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50. A domain decomposition algorithm for nonlinear interface problem

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1. Introduction. In this paper, we are interested in the numerical solution of a nonlinear elliptic problem by a nonoverlapping domain decomposition technique. The model problem under consideration takes the standard form

For a given $f \in L^2(\Omega)$, find $u \in V$ such that

$$\sum_i \Big\{ \int_{\Omega_i} \nabla u \cdot \nabla v \, dx \; + \int_{\Omega_i} (u_i^3 - f) v \, dx \Big\} \; = \; 0 \quad , \quad \forall \; v \in V \; , \tag{1.1}$$

where V is the usual Sobolev space

$$V = \left\{ v \in H^1(\Omega) \; , \; v = 0 \text{ on } \partial\Omega_D \right\} \; ,$$

defined over a given domain $\Omega = \cup_{i=1}^N \Omega_i$ of $\mathbb{R}^2$.
In any case, even if Ω is partitioned into nonoverlapping subdomains Ω_i (see Figure 4.1), the nonlinear problem (1.1) is not reduced to independent subproblems set on each subdomain Ω_i because elements of the space V are constrained to be continuous across the different interfaces $\partial\Omega_i \cap \partial\Omega_j$. Most nonoverlapping domain decomposition techniques handle this constraint by a standard Newton's algorithm in which all linearized subproblems are solved by iterative substructuring methods (see [4], [5]).

The purpose of this paper is to propose and study another numerical strategy well adapted to nonlinear problems. The resulting discrete problem of (1.1) is reduced to an interface problem via a nonlinear Steklov-Poincaré operator [8]. Modified Newton iterations are used to treat the nonlinear aspect of the interface problem. We extend the results obtained in [7] to the case of multidomain decomposition. We prove that this algorithm converges independently of the discretization step h. Numerical results are given to illustrate the efficiency of this approach. Moreover, the proposed algorithm is compared to the so called Newton conjugate gradient algorithm introduced in [4].

2. A generalized nonlinear interface problem. We begin with some notation used hereafter. Let us introduce the boundaries (see Figure 4.1)

$$\begin{aligned} \partial\Omega &= \Gamma_D \cup \Gamma_N \; , \quad \text{external Dirichlet and Neumann boundaries,} \\ \partial\Omega_{D_i} &= \Gamma_D \cap \partial\Omega_i \; , \quad \text{local Dirichlet boundary,} \\ \Gamma_i &= \partial\Omega_i \setminus \partial\Omega, \quad \text{local interface,} \\ \Gamma &= \cup_i \Gamma_i, \quad \text{global interface,} \end{aligned}$$

with $\Gamma_D \neq \{\emptyset\}$. The global interface Γ is made of N_f faces Γ_{ij} separating the domain Ω_i from the domain Ω_j. In this decomposition, we neglect corners. This is ligitimate if there are no corners (partition in strip) or if the interfaces are discretized by *mortar* elements which define discrete traces on faces and not on corners [9].
On this geometry, we introduce the spaces

$$V_i = \left\{ v \in H^1(\Omega_i) \; , \; v = 0 \text{ on } \partial\Omega_{D_i} \right\}, \quad V_i^0 \quad = \quad \left\{ v \in H^1(\Omega_i) \; , \; v = 0 \text{ on } \partial\Omega_{D_i} \cup \Gamma_i \right\}.$$

In a domain decomposition framework, the variational problem (1.1) is reduced to an interface problem whose unknown is the trace φ of u on the interface Γ. Indeed, if we knew φ on Γ

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and if we restrict ourselves to the test functions v in spaces V_i^0, then we observe that u_i is the solution of the following variational problem

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega_i} (\nabla u_i \cdot \nabla v_i + u_i^3) v \, dx \;=\; \int_{\Omega_i} f_i v \, dx \quad , \quad \forall \, v \in \; V_i^0 \; , \\ u_i - \varphi \;\in\; V_i^0 \; . \end{array} \right. \tag{2.1}$$

Introducing the Lagrange multiplier of the constraint $v_i \in V_i^0$, (2.1) can be written as

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega_i} (\nabla u_i \cdot \nabla v_i + u_i^3 \, v) \, dx \;=\; \int_{\Gamma_i} \lambda_i v \, ds + \int_{\Omega_i} f_i v \, dx \; , \quad \forall \, v \in V_i \; , \\ u_i = \; \varphi \quad \text{on} \quad \Gamma_i, \quad \lambda_i \in H^{-1/2}(\Gamma_i) \; . \end{array} \right. \tag{2.2}$$

To each $\varphi \in TrV$ (Tr is the operator mapping functions in V to their traces on Γ), we can then associate these multipliers $\lambda_i(\varphi, f)$, the corresponding map being the so-called Steklov-Poincaré operator. Then, by addition, the correct value of φ is the solution of the following interface problem.

$$\sum_i \int_{\Gamma_i} \lambda_i(\varphi, f) ds = 0, \quad \forall v \in V. \tag{2.3}$$

We want to approximate problem (2.3) with a *mortar* finite element method (see [3]). For this purpose, for each face Γ_{ij}, we introduce an approximation space W_{ijh}. We then define the trace space W_h and the local interface scalar product $< \cdot, \cdot >_{\Gamma_i}$:

$$W_h = \prod_{ij=1,N_f} W_{ijh}, \qquad < v, w >_{\Gamma_i} = \sum_{ij \ni i} \int_{\Gamma_i} vwds.$$

We denote by Tr_{ih} the discrete trace operator defined from V_{ih} into W_h and which to a given $v_{ih} \in V_{ih}$ associates its L^2 projection $Tr_{ih} v_{ih}$ onto W_h. With this notation, the definition of the global approximation space V_h is

$$V_h \;=\; \{ v_h = (v_{ih})_i \in \prod_i V_{ih} \; , \; \text{s.t.} \quad Tr_{ih} v_{ih} = Tr_{jh} v_{jh} \; , \; \forall \, i < j \} \; .$$

The generalisation to the discrete level of problem (2.3) is then immediate. Let $\varphi \in W_h$, be given, for $1 \leq i \leq N$, find $u_{ih}(\varphi, f) \in V_{ih}$, $\lambda_{ih} \in W_h|_{\Gamma_i}$ solution to

$$\int_{\Omega_i} \nabla u_{ih}(\varphi, f) \nabla v_{ih} + u_{ih}^3(\varphi, f) v_{ih} \, dx \quad = \quad < \lambda_{ih}, Trv_{ih} >_{\Gamma_i} + \int_{\Omega_i} f v_{ih} \, dx, \tag{2.4}$$

$$< Tru_{ih}(\varphi, f) - \varphi) q_h >_{\Gamma_i} \quad = \quad 0 \; , \; \forall \, q_h \in M_h \; ; \forall \, v_{ih} \in V_{ih}, \tag{2.5}$$

where M_h is the approximation space of $H^{-1/2}(\Gamma)$ (see [2] for the definition of M_h). The discrete Steklov-Poincaré operator (see [7]) which to φ associates λ_{ih} the generalized normal derivative of $u_{ih}(\varphi, f)$ on Γ_i is defined as follows:

$$\begin{array}{rcl} S_{ih} : W_h|_{\Gamma_i} & \longrightarrow & M_h|_{\Gamma_i} \; , \\ Tr_{ih} u_{ih}(\varphi, f) & \mapsto & \lambda_{ih} \; , \end{array}$$

where $(u_{ih}(\varphi, f), \lambda_{ih})$ is the solution of (2.4)-(2.5).

Theorem 2.1 *Assume $h \leq h_0$, if $\varphi \in W_h$ is a solution of the following interface problem*

$$\sum_{i=1}^{N} < S_{ih} \varphi, Trv_{ih} >_{\Gamma_i} = 0, \; \forall \, v_h \in V_h, \tag{2.6}$$

then Problem (1.1) and the interface Problem (2.6) are equivalent.

Proof. For the proof, the reader is referred to [7] when Ω is decomposed into two nonoverlapping subdomains. The extension to the case of multidomain is straightforward. ■

Let us notice that S_{ih} is a C^1 mapping from the Banach space $(W_h|_{\Gamma_i}\ ;\ \|\cdot\|_{\frac{1}{2};\Gamma})$ with values in the Banach space $(M_h|_{\Gamma_i};\ \|\cdot\|_{-\frac{1}{2};\Gamma})$, with $DS_{ih}(\varphi, f) \in \mathcal{L}(W_h|_{\Gamma_i}; M_h|_{\Gamma_i})$ defined by:
$DS_{ih}(\varphi, f)\psi = \mu_{ih}$ where $(v_{ih}, \mu_{ih}) \in V_{ih} \times M_h|_{\Gamma_i}$ verifies

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega_i} \nabla v_{ih}\nabla\eta_{ih} + 3u_{ih}^2(\varphi, f)v_{ih}\eta_{ih}\, dx =< \mu_{ih}, Tr\eta_{ih} >_{\Gamma_i},\ \forall\eta_{ih} \in V_{ih}, \\ < Trv_{ih} - \psi, q_h >_{\Gamma_i} = 0,\ \forall q_h \in M_h. \end{array}\right. \tag{2.7}$$

Here $(u_{ih}(\varphi, f), \lambda_{ih})$ is solution to Problem (2.4)-(2.5). Please remark that $DS_{ih}(0, 0)$ is the classical discrete Steklov-Poincaré operator (see [1]) and that

$$\begin{array}{rcl} DS_{ih}^{-1}(0,0) : M_h|_{\Gamma_i} & \longrightarrow & W_h|_{\Gamma_i}\ , \\ \mu_{ih} & \mapsto & Tr_{ih}v_{ih}\ , \end{array}$$

where v_{ih} verifies the first equation of (2.7) with $u_{ih} = 0$.

3. A modified Newton algorithm for interface problem. The solution algorithm that we propose for solving (2.6) is a Modified Newton method, with preconditioner M. It writes

- for $\varphi^0 \in W_h$ given and φ^n known, define φ^{n+1} as the solution of
- $\varphi^{n+1} = \varphi^n - \rho MS\varphi^n$

where

$$M\ =\ \sum_i (\alpha_i\, Id|_\Gamma)\, S_{ih}^{-1}(0,0)\, (\alpha_i\, Id|_\Gamma)^t \qquad \text{and } S_h = \Big(\sum_{i=1}^N S_{ih}\varphi^n\Big).$$

Above, α_i defined face by face and such that

$$\left\{\begin{array}{lcl} \alpha_l|_{\Gamma_{ij}} & = & 0 \qquad \text{if } l \neq i \text{ and } l \neq j, \\ (\alpha_i\ +\ \alpha_j)|_{\Gamma_{ij}} & = & 1, \end{array}\right.$$

and ρ is a positive parameter which will be specified later.
Modified Newton iterations can be rephrased in a parallel way as follows:

- Let φ^n be given on Γ. Then on each subdomain solve in parallel (2.4)-(2.5), with $\varphi = \varphi^n$ in order to compute

$$L_i = S_{ih}\varphi^n, \qquad \text{and} \quad \text{set } L(\varphi^n) = \sum_i L_i. \tag{3.1}$$

- On each subdomain, compute $Tr_{ih}v_{h_i}$ where v_{ih} is the solution of

$$\int_{\Omega_i} \nabla v_{ih}\nabla\eta_{ih}dx =< L(\varphi^n), \alpha_i Tr\eta_{ih} >_{\Gamma_i}, \forall\, \eta_{ih} \in V_{ih}. \tag{3.2}$$

- set $\varphi^{n+1} = \varphi^n - \rho\displaystyle\sum_{i=1}^N \alpha_i Tr_{ih}v_{ih}.$ (3.3)

Please remark that the linear preconditioner in the above algorithm (3.1)-(3.3) is determined for the nonlinear interface problem obtained after elimination of interior unknowns. Another approach, the so called Newton Preconditioned Conjugate Gradient method, is to use the Newton algorithm on the global problem (1.1) in which all linearized subproblems are solved by a domain decomposition solver based on a Preconditioned Conjugate Gradient algorithm on the interface Γ (see [4]). Concerning the Modified Newton algorithm the main result of this section is:

Theorem 3.1 *For all $h \leq h_0$, let φ be the solution to Problem (2.6). There exists a neighborhood $\mathcal{V}(\varphi) \subset W_h$ and a parameter $0 < \rho$ independent of h such that for all $\varphi^0 \in \mathcal{V}(\varphi)$ Modified Newton iterations (3.1)-(3.3) converge towards φ.*

The proof of Theorem 3.1 is classical. Define the iteration mapping $G_\rho : W_h \to W_h$ which to ψ associates $\psi - \frac{\rho}{2} MS\psi$. We want to show that for a certain norm on the finite dimensional space W_h, the mapping G_ρ is locally a contraction. The key property to be established is that the eigenvalues of the derivative of G_ρ are non negative, thus it will be possible to choose ρ such that G_ρ is a contracting mapping. Now, let us give some intermediate results useful for the proof of Theorem 3.1.

Lemma 3.1 *The trace operators Tr_{ih} are linear uniformly with respect to h, surjective and continuous from V_{ih} into $W_h|_{\Gamma_i}$. For all $\psi \in W_h|_{\Gamma_i}$, there exists at least an element $Tr_{ih}^{-1}\psi$ in V_{ih} and a constant $C > 0$ independent of h verifying: $Tr_{ih}\Big(Tr_{ih}^{-1}\psi\Big) = \psi$ and $\|Tr_{ih}^{-1}\psi\|_{V_i} \leq C\|\psi\|_{\frac{1}{2},\Gamma}$.*

Our motivation now is to define a discrete scalar product on W_h such that the operator MS is positive. So let us set

$$V_{ih}^0 = \{v_{ih} \in V_{ih}; Tr_{ih}v_{ih} = 0\}$$

and define for all $\psi \in W_h$ the function $\theta_{ih}(\psi) \in V_{ih}$ solution to

$$\left\{ \begin{array}{l} \int_{\Omega_i} \nabla\theta_{h_i}(\psi)\nabla\phi_{h_i}\, dx = 0 \quad \forall \phi_{h_i} \in V_{ih}^0 \\ Tr_{ih}\theta_{ih}(\psi) = \psi \text{ on } \Gamma_i. \end{array} \right. \tag{3.4}$$

We then define the discrete scalar products $(\cdot,\cdot)_h$ on $W_h \subset H^{\frac{1}{2}}(\Gamma)$ by:

$$(\psi,\varphi)_h = \sum_i \int_{\Omega_i} \nabla\theta_{ih}(\psi)\nabla Tr_{ih}^{-1}\varphi\, dx = \sum_i \int_{\Omega_i} \nabla\theta_{ih}(\psi)\nabla\theta_{ih}(\varphi)\, dx\ , \tag{3.5}$$

since $Tr_{ih}^{-1}\varphi - \theta_{ih}(\psi) \in V_{ih}^0$.

Lemma 3.2 *The discrete scalar products $(\cdot,\cdot)_h$ are uniformly with respect to h equivalent to the standard scalar product of $H^{\frac{1}{2}}(\Gamma)$ in W_h.*

For the proof, of Lemma 3.1 and Lemma 3.2, the reader is referred to [6].

Lemma 3.3 *There exists $\beta > 0$ independent of h such that $\forall \varphi \in W_h$ we have*

$$(M\, DS_h(\varphi,f)\psi,\psi)_h \geq \beta \quad \forall\, \psi \in W_h,$$

with

$$DS_h(\varphi,f) = \sum_i DS_{ih}(\varphi,f).$$

Proof. Let $\mu_{ih} \in M_h|_{\Gamma_i}$ and $\mu \in M_h$ be defined by $\mu_{ih} = DS_{ih}(\varphi,f)\psi$, $\mu = DS_h(\varphi,f)\psi$ respectively, and let $w_{ih} \in X_{ih}$ be solution to

$$\int_{\Omega_i} \nabla w_{ih}\nabla\eta_{ih}\, dx =< \mu, Tr_{ih}\eta_{ih} >_{\Gamma_i}\ \forall\eta_{ih} \in V_{ih}. \tag{3.6}$$

We have

$$
\begin{aligned}
(M\,DS_h(\varphi,f)\psi,\psi)_h &= \sum_{i=1}^{N}\int_{\Omega_i}\nabla\theta_{ih}(\psi)\nabla Tr_{ih}^{-1}(M\mu)\,dx \\
&= \sum_{i=1}^{N}\int_{\Omega_i}\nabla\theta_{ih}(\psi)\nabla Tr_{ih}^{-1}\Big(\sum_j \alpha_j Tr_{jh}w_{jh}\Big)dx \\
&= \sum_{i=1}^{N} < \mu_{ih}, Tr\theta_{ih}(\psi) >_{\Gamma_i}
\end{aligned}
$$

Finally, Problem (2.7) provides

$$
(M\,DS_h(\varphi,f)\psi,\psi)_{h_i} = \sum_{i=1}^{N}\int_{\Omega_i}\nabla v_{ih}\nabla\theta_{ih}(\psi) + 3u_{ih}^2(\varphi,f)v_{ih}\theta_{ih}(\psi)\,dx. \tag{3.7}
$$

From the identity

$$
\sum_{i=1}^{N}\int_{\Omega_i}\nabla\Big(v_{ih}-\theta_{ih}(\psi)\Big)\nabla\eta_{ih} + 3u_{ih}^2(\varphi,f)v_{ih}\eta_{ih}\,dx = 0\,,\forall\eta_{ih}\in V_{ih}^0, \tag{3.8}
$$

written with $\eta_{ih} = v_{ih} - \theta_{ih}(\psi) \in V_{ih}^0$ we get

$$
\begin{aligned}
&\textstyle\sum_{i=1}^{N}\int_{\Omega_i}\nabla v_{ih}\nabla\theta_{ih}(\psi) + 3u_{ih}^2(\varphi,f)v_{ih}\theta_{ih}(\psi)\,dx = \\
&\textstyle\sum_{i=1}^{N}\int_{\Omega_i}|\nabla v_{ih}|^2 + |\nabla\theta_{ih}(\psi)|^2 - \nabla v_{ih}\nabla\theta_{ih}(\psi) + 3u_{ih}^2(\varphi,f)v_{ih}^2\,dx.
\end{aligned} \tag{3.9}
$$

The identity $0 \le \frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2 = a^2 + b^2 - ab$ implies that the right hand side of (3.9) is bounded from below by $\sum_{i=1}^{N}\int_{\Omega_i}|\nabla\theta_{ih}(\psi)|^2\,dx$. From Lemma 3.1 we have the desired estimate. Lemma 3.3 is proved. ■

Proof. **of Theorem 3.1** We show that $DG_\rho$the derivative of G_ρ is bounded by a constant less than one in a neighborhood of φ. It is well known that for an $0 < \delta$ given, there exists a norm $|||\cdot|||$ on W_h such that for the induced norm for the operators we have $|||DG_\rho(\varphi)||| \le \sigma\big(DG_\rho(\varphi)\big) + \delta$, where σ denotes the spectral radius. Lemma 3.2 and Lemma 3.3 imply that $M\,DS_h(\varphi,f)$ has positive eigenvalues in W_h uniformly bounded from below with respect to h. Thus we have

$$
k = \sigma\Big(I - \rho M\,DS_h(\varphi,f)\Big) = 1 - \rho\sigma\Big(M\,DS_h(\varphi,f)\Big).
$$

The stability of $DS_{ih}(\varphi,f)$ and $DS_{ih}^{-1}(0,0)$ provides the existence of $0 < \rho$ independent of h such that $k < 1$. Then a classical Banach fixed point theorem applies and thus Theorem 3.1 is proved. ■

4. Numerical results. In this section we describe some numerical results obtained with the Preconditioned Modefied Newton (PMN) algorithm (3.1)-(3.3). This resulst are done for various mesh sizes and various numbers of subdomains in the case of nonmatching grids. the corresponding physical problem is the nonlinear elliptic problem $-\Delta u + u^3 = f \quad \text{in}\,\Omega = (0,1)\times(0,1)$ where the source term f is a Gaussian function centred at the point (1,1) and the Dirichlet boundary conditions are prescribed on the side $x_2 = 0$ (see Figure 4.1).

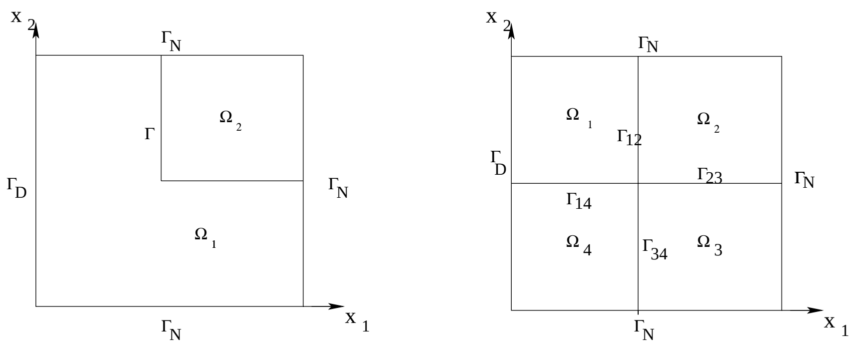

Figure 4.1: Decomposition in 2 and 4 subdomains

Remark 4.1 *The modified Newton algorithm requires on each subdomain the successive solution of a Dirichlet and of a Neumann problem (preconditioner). In the abscence of a Dirichlet boundary conditions on $\partial\Omega_i \setminus \Gamma$, the Neumann problem is not well-posed. In such situations, we replace in the factorization of the finite-element matrix of problem (3.2), all the singular pivots by an averaged strictly positive pivot.*

First we present some numerical results obtained with the PMN algorithm (3.1)-(3.3) in two-domains case with a fixed value of the relaxation parameter ρ . For the optimal value of the relaxation parameter ρ, the results could be different. Next, the obtained results with Newton Preconditioned Conjugate Gradient (Newton-PCG) algorithm for the same test case are given.

In Table 1 the number of iterations necessary for Modified Newton iterations to converge (with a level of precision of 10^{-6}), and the values of parameter ρ are reported as functions of degrees of freedom. Please remark that the number of iterations for reaching convergence with a constant ρ are independent of h.

d.o.f in $\Omega_1 \cup \Omega_2$	ρ	*number of iter.*
102	0.16	32
354	0.155	34
1314	0.16	34

Table 1: evolution of ρ and the number of iterations of Modified Newton algorithm

d.o.f in $\Omega_1 \cup \Omega_2$	*Newton iter.*	*PCG iter. on* Γ	*total nb. of iter.*
102	6	6	36
354	6	6	36
1314	7	6	42

Table 2: evolution of the number of iterations of Newton-PCG algorithm.

Table 2 shows that the Newton-PCG algorithm converges (with a level of precision of 10^{-6} for the Newton algorithm and for the PCG algorithm) at a rate which is independent of the mesh size h.

Modified Newton algorithm is proved to converge independently of the discretization step, which is confirmed by our numerical tests. Moreover, the potential parallelism offered by this algorithm is easy to exploit on the contrary of the Newton-PCG. Nevertheless, its practical implementation still faces the problem of the optimal choice of the parameter ρ.

We have tested the dependency over h in the case where Ω is decomposed into four geometrically identical subdomains (see Figure 4.1). There is a slight dependence on h due to the presence of cross points in our decomposition (see Table 3).

step	*nb. of iter.*	ρ
h	64	0.15
h/2	68	0.14

Table 3: Test over the mesh size h (**p=4**)

Here, we study the convergence rate of the PMN algorithm (3.1)-(3.3) with respect to the number of subdomains p. We consider the case where the domain Ω has been decomposed into two and four subdomains (see Figure 4.1). The number of degrees of freedom in Ω varies with p because each interface node is treated in our approach as two independent nodes.

p	*number of iter.*	*d.o.f in* Ω
2	34	1314
4	60	1350

Table 4: Test over the number of subdomains **p**.

In terms of iteration count, Table 4 and show that the smaller the number of subdmains the faster the PMN convergence. Indeed, the diameter d of each subdomain has a direct influence on the condition number of our operator.

5. Conclusion. A Modified Newton method for a domain decomposed nonlinear elliptic problem has been introduced and studied. For a small number of subdomains and very fine grids, this approach leads to efficient numerical algorithm even in the case of nonmatching grids. Indeed with the choice of adequate preconditioners such as the one introduced in §3, the method is proved to converge independently of the discretization step, which is confirmed by our numerical tests. Nevertheless, the Preconditioned Modified Newton algorithm does not scale well with the diameter of the subdomains. The addition of an unstructured coarse grid solver when using decompositions with a large number of subdomains is actually under consideration.

REFERENCES

[1] V. I. Agoshkov. Poincaré-Steklov operators and domain decomposition methods in finite dimensional spaces. In R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Philadelphia, PA, 1988. SIAM.

[2] F. Ben Belgacem. The mortar finite element method with Lagrange multipliers. *Numer. Math.*, 84(2):173–197, 1999.

[3] C. Bernardi, Y. Maday, and A. T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In H. Brezis and J.-L. Lions, editors, *Collège de France Seminar*. Pitman, 1994. This paper appeared as a technical report about five years earlier.

[4] P. De Roeck Yann-Hervé, Le Tallec and V. M. Marina. A domain-decomposition solver for nonlinear elasticity. *Comp. Meth. Appli. Mech. Eng.*, 99:187–207, 1992.

[5] M. Dryja and W. Hackbusch. On the nonlinear domain decomposition method. *BIT*, pages 296–311, 1997.

[6] P. Le Tallec and T. Sassi. Domain decomposition with nonmatching grids: Augmented Lagrangian approach. *Math. Comp.*, 64(212):1367–1396, 1995.

[7] J. Pousin and T. Sassi. Domain decomposition with non matching grids and adaptive finite element techniques. *East-West J. Numer. Math.*, 8:243–246, 2000.

[8] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford Science Publications, 1999.

[9] P. L. Tallec, T. Sassi, and M. Vidrascu. Three-dimensional domain decomposition methods with nonmatching grids and unstructured coarse solvers. In D. Keyes and J. Xu, editors, *the seventh international symposium on Domain Decomposition Methods for Partial Differential Equations.*, pages 133–139. AMS, 1994.