(2)})[S^{(2)} + A^{(2)}]^{-1} c_p^{(2)} \end{bmatrix} \tag{2.14}$$

This manipulation on the interface relations simply correspond to a left multiplication of the linear system (2.10) by the following preconditioner:

$$\begin{bmatrix} C^{(1)} & I \\ -C^{(2)} & I \end{bmatrix} \tag{2.15}$$

Different choices can be considered for the matrices $C^{(1)}$ and $C^{(2)}$, but a natural choice consist in $C^{(1)} = A^{(1)}$ and $C^{(2)} = A^{(2)}$. Indeed, with this choice, the constraints on the interface becomes:

$$\lambda^{(1)} + \lambda^{(2)} - (A^{(1)} + A^{(2)})x_p^{(1)} \quad = \quad 0 \tag{2.16}$$
$$\lambda^{(1)} + \lambda^{(2)} - (A^{(1)} + A^{(2)})x_p^{(2)} \quad = \quad 0 \tag{2.17}$$

and the diagonal block of the matrix of the linear system reduces to the identity block:

$$\begin{bmatrix} I & I - (A^{(1)} + A^{(2)})[S^{(2)} + A^{(2)}]^{-1} \\ I - (A^{(1)} + A^{(2)})[S^{(1)} + A^{(1)}]^{-1} & I \end{bmatrix} \begin{bmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{bmatrix} = \begin{bmatrix} (A^{(1)} + A^{(2)})[S^{(2)} + A^{(2)}]^{-1} c_p^{(2)} \\ (A^{(1)} + A^{(2)})[S^{(1)} + A^{(1)}]^{-1} c_p^{(1)} \end{bmatrix} \tag{2.18}$$

3. Optimal Discrete Transmission Conditions. In the context of the additive Schwarz method with no overlap, it is shown in [9], [3] that the best choice for the continuous augmented operators $\mathcal{A}^{(s)}$, $s = 1, 2$ corresponds to the continuous transparent operators, which are not partial differential operators. Different techniques of approximation based on two dimensional Fourier analysis of Steklov-Poincaré operator in an half space have been analyzed in the recent years [4], [1] [2], [7].

In the following, a new analysis is performed directly on the discrete problem and shows that the optimal convergence of a two-Lagrange multiplier algorithm is obtained with a choice of the augmented term $A^{(s)}$, $s = 1, 2$ equal to the complete outer Schur complement. The extension to the case of a one-way splitting is analyzed.

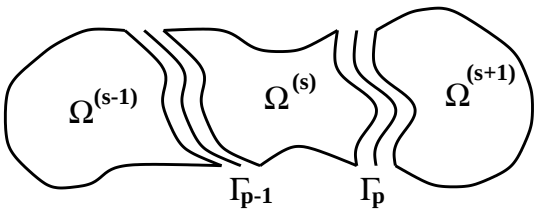

Figure 3.1: One-way decomposition without cross-points

3.1. Two-domain splitting. Eliminating of the inner unknowns of outer subdomain, $x_i^{(q)}$, $q = 1, 2$, $q \neq s$ in sysstem (2.2) leads to:

$$\begin{bmatrix} K_{ii}^{(s)} & K_{ip}^{(s)} \\ K_{pi}^{(s)} & K_{pp}^{(s)} + S^{(q)} \end{bmatrix} \begin{bmatrix} x_i^{(s)} \\ x_p \end{bmatrix} = \begin{bmatrix} b_i^{(s)} \\ b_p^{(s)} + c_p^{(q)} \end{bmatrix} \tag{3.1}$$

where $S^{(q)}$ and $c_p^{(q)}$ denote the Schur complement and condensed right hand side like in section 2.2. Equation (3.1) suggests that the optimal augmented term to add to local admittance matrix $K^{(s)}$ on interface is $S^{(q)}$, since then system (3.1) is similar to the augmented local problem:

$$\begin{bmatrix} K_{ii}^{(s)} & K_{ip}^{(s)} \\ K_{pi}^{(s)} & K_{pp}^{(s)} + A^{(s)} \end{bmatrix} \begin{bmatrix} x_i^{(s)} \\ x_p^{(s)} \end{bmatrix} = \begin{bmatrix} b_i^{(s)} \\ b_p^{(s)} + \lambda^{(s)} \end{bmatrix} \tag{3.2}$$

Theorem 3.1 *In a case of a two-domain splitting, the simple (Jacobi) iterative algorithm for 2-Lagrange multiplier with augmented term equal to the complete outer Schur complement defined as in equation (3.1) converges in one iteration at most.*

Proof. Choosing augmented local terms $A^{(s)} = S^{(q)}$, $s = 1, 2$, $q = 1, 2$, $s \neq q$ makes the matrix of condensed interface system (2.18) equal to identity. ■

3.2. One-way Splitting. Consider a one-way splitting of the domain as in Figure 3.1 and note by subscripts i, $p-1$ and p the degrees of freedom located inside subdomain $\Omega^{(s)}$, on left interface Γ_{p-1} and right interface Γ_p. Then, the contribution of subdomain $\Omega^{(s)}$ to admittance matrix and right-hand side can be written:

$$K^{(s)} = \begin{bmatrix} K_{ii}^{(s)} & K_{ip-1}^{(s)} & K_{ip}^{(s)} \\ K_{p-1i}^{(s)} & K_{p-1p-1}^{(s)} & 0 \\ K_{pi}^{(s)} & 0 & K_{pp}^{(s)} \end{bmatrix} \quad , \quad b^{(s)} = \begin{bmatrix} b_i^{(s)} \\ b_{p-1}^{(s)} \\ b_p^{(s)} \end{bmatrix} \tag{3.3}$$

The global system of equations can be reduced on the interfaces by elimination of inner degrees of freedom. The contribution of subdomain $\Omega^{(s)}$ to the condensed matrix and right-hand side is as follows:

$$\begin{bmatrix} S_{p-1p-1}^{(s)} & S_{p-1p}^{(s)} \\ S_{pp-1}^{(s)} & S_{pp}^{(s)} \end{bmatrix} = \\ = \begin{bmatrix} K_{p-1p-1}^{(s)} - K_{p-1i}^{(s)}[K_{ii}^{(s)}]^{-1}K_{ip-1}^{(s)} & -K_{p-1i}^{(s)}[K_{ii}^{(s)}]^{-1}K_{ip}^{(s)} \\ -K_{pi}^{(s)}[K_{ii}^{(s)}]^{-1}K_{ip-1}^{(s)} & K_{pp}^{(s)} - K_{pi}^{(s)}[K_{ii}^{(s)}]^{-1}K_{ip}^{(s)} \end{bmatrix} \\ \begin{bmatrix} c_{p-1}^{(s)} \\ c_p^{(s)} \end{bmatrix} = \begin{bmatrix} b_{p-1}^{(s)} & - & K_{p-1i}^{(s)} & [K_{ii}^{(s)}]^{-1} & b_i^{(s)} \\ b_p^{(s)} & - & K_{pi}^{(s)} & [K_{ii}^{(s)}]^{-1} & b_i^{(s)} \end{bmatrix} \tag{3.4}$$

The global condensed problem on interfaces is a block 3-diagonal system obtained by assembling local contribution of each subdomain:

$$\begin{bmatrix} ... & ... & 0 & 0 \\ S^{(s-1)}_{p-1p-2} & S^{(s-1)}_{p-1p-1} + S^{(s)}_{p-1p-1} & S^{(s)}_{p-1p} & 0 \\ 0 & S^{(s)}_{pp-1} & S^{(s)}_{pp} + S^{(s+1)}_{pp} & S^{(s+1)}_{pp+1} \\ 0 & 0 & ... & ... \end{bmatrix} , \begin{bmatrix} ... & & \\ c^{(s-1)}_{p-1} & + & c^{(s)}_{p-1} \\ c^{(s)}_{p} & + & c^{(s+1)}_{p} \\ ... & & \end{bmatrix} \quad (3.5)$$

If the system (3.5) is factorized by successive condensation of matrix and right-hand side starting from both ends up to block associated with subdomain $\Omega^{(s)}$, the following final condensed problem is obtained in one subdomain:

$$\begin{bmatrix} S^{-}_{p-1p-1} + S^{(s)}_{p-1p-1} & S^{(s)}_{p-1p} \\ S^{(s)}_{pp-1} & S^{(s)}_{pp} + S^{+}_{pp} \end{bmatrix} \begin{bmatrix} x_{p-1} \\ x_p \end{bmatrix} = \begin{bmatrix} c^{-}_{p-1} & + & c^{(s)}_{p-1} \\ c^{(s)}_{p} & + & c^{+}_{p} \end{bmatrix} \quad (3.6)$$

The condensed right and left blocks and right-hand sides of system (3.6) that are noted with plus and minus super-script are defined by the following recurrence relations:

$$\begin{array}{lcllll} S^{+}_{p-1p-1} & = & S^{(s)}_{p-1p-1} & - & S^{(s)}_{p-1p} & [S^{(s)}_{pp} + S^{+}_{pp}]^{-1} S^{(s)}_{pp-1} \\ S^{-}_{pp} & = & S^{(s)}_{pp} & - & S^{(s)}_{pp-1} & [S^{-}_{p-1p-1} + S^{(s)}_{p-1p-1}]^{-1} S^{(s)}_{p-1p} \\ c^{+}_{p-1} & = & c^{(s)}_{p-1} & - & S^{(s)}_{p-1p} & [S^{(s)}_{pp} + S^{+}_{pp}]^{-1} [c^{(s)}_{p} + c^{+}_{p}] \\ c^{-}_{p} & = & c^{(s)}_{p} & - & S^{(s)}_{pp-1} & [S^{-}_{p-1p-1} + S^{(s)}_{p-1p-1}]^{-1} [c^{-}_{p-1} + c^{(s)}_{p-1}] \end{array} \quad (3.7)$$

Equation (3.6) suggests that the optimal augmented term to add to local admittance matrix $K^{(s)}$ on left or right interface is respectively S^{-}_{p-1p-1} and S^{+}_{pp}, since then, if $\Omega^{(s)}$ is the only subdomain with non zero right-hand side, $c^{-}_{p-1} = 0$ and $c^{+}_{p} = 0$, and system (3.6) is exactly the condensation of the augmented local problem:

$$\begin{bmatrix} K^{(s)}_{ii} & K^{(s)}_{ip-1} & K^{(s)}_{ip} \\ K^{(s)}_{p-1i} & K^{(s)}_{p-1p-1} + S^{-}_{p-1p-1} & 0 \\ K^{(s)}_{pi} & 0 & K^{(s)}_{pp} + S^{+}_{pp} \end{bmatrix} \begin{bmatrix} x^{(s)}_{i} \\ x^{(s)}_{p-1} \\ x^{(s)}_{p} \end{bmatrix} = \begin{bmatrix} b^{(s)}_{i} \\ b^{(s)}_{p-1} \\ b^{(s)}_{p} \end{bmatrix} \quad (3.8)$$

Theorem 3.2 *In a case of a one-way splitting, the simple (Jacobi) iteration algorithm for 2-Lagrange multiplier with augmented term equal to complete outer Schur complement defined as in equation (3.8) converges in (number of subdomain - 1) iterations at most.*

Proof. If $\Omega^{(s)}$ is the only subdomain with non zero right-hand side, equations (3.6) and (3.8) mean that the first iteration with null initial Lagrange multipliers gives exact solution in Ω_s and zero in the other subdomains.
Since λ and x are zero everywhere except on Γ_{p-1} and Γ_p, the initial gradient is non zero on adjacent interfaces only:

$$\begin{array}{l} g^{(s-1)}_{p-1} = \lambda^{(s-1)}_{p-1} + \lambda^{(s)}_{p-1} - (S^{-}_{p-1p-1} + S^{+}_{p-1p-1}) x^{(s)}_{p-1} = -(S^{-}_{p-1p-1} + S^{+}_{p-1p-1}) x^{(s)}_{p-1} \\ g^{(s+1)}_{p} = \lambda^{(s+1)}_{p} + \lambda^{(s)}_{p} - (\ S^{-}_{pp} + S^{+}_{pp}\ )\ \ x^{(s)}_{p} = -(\ S^{-}_{pp} + S^{+}_{pp}\ )\ \ x^{(s)}_{p} \end{array} \quad (3.9)$$

By condensation of equation (3.6) it comes that initial solution on interface Γ_{p-1} satisfies:

$$(S^{-}_{p-1p-1} + S^{(s)}_{p-1p-1} - S_{p-1p}[S^{(s)}_{pp} + S^{+}_{pp}]^{-1} S_{pp-1}) x^{(s)}_{p-1} = \quad (3.10)$$

$$c^{(s)}_{p-1} - S_{p-1p}[S^{(s)}_{pp} + S^{+}_{pp}]^{-1} c^{(s)}_{p} \quad (3.11)$$

Under the assumption that right-hand-side is non zero in Ω_s only, $c_p^{(s)} = c_p^+$. So, from definition of condensed matrices and right-hand sides (3.7) it derives from equation (3.11) that:

$$(S_{p-1p-1}^- + S_{p-1p-1}^+) x_{p-1}^{(s)} = c_{p-1}^+ \tag{3.12}$$

and so:

$$g_{p-1}^{(s-1)} = - (S_{p-1p-1}^- + S_{p-1p-1}^+) x_{p-1}^{(s)} = -c_{p-1}^+ \tag{3.13}$$

A similar result is obtained for $g_p^{(s+1)}$.
The Jacobi algorithm on the condensed interface problem consists in updating λ by $\lambda - g$. So, at the second iteration, both subdomains $s-1$ and $s+1$ will have their complete condensed right-hand side, as well as subdomain s for which $\lambda_{p-1}^{(s)}$ and $\lambda_p^{(s)}$ will remain unchanged and equal to zero. After the second iteration, the solution in the three subdomain will be the exact restriction of the solution of the global problem.
It is easy to see now that, at iteration 2, the situation on interface Γ_{p-2} between subdomains Ω_{s-2} and Ω_{s-1} is exactly the same as, at iteration 1, on interface Γ_{p-1} between subdomains Ω_{s-1} and Ω_s. So, exact condensed right-hand side will be passed to subdomain Ω_{s-2} when updating λ at iteration 2.
On the other hand, if λ is such that in two neighboring subdomains Ω_{s-1} and Ω_s with interface Γ_{p-1}, the local condensed right-hand sides are complete, then $x_{p-1}^{(s)} = x_{p-1}^{(s)} = x_{p-1}$. Condensation on interface Γ_{p-1} of equation (3.6) gives:

$$(S_{p-1p-1}^- + S_{p-1p-1}^+)x_{p-1} = c_{p-1}^- + c_{p-1}^+ \tag{3.14}$$

So, if $\lambda_{p-1}^{(s-1)} + \lambda_{p-1}^{(s)} = c_{p-1}^- + c_{p-1}^+$ then:

$$g_{p-1}^{(s-1)} = g_{p-1}^{(s)} = \lambda_{p-1}^{(s-1)} + \lambda_{p-1}^{(s)} - (S_{p-1p-1}^- + S_{p-1p-1}^+)x_{p-1} = 0 \tag{3.15}$$

This is exactly the situation between Ω_{s-1} and Ω_s as well as between Ω_s and Ω_{s+1} at iteration 2. This means that the gradient will be zero on all the interfaces of these subdomains at iteration 2.
In the same way, it can be proved by recurrence that each Jacobi iteration will propagate the complete condensed right-hand side one subdomain further on the left and on the right while leaving the values of λ unmodified in all subdomains where the condensed right-hand side is already complete.
So, Jacobi method will converge in at most (number of subdomain - 1) iterations if the initial right-hand side is non zero in only one subdomain. As any general right-hand side can be decomposed in the sum of right-hand sides that are non zero in one subdomain only, and since the Jacobi procedure is additive, the same result holds for any case. ■

4. Approximation of Optimal Discrete Transmission Conditions. Unfortunately, the optimal choice derive in the previous section can not be done in practice since the computational cost of the complete Schur complement matrix is too expensive. A first natural step to reduce the cost consists in approximating the complete Schur complement with the Schur complement of the neighboring subdomains. Nevertheless, even with this approximation, the matrix $A^{(s)}$ is still dense and adding it to the local matrix $K^{(s)}$ increases its bandwidth a lot. So, rather than to consider the exact Schur complement, we can consider its approximation with a sparse matrix. One method of choice to approximate this dense matrix is based on the Sparse Approximate Inverse (SPAI) method. It consists in approximating the inverse of a $N \times N$ matrix A by a sparse matrix M which minimizes the Frobenius norm of $AM - I$ [10].
The ultimate step consists in approximating the Schur complement matrix by its first term

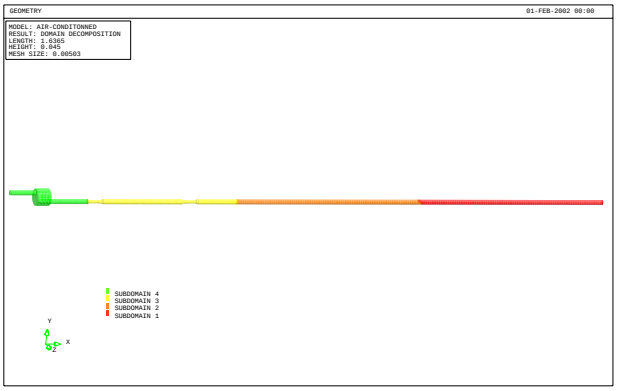

Figure 5.1: Decomposition of the air-cooling tube into four subdomains

Number of Subdomains	Exact Schur Complement		Approximation of Schur Complement		Absorbing Interface Conditions	
	Complete Schur Complement	Neighbor Schur Complement	Sparse Approx.	Lumped Approx.	Taylor	Optimized
2	1	1	12	10	92	86
4	3	4	27	30	155	137
6	5	8	41	46	212	174
8	7	12	56	77	311	247

Table 5.1: Number of iterations for different regularization matrix and different number of subdomains for the air-cooling tube problem

$K_{pp}^{(s)}$ like in the lumped preconditioner for the FETI method [6]. Such an approximation is extremely easy to implement and since $K_{pp}^{(s)}$ and $K_{pp}^{(q)}$ have the same sparse structure, the sparse structure of the local subdomain matrix is not modified.

5. Numerical Experiments. A three dimensional simulation of the noise level distribution in an air-conditionned tube is performed. Figure 5.1 shows the decomposition of the initial mesh into four subdomains. It is important to notice that the interface between the subdomains is irregular. The problem is characterized by a reduced frequency $\omega a = 75.60$ which corresponds, with the relation $\omega = 2\pi F/c$ with c the sound celerity in the fluid, a the length of the tube, to a frequency F of 2500 Hz. The length a is equal to 1.6365 and the diameter to 0.045. When using zeroth order Taylor conditions and a decomposition into 16 subdomains, the method needs 100 iterations to converge, whereas when using the "lumped" approximation of the Schur complement the method converges in 10 iterations. The SPAI approximation gives slightly faster convergence than the lumped for larger number of subdomains. The stopping criterion on the relative global error is set to 10^{-9}. More results are reported Table (5.1).

6. Conclusions. A general algebraic presentation of two-Lagrange multiplier domain decomposition method has been introduced. Optimal transmission conditions have been derived from this algebraic analysis. Since the optimal augmented operator in a subdomain is the Schur complement of the outer domain, it is not possible to compute it in practice. Promising results have been obtained using simple approximation techniques for this Schur

complement. The key issue to improve the method presented in this paper lies in the design of good sparse approximation method.

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