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12. Partition of Unity Coarse Spaces: Enhanced Versions, Discontinuous Coefficients and Applications to Elasticity

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1. Introduction. In this paper, we consider overlapping Schwarz methods for finite element discretizations for certain elliptic problems. In order to make Schwarz methods scalable with respect to the number of subdomains, we add an appropriate coarse problem to the algorithms. The main purpose of this paper is to introduce new coarse spaces for overlapping Schwarz methods. The proposed coarse spaces are based on partitions of unity and on local functions of low energy. The set of local functions must contain the kernel of the discrete operator when restricted to the overlapping subdomains. For instance, for linear elasticity, it must include the local rigid body motions, and for Poisson equation it should include the constant function. We consider several classes of choices of partition of unity. We consider PU based on the kind of partition of unity used in the theoretical analysis of standard additive Schwarz methods, as well as PU based on the class of additive Schwarz methods based on harmonic extensions. The condition number of the algorithms grows only linearly or quasi-linearly with respect to the relative size of the overlap. And for certain choices of partition of unity, the methods are robust also with respect to jumps of coefficients.

Work on two-level methods on unstructured meshes is not new. Several different approaches have been introduced and some can be found in [4, 5, 7, 9, 8, 11, 12, 14, 17, 16, 1, 18, 22, 21] and papers cited therein. Related works to ours [6, 20, 19], based on two-level agglomeration techniques, can be found in [4, 14, 15]. Their algorithms and analysis use a class of partition of unity coarse space based on agglomeration smoothing techniques. In this paper, we consider coarse spaces that combine the partition of unity and low energy functions associated to the (overlapping) subdomains in order to design a new coarse spaces for elliptic problems. The partition of unity is used: 1) to localize the coarse basis functions to the subdomains, and 2) to force the coarse basis functions to have a smooth decaying to zero near the boundary of the subdomains. The low energy local functions associated to the subdomains allow us to have good approximation properties for the coarse space. The proposed coarse spaces, have several advantages over traditional coarse spaces: 1) it is applicable to discretizations on unstructured meshes, 2) it is algebraic (see below), 3) the associated algorithms do not require that the subdomains be connected or that the boundary of the subdomains be smooth, 4) the coarse basis functions of the PU coarse space are constructed explicitly and without the use of exact local solvers, 5) the stencil of the coarse matrix is sparser than the traditional ones, and 6) the support of the coarse basis functions is localized on the subdomains, and therefore easy in communication if implemented in a distributed memory parallel machine.

The preconditioners to be considered here are algebraic in the sense that the preconditioners are built in terms of the graph of the sparse matrix and the mesh partition. In this paper we provide some unified mathematical analysis to the finite element problems considered in this paper.

2. Elliptic Problems and Discretization. In this paper we consider two problems: the two-dimensional linear elasticity and the scalar transmission problem.

2.1. Linear Elasticity. We consider an isotropic elastic material in the configuration region $\Omega \subset \Re^2$. Let us denote $u^* = (u_1^*, u_2^*)^t$ to be the displacement and $f = (f_1, f_2)^t$ the body

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force. Let the region Σ be Ω or a subregion of Ω, and the spaces $\vec{H}^1(\Sigma)$, $\vec{L}_2(\Sigma)$, and $\vec{H}_0^1(\Sigma)$ to be the spaces $(H^1(\Sigma), H^1(\Sigma))^t$, $(L_2(\Sigma), L_2(\Sigma))^t$, and $(H_0^1(\Sigma), H_0^1(\Sigma))^t$, respectively. The weak formulation of the static theory of linear elasticity with zero boundary displacement condition is given as follows:

Find $u^* \in \vec{H}_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \quad \forall v \in \vec{H}_0^1(\Omega), \tag{2.1}$$

where

$$a(u^*, v) = \int_\Omega (\mu E(u^*) : E(v) + \lambda \text{div}(u^*) \, \text{div}(v)) \; dx \, ,$$

$$f(v) = \int_\Omega f \cdot v \, dx \;\; \text{for} \;\; f \in \vec{L}^2(\Omega),$$

and

$$E(v) = \frac{1}{2} \left(\nabla v + (\nabla v)^t \right) .$$

The positive constants μ and λ are called the Lamé constants. It is well-known that $a(\cdot, \cdot)$ is elliptic and bounded, and therefore the system (1.1) is well posed [2, 3].

For simplicity, let Ω be a bounded polygonal region in $\Re^2$ with a diameter of size $O(1)$. The extension of the results to $\Re^3$ can also be carried out using similar ideas. Let $T^h(\Omega)$ be a shape regular, quasi-uniform triangulation of grid size $O(h)$ of Ω, and $V \subset \vec{H}_0^1(\Omega)$ be the finite element space consisting of continuous piecewise linear functions associated with the triangulation $T^h(\Omega)$ and zero Dirichlet boundary condition. The extension of the results for the case of local quasi-uniform triangulation is also straightforward.

We are interested in solving the discrete problem associated to (1.1): Find $u \in V$ such that

$$a(u, v) = f(v), \quad \forall \; v \in V. \tag{2.2}$$

Since $V \subset \vec{H}_0^1(\Omega)$, the discrete version is also well-posed.

2.2. Transmission Problem. We also consider a finite element problem, the scalar transmission problem with zero Dirichlet boundary condition. Find $u^* \in H_0^1(\Omega)$, such that

$$a(u^*, v) = f(v), \quad \forall \; v \in H_0^1(\Omega), \tag{2.3}$$

where now

$$a(u^*, v) = \int_\Omega \rho(x) \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) = \int_\Omega f v \, dx \;\; \text{for} \;\; f \in L^2(\Omega).$$

We assume the coefficient ρ satisfy $0 < \rho_{\min} \le \rho(x) \le \rho_{\max}$ and is constant and equal to ρ_i inside each substructure Ω_i. We allow the coefficient ρ to have highly discontinuity across substructures. Here, we let $V \subset H_0^1(\Omega)$ be the finite element space consisting of continuous piecewise linear functions associated with the triangulation $\mathcal{T}_h(\Omega)$ with zero Dirichlet boundary condition. We introduce the discrete problem (2.2) with the $a(\cdot, \cdot)$ and V given in this subsection.

Throughout this paper, C is positive generic constant that do not depend of any of the mesh parameters, the number of subdomains, and the parameters λ and μ. All the domains and subdomains are assumed to be open; i.e., boundaries are not included in their definitions. We will use a unified notation for both problems since the techniques used to design and analyze the algorithms are essentially the same.

3. Algebraic Subregions. Given the domain Ω and the triangulation $T^h(\Omega)$, we assume that a domain partition has been applied and resulted in N substructures (non-overlapping subregions) $\Omega_i, i = 1, \dots N$, of size $O(H)$, such that

$$\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i, \qquad \Omega_i \cap \Omega_j = \emptyset, \quad \text{for} \quad j \neq i.$$

We define the overlapping subdomains Ω_i^δ as follows: Let Ω_i^1 be the one-overlap element extension of Ω_i, where $\Omega_i^1 \supset \Omega_i$ is obtained by including all the immediate neighboring elements $\tau_h \in T^h(\Omega)$ of Ω_i such that $\overline{\tau}_h \cap \overline{\Omega}_i \neq \emptyset$. Using the idea recursively, we can define a δ-extension overlapping subdomains Ω_i^δ

$$\Omega_i \subset \Omega_i^1 \subset \cdots \Omega_i^\delta.$$

Here the integer $\delta \geq 1$ indicates the level of element extension and δh is the approximate length of the extension. We note that this extension can be coded easily through the knowledge of the adjacent matrix associated to the mesh and the partition.

We want to design coarse spaces based on a class of partition of unity for which has been used as a very powerful tool in the theoretical analysis of Schwarz type domain decomposition methods. We note however that the partition of unity functions on this class do not necessarily vanish on $\partial\Omega$. Hence, they cannot be used straightforwardly as coarse basis functions since they should satisfy zero Dirichlet boundary conditions for Dirichlet type boundary problems. Hence, for the coarse basis functions that touch $\partial\Omega$, we modify them so that they have a controlled decaying to zero near $\partial\Omega$. To obtain such coarse basis functions, we next introduce a Dirichlet boundary treatment. Let Ω_B^1 be one layer of elements near the Dirichlet boundary $\partial\Omega$ and then define recursively,

$$\Omega_B^1 \subset \Omega_B^2 \cdots \Omega_B^\delta$$

with δ levels of extension by adding recursively neighboring elements.

To define and analyze the new methods, we introduce some notations. We subdivide Ω_i^δ as follow: Let $\gamma_i^\delta = \partial\Omega_i^\delta \backslash \partial\Omega, i = 1, \cdots, N$; i.e., the part of the boundary of Ω_i^δ that does not belong to the physical boundary of Ω, and let $\gamma_B^\delta = \partial\Omega_B^\delta \backslash \partial\Omega$. We define the interface overlapping boundary Γ^δ as the union of all the γ_i^δ and γ_B^δ; i.e., $\Gamma^\delta = \cup_{i=B,1}^N \gamma_i^\delta$. We also need the following subsets of Ω_i^δ

- $\Gamma_i^\delta = \Gamma^\delta \cap \Omega_i^\delta$ (local interface)
- $N_i^\delta = \Omega_i^\delta \backslash (\cup_{j \neq i} \Omega_j^\delta \cup \Omega_B^\delta \cup \Gamma_i^\delta)$ (non-overlapping region)
- $O_i^\delta = \Omega_i^\delta \backslash (N_i^\delta \cup \Gamma_i^\delta)$ (overlapping region)

In order to consider discontinuous coefficients for the scalar transmission problem, we introduce the following notations. Let the interior local interface $\Gamma_i^0 = \Gamma_i^\delta \cap \Omega_i$ be the part of Γ_i^δ which is inside of substructure Ω_i. We also introduce complementary local interface $\Gamma_i^c = \Gamma_i^\delta \backslash \overline{\Gamma}_i^0$. We note for later use that nodes on $\overline{\Gamma}_i^0 \cap \overline{\Gamma}_i^c$ also belong to $\partial\Omega_i \backslash \partial\Omega$. We subdivide the regions O_i^δ into subregions $O_{ij}^\delta = O_i^\delta \cap \Omega_j$ where the substructures Ω_j are neighbors of Ω_i. We note that the coefficient ρ is constant inside each of the subregion O_{ij}^δ and N_i^δ.

4. Additive Overlapping Schwarz Methods. We next describe the PU coarse spaces and introduce the corresponding overlapping additive Schwarz method and a hybrid Schwarz method. We first consider the local problems.

4.1. The Local Problems. We introduce local spaces as

$$V_i^\delta = V \cap \vec{H}_0^1(\Omega_i^\delta) \quad \text{or} \quad (V_i^\delta = V \cap H_0^1(\Omega_i^\delta)) \qquad i = 1, 2, \cdots, N,$$

extended by zero to $\Omega \backslash \Omega_i^\delta$. It is easy to verify that

$$V = V_1^\delta + V_2^\delta + \cdots + V_N^\delta. \tag{4.1}$$

The property (4.1) gives robustness for the preconditioners defined in this paper in the sense that a convergence is always attained independently of the quality of the partitioning. The coarse space is only introduced to accelerate the convergence of the iterative method. This is an advantage over some iterative substructuring methods in which are based on the requirement that all the substructures Ω_i must be connected. We point out that the space decomposition given by (4.1) is not a direct sum if $\delta > 1$. This increases robustness for the methods when the Ω_i have rough (*zigzag*) boundaries. By extending the substructures to Ω_i^δ, we allow the possibility of decomposing a function of V as a sum of functions of V_i^δ without the *zigzag* behavior. So it is possible to obtain low energy decompositions, and hence better lower bounds for the condition number of the preconditioners.

We define the local projections (local problems) $P_i^\delta : V \rightarrow V_i^\delta$ as follows:

$$a(P_i^\delta u, v) = a(u, v), \quad \forall v \in V_i^\delta. \tag{4.2}$$

We next introduce the PU coarse space V_0^δ for the linear elasticity and for the transmission problem.

4.2. Partition of Unity. We next construct a partition of unity θ_i^δ such that $\theta_i^\delta \in V_i^\delta$, $0 \leq \theta_i^\delta(x) \leq 1$, $|\nabla \theta_i^\delta(x)| \leq C/(\delta h)$ in the interior of the elements, and $\sum_{i=B,1}^N \theta_i^\delta \equiv 1$. Such construction is natural, algebraic and easy to implement. We first construct the function $\hat{\theta}_B^\delta \in V_i^\delta$ as follows. We let $\hat{\theta}_B^\delta(x) = 1$ for nodes x on $\partial\Omega$. For the first layer of neighboring nodes x of $\partial\Omega$ we let $\hat{\theta}_B^\delta(x) = (\delta - 1)/\delta$. For the second layer of neighboring nodes x of $\partial\Omega$ we let $\hat{\theta}_B^\delta(x) = (\delta - 2)/\delta$, and recursively until $k = \delta - 1$, we let $\hat{\theta}_B^\delta(x) = (\delta - k)/\delta$ for the $(k)st$ layer of neighboring nodes x of $\partial\Omega$. For the reminaing nodes x of $\overline{\Omega}$ we let $\hat{\theta}_B^\delta(x) = 0$. Similarly, for $i = 1, \cdots, N$, we let $\hat{\theta}_i^\delta(x) = 1$ for nodes x of $\overline{\Omega}_i \backslash \Omega_B^\delta$. For the first layer of neighboring nodes x of $\overline{\Omega}_i \backslash \Omega_B^\delta$ we let $\hat{\theta}_i^\delta(x) = (\delta - 1)/\delta$, and recursively until $k = \delta - 1$, we let $\hat{\theta}_i^\delta(x) = (\delta - k)/\delta$ for the $(k)st$ layer of neighboring nodes x of $\overline{\Omega}_i \backslash \Omega_B^\delta$. For the remaining nodes x of $\overline{\Omega}$ we let $\hat{\theta}_i^\delta(x) = 0$. It is easy to verify that $0 \leq \hat{\theta}_i^\delta(x) \leq 1$, and for quasi-uniform triangulation $|\nabla \hat{\theta}_i^\delta(x)| \leq C/(\delta h)$ in the interior of the elements. The partition of unity θ_i^δ is defined as

$$\theta_i^\delta = I_h\left(\frac{\hat{\theta}_i^\delta}{\sum_{j=B,1}^N \hat{\theta}_j^\delta}\right).$$

Here I_h is the regular pointwise linear interpolation operator from the continuous functions to piecewise linear and continuous functions. It is easy to verify that $\sum_{i=B,1}^N \theta_i^\delta(x) = 1$, $0 \leq \theta_i^\delta(x) \leq 1$, and $|\nabla \theta_i^\delta(x)| \leq C/(\delta h), \forall x \in \overline{\Omega}$.

4.3. PU Coarse Space for Linear Elasticity. We next consider the key ingredient for designing the coarse space for linear elasticity: the local rigid body motions. We let

$$R\vec{M}(\Sigma) = \{v \in \vec{L}_2(\Sigma) : v = c + b(x_2, -x_1)^t, c \in \Re^2, b \in \Re\}$$

be the space of rigid body motions functions on Σ. An important property of the space $R\vec{M}(\Sigma)$, and which plays an important role in the design and analysis of the algorithms, is described as follows. If $v \in R\vec{M}(\Sigma)$ then $a_\Sigma(v, v) \equiv 0$. In addition, in certain extent the converse is also true; if $a_\Sigma(v,v) \equiv 0$ and Σ is connected, then $v \in \vec{RM}(\Sigma)$. Here, the bilinear form $a_\Sigma(\cdot,\cdot)$ on $\vec{H}^1(\Sigma) \times \vec{H}^1(\Sigma)$ is given by

$$a_\Sigma(u,v) = \int_\Sigma (2\mu E(u) : E(v) + \lambda \mathrm{div}(u)\, \mathrm{div}(v))\ dx.$$

A PU coarse space V_0^δ is defined as

$$V_0^\delta = \{\sum_{i=1}^N \vec{I}_h \left(R\vec{M}(\Omega_i^\delta)\theta_i^\delta \right)\} = \{\sum_{i=1}^N \vec{I}_h \left([c_i + b_i(x_2, -x_1)^t]\theta_i^\delta \right), \forall c_i \in \Re^2, \forall b_i \in \Re\}. \quad (4.3)$$

Here, the interpolator $\vec{I}_h = (I_h, I_h)^t$ is the regular componentwise pointwise linear interpolation operator. We note that we do not include the θ_B^δ in the sum of (4.3), and therefore the number of degrees of freedom of V_0^δ is $3N$. The function θ_B^δ is needed only to define the others functions θ_i^δ, $i = 1, \cdots, N$.

We define the global projection (global problem) $P_0^\delta : V \to V_0^\delta$ as follows:

$$a(P_0^\delta u, v) = a(u,v), \quad \forall v \in V_0^\delta. \quad (4.4)$$

4.4. PU Coarse Space for the Scalar Transmission Equation. The PU coarse space V_0^δ for the transmission problem is defined as

$$V_0^\delta = \{\sum_{i=1}^N I_h(c_i\theta_i^\delta), \forall c_i \in \Re\},$$

and the global projection P_0^δ by (4.4), where of course V and $a(\cdot,\cdot)$ are the ones related to the transmission. We note here also as in the elasticity case, the constant functions c_i are the kernel of the operator $a_{\Omega_i^\delta}(\cdot,\cdot)$.

4.5. Enhanced PU Coarse Spaces. We can also make richer the coarse spaces V_0^δ. We do this by redefining V_0^δ as

$$V_0^\delta = \{\sum_{i=1}^N \vec{I}_h \left([c_i + b_i(x_2, -x_1)^t + f_i(x)]\theta_i^\delta \right), \forall c_i \in \Re^2, \forall b_i \in \Re, f_i \in V_i^E(\Omega_i^\delta)\},$$

for the finite elasticity problem, and

$$V_0^\delta = \{\sum_{i=1}^N I_h \left([c_i + f_i(x)]\theta_i^\delta \right), \forall c_i \in \Re, \forall f_i \in V_i^E\}$$

for the scalar transmission problem. For each subdomain Ω_i^δ, we let the space $V_i^E(\Omega_i^\delta)$ be defined as the vector space generated by few lowest finite element eigenmodes associated to operator $a_{\Omega_i^\delta}(\cdot,\cdot)$ without assuming Dirichlet boundary condition on $\partial\Omega_i^\delta$. Another possibility and also cheaper to construct is to choose $V_i^E(\Omega_i^\delta)$ as the vector space of polynomial functions of small degrees.

4.6. Preconditioners. We consider two preconditioners:

- The two-level overlapping additive Schwarz operator [10] given by

$$P_{as}^\delta = \sum_{i=0}^N P_i^\delta,$$

- The hybrid Schwarz operator [16, 13] given by

$$P_{hyb}^\delta = P_0^\delta + (I - P_0^\delta)(\sum_{i=1}^N P_i^\delta)(I - P_0^\delta).$$

4.7. Condition Number. It is possible to show that the solution of (2.2) is the solution of the preconditioned system $P_{as}^{\delta}u = g_{as}$ ($P_{hyb}^{\delta}u = g_{hyb}$), for an appropriate right hand side g_{as} (g_{hyb}); see [13]. These preconditioned systems are typically solved by the conjugate gradient method, without further preconditioning, using $a(\cdot,\cdot)$ as the inner product. The preconditioned systems presented in this paper are applicable to any unstructured mesh and partitioning. The notions of subdomains, the classification of the regions O_i^{δ} and N_i^{δ} and the interfaces Γ_i^{δ}, etc., can all be defined in terms of the graph of the sparse matrix. The two algorithms (preconditioners) will converge even if the substructures Ω_i are not connected. For the next theoretical result [19, 20], we assume that the substructures Ω_i have nice aspect ratios and are connected.

Theorem 4.1 *There exists a constant* $C > 0$ *such that*

- *Linear Elasticity*

$$\kappa(P_{hyb}^{\delta}) \le \kappa(P_{as}^{\delta}) \le C(1+\lambda/\mu)(1+\frac{H}{\delta h}). \tag{4.5}$$

- *Transmission Problem*

$$\kappa(P_{hyb}^{\delta}) \le \kappa(P_{as}^{\delta}) \le Cc(\rho)(1+\frac{H}{\delta h}). \tag{4.6}$$

The constant C *does not depend on* h, δ, H, λ, *and* μ. *The constant* $c(\rho) \le C\max_{ij}\frac{\rho_i}{\rho_j}$, *where the pairs* ij *run over all* ij *combinations such that* $\overline{\Omega}_i \cap \overline{\Omega}_j \neq \emptyset$.

We note that the discretization considered in this paper gives satisfactory (second order accurate) convergent finite element approximation to the elasticity problem when λ/μ is not large. It can be shown [2, 3] that the a priori error estimate of this finite element method deteriorates as $\lambda \gg \mu$; this phenomenon is called *locking effect* or *volume locking.* We note that the upper bound estimate of the preconditioners presented here also follows similar patterns. Here also, we cannot remove the λ/μ dependence on the upper bound estimates for the conditioning number of the preconditioned systems. To see this we use the following arguments: If $\mathrm{div}(u) = 0$ and λ is close to ∞, the only way to obtain a decomposition stable with respect to λ is to have all the $\mathrm{div}(u_i) = 0$. However, it is easy to see that $\mathrm{div}(u_0) = 0$ implies that u_0 vanishes. Hence, there is no global communication and therefore the condition number must have a H dependence on the upper bound estimation. Hence fortunately, the preconditioners considered here in this paper are effective exactly when the discretization is accurate. For incompressible ($\lambda = \infty$) or almost incompressible materials, other discretizations based on hybrid or non-conforming finite elements approximations [2, 3] are more appropriate and they will not be considered here.

For the transmission problem, the upper bound (4.6) is satisfactory if the jumps on the coefficient ρ are moderate. Later in the paper we design better coarse spaces for highly discontinuity in the coefficients.

5. AS Methods with Harmonic Overlap (ASHO). We next introduce the PU coarse spaces for the ASHO methods.

5.1. Local Problems for the ASHO Methods. We define $\widetilde{V}_i^{\delta}$ as the subspace of V_i^{δ} consisting of functions that are discrete harmonic at all nodes interior to O_i^{δ}, i.e. $u \in \widetilde{V}_i^{\delta}$, if for all nodes $x_k \in O_i^{\delta}$,

$$a(u, \phi_{x_k}) = 0.$$

Here, $\phi_{x_k} \in V$ is the regular componentwise finite element basis function associated with a node x_k.

We define $\widetilde{V}^{\delta}$ as a subspace of V defined as

$$\widetilde{V}^{\delta} = \widetilde{V}_1^{\delta} + \widetilde{V}_2^{\delta} + \cdots + \widetilde{V}_N^{\delta}.$$

We note that the above sum is not a direct sum and $\widetilde{V}_i^\delta \subset V_i^\delta$. We define $\widetilde{P}_i^\delta : \widetilde{V}^\delta \to \widetilde{V}_i^\delta$ to be the projection operators such that, for any $u \in \widetilde{V}^\delta$

$$a(\widetilde{P}_i^\delta u, v) = a(u, v), \quad \forall v \in \widetilde{V}_i^\delta.$$

We next introduce a PU coarse space $\widetilde{V}_0^\delta$ for the ASHO method.

5.2. A PU Coarse Space for ASHO Methods. For the finite elasticity, we define the PU coarse space $\widetilde{V}_0^\delta \subset \widetilde{V}^\delta$, by simply modifying the basis functions

$$\varphi_i^\delta = \vec{I}_h \left([c_i + b_i(x_2, -x_1)^t]\theta_i^\delta \right) \quad \text{or} \quad (\varphi_i^\delta = c_i \theta_i^\delta)$$

to $\widetilde{\varphi}_i^\delta$. The $\widetilde{\varphi}_i^\delta$ are defined to be equal to the φ_i^δ except on O_i^δ. On O_i^δ we make the $\widetilde{\varphi}_i^\delta$ discrete harmonic in the $a(\cdot,\cdot)$ inner product. The PU coarse space $\widetilde{V}_0^\delta$ is defined as the linear combination of the coarse basis functions $\widetilde{\varphi}_i^\delta, i = 1, \cdots, N$. We introduce $\widetilde{P}_0 : \widetilde{V}^\delta \to \widetilde{V}_0^\delta$ as the operator such that, for any $u \in \widetilde{V}^\delta$,

$$a(\widetilde{P}_0^\delta u, v) = a(u, v), \quad \forall v \in \widetilde{V}_0^\delta. \tag{5.1}$$

Then, the two-level additive and hybrid ASHO with the PU coarse problem $\widetilde{P}_0^\delta$ are defined as

$$\widetilde{P}_{as}^\delta = \sum_{i=0}^N \widetilde{P}_i^\delta, \quad \text{and} \quad \widetilde{P}_{hyb}^\delta = \widetilde{P}_0^\delta + (I - \widetilde{P}_0^\delta)(\sum_{i=1}^N \widetilde{P}_i^\delta)(I - \widetilde{P}_0^\delta).$$

The following bounds can be obtained [19].

Theorem 5.1 *On the space $\widetilde{V}_\delta$, we have*

$$\kappa(\widetilde{P}_{hyb}^\delta) \le \kappa(\widetilde{P}_{as}^\delta) \le \kappa(P_{as}^\delta)$$

5.3. A Robust PU Coarse Space for ASHO Methods. We next construct the coarse basis functions $\widetilde{\varphi}_i^\delta$ that make the ASHO methods robust with respect to the jumps of the coefficients ρ.

We redefine $\widetilde{\varphi}_i^\delta \in \widetilde{V}_i^\delta$ as follows. Nodes x_k on $(\Gamma_i^0 \cup N_i^\delta$, we define $\widetilde{\varphi}_i^\delta(x_k) = 1$. Nodes x_k on Γ_i^c, we let $\widetilde{\varphi}_i^\delta(x_k) = 0$. A node x_k on $\overline{\Gamma}_i^0 \cap \overline{\Gamma}_i^c$ also belongs to the $\partial\Omega_i \backslash \partial\Omega$. Hence, x_k belongs to $\overline{\Omega}_i$ and to some neighboring substructures $\overline{\Omega}_j$ and we define

$$\widetilde{\varphi}_i^\delta(x_k) = \frac{\rho_i^\beta}{\rho_i^\beta + \sum_j \rho_j^\beta},$$

where $\beta \ge 1/2$. Nodes x_k on $\overline{\Omega} \backslash \Omega_i^\delta$ we let $\widetilde{\varphi}_i^\delta(x_k) = 0$. It remains only to define $\widetilde{\varphi}_i^\delta(x_k)$ at nodes in O_i^δ. There, we make $\widetilde{\varphi}_i^\delta$ to be discrete harmonic on the $a(\cdot,\cdot)$ inner product.

Theorem 5.2 *On the space $\widetilde{V}^\delta$ we have*

$$\kappa(\widetilde{P}_{hyb}^\delta) \le \kappa(\widetilde{P}_{as}^\delta) \le C \left(1 + \frac{H}{\delta h} + \log(\frac{H}{\delta h}) \log(\delta) \right).$$

The constant C does not depend on h, δ, H, and ρ.

Proof. We here give a sketch of the proof. We define $\hat{\varphi}_i^\delta \in V_i^\delta$ as follows. Nodes x_k on $\Gamma_i^0 \cup N_i^\delta$, we define $\hat{\varphi}_i^\delta(x_k) = 1$. Nodes x_k on Γ_i^c, we let $\hat{\varphi}_i^\delta(x_k) = 0$. Nodes x_k on $\partial\Omega_i \backslash \partial\Omega$ we define

$$\hat{\varphi}_i^\delta(x_k) = \frac{\rho_i^\beta}{\rho_i^\beta + \sum_j \rho_j^\beta},$$

where $\beta \geq 1/2$, where the indices $j \neq i$ are the domains Ω_j for which $x_k \in (\partial\Omega_j \backslash \partial\Omega)$. Nodes x_k on $\overline{\Omega} \backslash \Omega_i^\delta$ we let $\hat{\varphi}_i^\delta(x_k) = 0$. It remains only to define $\hat{\varphi}_i^\delta(x_k)$ at nodes on the O_{ij}^δ. There, we make $\hat{\varphi}_i^\delta$ to be discrete harmonic.

There is an important distinction between the functions $\hat{\varphi}_i^\delta$ and $\widetilde{\varphi}_i^\delta$. The function $\hat{\varphi}_i^\delta$ is discrete harmonic on the regions O_{ij}^δ while the function $\widetilde{\varphi}_i^\delta$ is discrete harmonic (in the $a(\cdot,\cdot)$ inner product) on the region O_i^δ. We note that in each region O_{ij}^δ, the coefficient ρ is constant and therefore $\hat{\varphi}_i^\delta$ is discrete harmonic (in the H_1-seminorm) on the regions O_{ij}^δ. Because ρ is constant on the regions O_{ij}^δ we can borrow several previous results developed for RASHO [6] and for elliptic problems with discontinuous coefficients [9, 17, 12, 18, 8] to obtain

$$\kappa(\hat{P}_{as}) \leq C\left(1 + \frac{H}{\delta h} + \log(\frac{H}{\delta h})\log(\delta)\right),$$

where

$$\hat{P}_{as} = \hat{P}_0^\delta + \sum_{i=1}^{N} P_i^\delta,$$

and the global projection $\hat{P}_0^\delta : V \rightarrow \hat{V}_0^\delta$ is defined as

$$a(\hat{P}_0^\delta u, v) = a(u, v), \quad \forall v \in \hat{V}_0^\delta.$$

Here, the coarse space $\hat{V}_0^\delta$ is the space generated by the coarse basis functions $\hat{\varphi}_i^\delta$. Finaly, we use similar arguments as in [20], where we use that a function on $\widetilde{V}_i^\delta$ has smaller or equal $a(\cdot,\cdot)$ norm than a function on V_i^δ with the same values on Γ_i^δ, to obtain

$$\kappa(\widetilde{P}_{as}) \leq \kappa(\hat{P}_{as}).$$

■

6. Remarks about ASHO Methods.

We next show that the explicit elimination of the variables associated with the overlapping nodes is not needed in order to apply $\widetilde{P}^\delta$ to any given vector $v \in \widetilde{V}^\delta$.

Lemma 6.1 *For any $u \in \widetilde{V}^\delta$, we have*

$$\widetilde{P}_i^\delta u = P_i^\delta u, \quad i = 1, \cdots, N.$$

Proof. If $u \in \widetilde{V}^\delta$ then

$$a(P_i^\delta u, \phi_{x_k}) = a(u, \phi_{x_k}) = 0, \quad \forall x_k \in O_i^\delta.$$

Hence, $P_i^\delta u \in \widetilde{V}_i^\delta$. Here, $\phi_{x_k} \in V_i^\delta$ are the regular basis functions associated to the nodes x_k. To complete the proof of the lemma, we just need to verify that

$$a(P_i^\delta u, v) = a(u, v), \quad \forall v \in \widetilde{V}_i^\delta. \tag{6.1}$$

To verify (6.1), we use the definition of P_i^δ (4.2) and that $\widetilde{V}_i^\delta$ is a subset of V^δ. ■

We note that the solution u of (2.2) is not in the subspace $\widetilde{V}^\delta$, therefore, the operators $\widetilde{P}_{as}^\delta$ and $\widetilde{P}_{hyb}^\delta$ cannot be used to solve the linear system (2.2) directly. We will need to modify the right-hand side of this system. A reformulated problem will be presented in Lemma 6.2 below. Using the matrix notations, the next lemma shows how to modify the system (2.2) so that its solution belongs to $\widetilde{V}^\delta$. Let $O^\delta = \cup_i O_i^\delta$. Let W_O^δ be the set of nodes associated to the degree of freedom of V^δ in O^δ. We define the restriction operator, or a matrix, R_{O^δ}: $W \to W$ as follows

$$(R_{O^\delta} v)(x_k) = \begin{cases} v_k & \text{if } x_k \in W_{O^\delta} \\ \\ 0 & \text{otherwise.} \end{cases}$$

The matrix representation of R_O^δ is given by a diagonal matrix with 1 for nodal points in the interior of O^δ and zero for the remaining nodal points. We denote by A the matrix associated to the problem (2.2). Using the restriction operator R_O^δ, we define the subdomain stiffness matrix as

$$A_{O^\delta} = R_{O^\delta}\ A\ R_{O^\delta}^T,$$

which can also be obtained by the discretization of the original finite element problem on O^δ with zero Dirichlet data on ∂O^δ and extended by zero outside of O^δ. We remark that O is a disconnected region where $\partial O = \Gamma_i^\delta \cup \partial\Omega$. Therefore, $A_{O^\delta} w = f$ can be solved locally and inexpensively.

It is easy to see that the following lemma holds; see [6].

Lemma 6.2 *Let u and f be the exact solution and the right-hand side of (2.2), and*

$$w = R_{O^\delta}^T A_{O^\delta}^+ R_{O^\delta} f. \tag{6.2}$$

Then $\widetilde{u} = u - w \in \widetilde{V}^\delta$ and satisfies the following modified linear system of equations

$$A\widetilde{u}^* = f - Aw = \widetilde{f}.$$

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