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51. Singular Function Enhanced Mortar Finite Element

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1. Introduction. We are interested in solving the following elliptic variational problem: Find $u^* \in H^1(\Omega)$, such that

$$\begin{cases} a(u^*, v) &= f(v) \quad \forall v \in H_0^1(\Omega) \\ u^* &= u_0^* \quad \text{on } \partial\Omega \end{cases}, \tag{1.1}$$

where

$$a(u^*, v) = \int_\Omega \nabla u^* \cdot \nabla v \, dx \quad \text{and} \quad f(v) = \int_\Omega f v \, dx.$$

We assume the function $f \in L^2(\Omega)$. We also assume the function u_0^* has an extension $H^2(\Omega)$, which we denote also by u_0^*. We let the domain Ω to be the L-shaped domain in $\Re^2$ with vertices $V_1 = \{0, 0\}, V_2 = \{1, 0\}, V_3 = \{1, 1\}, V_4 = \{-1, 1\}, V_5 = \{-1, -1\}$, and $V_6 = \{0, -1\}$. It is well-known that the solution u^* of (1.1) does not necessarily belong to $H^2(\Omega)$ due to the nonconvexity of the domain Ω at the corner V_1, and therefore, standard finite element discretizations do not give second order accurate schemes. Theoretical and numerical work on corner singularity are very well-known and several different approaches were proposed [4, 2, 5, 6, 7, 8, 9, 10]; see the references therein. The main goal of the paper is to design and analyze optimal accurate finite element discretizations based on mortar techniques [1, 11] and singular functions [8, 7]. The proposed methods are variation of the methods described in Chapter 8 of [10] where a smoothed cut-off singular function is added to the space of finite elements. There, a smoothed cut-off function is applied to make the singular function to satisfy the zero Dirichlet boundary condition. Here, instead, we use mortar finite element techniques on the boundary of $\partial\Omega$ to force, in a weak sense, the boundary condition. As a result, accurate and general schemes can be obtained for which they do not rely on costly numerical integrations and linear solvers.

2. Notations. We next introduce some notations and tools.

2.1. Triangulation. Let $\mathcal{T}^h(\Omega)$ be a standard finite element triangulation of $\overline{\Omega}$. We assume the triangulation $\mathcal{T}^h(\Omega)$ to be shape regular and quasi-uniform with grid size of $O(h)$. Let $V^h(\Omega)$, also denoted by V^h, be the space of continuous piecewise linear functions on $\mathcal{T}^h(\Omega)$; note that we have not assumed the functions of V^h to vanish on $\partial\Omega$.

2.2. Singular Functions and Regularity Results. We note that the solution u^* of (1.1) does not necessarily belong to $H^2(\Omega)$ even if f and u_0^* are very smooth. For instance, consider the primal singular function defined by $\psi^+(r, \theta) = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$. The function ψ^+ is smooth everywhere in Ω except near the non-convex corner V_1. It is easy to check that $\psi^+ \in H^{5/6-\epsilon}(\Omega)$ if, and only if, ϵ is positive and $-\triangle\psi^+ \equiv 0$ on Ω. We note that ψ^+ vanishes on the intervals $[V_1, V_2]$ and $[V_6, V_1]$, plus it is smooth on the remaining boundary of $\partial\Omega$.

Another function that will play an important role in our studies here is the dual singular function ψ^- defined as $\psi^-(r, \theta) = r^{-\frac{2}{3}} \sin(\frac{2}{3}\theta)$. We note that $-\Delta\psi^- \equiv 0$ and ψ^- vanishes on the intervals $[V_1, V_2]$ and $[V_6, V_1]$, and it is easy to check that $\psi^- \in H^{1/3-\epsilon}$ if, and only if, ϵ is positive.

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It is well-known [8, 7] that the solution of (1.1) has a unique representation

$$u^* = w_{u^*} + \lambda_{u^*}\psi^+, \tag{2.1}$$

where $w_{u^*} \in H^2(\Omega)$ and $\lambda_{u^*} \in \Re$, and the following regularity estimates hold:

$$\|w_{u^*}\|_{H^2(\Omega)} \leq C\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right), \tag{2.2}$$

and

$$|\lambda_{u^*}| \leq C\|f\|_{L^2(\Omega)}. \tag{2.3}$$

2.3. Mortar Functions on the Boundary. The boundary of our domain is given by $\partial\Omega = \bigcup_{m=1}^{6} \overline{D}_m$, where the open segments D_m are given by the intervals $D_1 = (V_1, V_2)$, $D_2 = (V_2, V_3)$, $D_3 = (V_3, V_4)$, $D_4 = (V_4, V_5)$, $D_5 = (V_5, V_6)$, and $D_6 = (V_6, V_1)$. For each interval $\overline{D}_m$, the triangulation $\mathcal{T}_h(D_m)$ is inherited from the triangulation $\mathcal{T}_h(\Omega)$. Let us denote the space $W_h(D_m)$ as the trace of V_h to $\overline{D}_m$; i.e.

$$W_h(D_m) = \{v \in C(\overline{D}_m) : v = w(\overline{D}_m), w \in V_h\}.$$

We also denote the space $W_h^0(D_m)$ as the functions of $W_h(D_m)$ which vanish at the two end points of $\overline{D}_m$. Thus, $W_h^0(D_m) = W_h(D_m) \cap H_0^1(D_m)$. The number of degrees of freedom of $W_h^0(D_m)$ are the number of interior nodes of $\mathcal{T}_h(D_m)$ which are equal aslo the number of degrees of freedom of the Lagrange multiplier spaces $M_h(D_m)$. In this paper, in the numerical experiments, we adopt the dual biorthogonal functions introduced in [11]. We note that the theory presented here also holds for the old mortars [1]. For each edge D_m, the mortar projection operator $\Pi_m : C(\overline{D}_m) \longrightarrow W_h(D_m)$ is defined by

$$v - \Pi_m v \in C_0(D_m), \quad \text{and} \quad \int_{D_m} (v - \Pi_m v)\mu_m ds = 0, \quad \forall \mu_m \in M_h(D_m). \tag{2.4}$$

It can be shown [1, 11] that

$$\|v - \Pi_m v\|_{H_{00}^{1/2}(D_m)} \leq Ch\|v\|_{H^{3/2}(D_m)}, \ \forall v \in H^{3/2}(D_m), \tag{2.5}$$

and

$$\inf_{\mu_m \in M_h(D_m)} \|v - \mu_m\|_{(H^{1/2}(D_m))'} \leq Ch\|v\|_{H^{1/2}(D_m)}, \ \forall v \in H^{1/2}(D_m). \tag{2.6}$$

3. Singular Function Enhanced Mortar Finite Element. We define the discrete global space V_h^+ as follows:

$$V_h^+ = \{v = w + \lambda\psi^+ : w \in V_h, \quad \lambda \in \Re, \text{ and } \Pi_m v = 0, m = 1, \cdots, 6\}.$$

Functions of the space V_h^+ vanish at the vertices $V_k, k = 1, \cdots, 6$ and satisfy zero Dirichlet boundary condition (in the weak discrete sense) on the intervals $\overline{D}_m$. It is easy to see that the degrees of freedom of the space V_h^+ are the λ and the nodal values of w at the interior nodes of $\mathcal{T}_h(\Omega)$; the values of w on the $\overline{D}_m$ are obtained via $w = -\lambda\Pi_m\psi^+$.

We next introduce the new finite element formulation using the primal singular function ψ^+ and mortar techniques in order to obtain an approximation for u^*. We then introduce two second order accurate approximations for the stress intensive factor (SIF) λ_{u^*} based on the dual singular function ψ^-.

3.1. Finite Element Formulation. Let us define $u_0 \in V_h$ as $u_0 = \Pi_m u_0^*$ on D_m, $m = 1, \cdots, 6$, and zero nodal values at the interior nodes of $\mathcal{T}_h(\Omega)$. We define the singular function enhanced mortar finite element method as follows:

Find $u = w_u + \lambda_u \psi^+$ such that $u - u_0 \in V_h^+$ and

$$a(u, v) = f(v), \quad \forall v \in V_h^+. \tag{3.1}$$

We prove later in this paper that u is a second order approximation to u^*. We note however that the λ_u and w_u separately are not second order approximations of λ_{u^*} and w_{u^*}, respectively. So, in the next subsections, we introduce two algorithms for obtaining second order approximations for the stress intensive factor (SIF) λ_{u^*}.

3.2. Extraction of SIF through a Smoothed Cut-off Function. Define $f = -\triangle u^*$ and $f^- = -\triangle s^-$, where $s^- = \rho\psi^-$. Here, the smoothed cut-off function $\rho(r)$ is defined in the polar coordinate system as

$$\rho(r) = \begin{cases} 1 & 0 \le r \le \frac{1}{4} \\ -192r^5 + 480r^4 - 440r^3 + 180r^2 - \frac{135}{4}r + \frac{27}{8} & \frac{1}{4} \le r \le \frac{3}{4} \\ 0 & \frac{3}{4} \le r \end{cases}.$$

It is easy to check that the function ρ has two continuous derivatives. By applying Green's formula twice [9], we obtain

$$\lambda_{u^*} = \frac{\int_\Omega (fs^- - f^-u^*) + \int_{\partial\Omega} s^-\partial_n s^- - u_0^*\partial_n s^-}{\pi},$$

and by using that s^- vanishes on $\partial\Omega$ we have

$$\lambda_{u^*} = \frac{\int_\Omega (fs^- - f^-u^*) - \int_{\partial\Omega} s^- u_0^*\partial_n s^-}{\pi}. \tag{3.2}$$

The discrete stress intensity factor is obtained as follows. We first solve (3.1) to obtain $u = w_u + \lambda_u \psi^+$, and then we plug this u as u^* in (3.2) to define the discrete stress intensity factor as

$$\lambda_u^h = \frac{\int_\Omega (fs^- - f^-u) - \int_{\partial\Omega} u_0^*\partial_n s^-}{\pi}. \tag{3.3}$$

3.3. Extraction of SIF without a Smoothed Cut-off Function. Similarly, we can use the same approach above for ψ^- as s^-. Using $-\triangle\psi^- \equiv 0$, we obtain

$$\lambda_{u^*} = \frac{\int_\Omega f\psi^- - \int_{\partial\Omega}(u_0^*\partial_n\psi^- - \psi^-\partial_n u^*)}{\pi}. \tag{3.4}$$

We note that we do not know the value of $\partial_n u^*$ and therefore, the formula (3.4) is not applicable for defining the discrete stress intensity factor. We remark that an approximation of $\partial_n u^*$ can be obtained via the saddle point formulation [11] of (3.1) but unfortunately we cannot prove that this approximation is of second order. We next introduce a new method that does not require the knowledge of $\partial_n u^*$.

We modify ψ^- to $\tilde{\psi}^-$, where $\tilde{\psi}^-$ vanishes on the whole $\partial\Omega$, $\tilde{\psi}^-$ and ψ^- have the same singular behavior in a neighbourhood of the origin, and $-\Delta\tilde{\psi}^- \equiv 0$. This is done as follows. We first solve $\delta\psi^- \in H^1(\Omega)$ such that

$$\begin{cases} a(\delta\psi^-, v) &= 0 & \forall v \in H_0^1(\Omega) \\ \delta\psi^- &= \psi^- & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Then, we define $\tilde{\psi}^- = \psi^- - \delta\psi^-$. We note that ψ^- has a H^2 extension to Ω and therefore, the solution of (3.5) is in the form of $\delta\psi^- = w_{\delta\psi^-} + \lambda_{\delta\psi^-}\psi^+$, where $w_{\delta\psi^-} \in H^2(\Omega)$. Hence, the singular behavior of $\tilde{\psi}^-$ near the origin is the same as of ψ^-, and we obtain

$$\lambda_{u^*} = \frac{\int_\Omega f\tilde{\psi}^- - \int_{\partial\Omega} u_0^*\partial_n\tilde{\psi}^-}{\pi}.$$

In the case the boundary value u_0^* vanishes on $\partial\Omega$, we have

$$\lambda_{u^*} = \frac{\int_\Omega f\tilde{\psi}^-}{\pi}. \tag{3.6}$$

We note that we do not know $\tilde{\psi}^-$ and therefore, a numerical approximation for $\tilde{\psi}^-$ must be obtained. We first define $\delta\psi_0^- \in V_h$ as $\delta\psi_0^- = \Pi_m\psi^-$ on the D_m and zero nodal values at the interior nodes of $\mathcal{T}_h(\Omega)$. We solve $\delta\psi_h^- - \delta\psi_0^- \in V_h^+$ such that

$$a(\delta\psi_h^-, v) = 0, \forall v \in V_h^+.$$

We let $\tilde{\psi}_h^- = \psi^- - \delta\psi_h^-$, and define the discrete stress intensity factor by

$$\hat{\lambda}_u^h = \frac{\int_\Omega f\tilde{\psi}_h^-}{\pi} = \frac{\int_\Omega f\psi^- - f\delta\psi_h^-}{\pi}. \tag{3.7}$$

We remark that $\hat{\lambda}_u^h$ can be obtained without computing the discrete solution u and can be used only if u_0^* vanishes on $\partial\Omega$.

4. Analysis.

In this section we analyze the proposed methods. We will prove optimality accuracy errors of the discrete solution u on the L_2 and H_1 norms. We also show that the two proposed discrete stress intensive factor formulas given by (3.3) and (3.7) are both second order approximations for λ_{u^*}.

4.1. Uniform ellipticity.

We note that $v \in V_h^+$ implies that v vanishes on D_1 and D_6. Therefore, using a standard Poincaré inequality, we have:

Lemma 4.1 *There exists a constant C that does not depend on h and v such that*

$$\|v\|_{H^1(\Omega)} \leq C|v|_{H^1(\Omega)}, \quad \forall v \in V_h^+. \tag{4.1}$$

4.2. Energy Discrete Error.

We note that proposed discretization (3.1) is non-conforming since the space V_h^+ is not included in $H_0^1(\Omega)$; functions in V_h^+ vanishes on D_m, $m = 2, \cdots 5$ only in a weak sense. To establish H_1 apriori error estimate, we use the Cea's lemma (the second Strang lemma) for non-conforming discretization [3]. We obtain

$$\|u^* - u\|_{H^1(\Omega)} \leq \inf_{v\in u_0+V_h^+} \|u^* - v\|_{H^1(\Omega)} + \sup_{z\in V_h^+} \frac{|a(u^*, z) - f(z)|}{\|z\|_{H^1(\Omega)}} =$$

$$\inf_{v\in u_0+V_h^+} \|u^* - v\|_{H^1(\Omega)} + \sup_{z\in V_h^+} \frac{|\int_{\partial\Omega} z\partial_n u^* ds|}{\|z\|_{H^1(\Omega)}}. \tag{4.2}$$

The first term of (4.2) is the **best aproximation error** and the second term is the **consistency error**.

4.2.1. Best Approximation Error. We next establish that the best approximation error in the energy norm is of optimal order.

Lemma 4.2 *The best approximation error is of order h,*

$$\inf_{v\in u_0+V_h^+} \|u^* - v\|_{H^1(\Omega)} \le Ch\left(\|f+\Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right). \tag{4.3}$$

Proof. Let $\tilde v$ be defined as

$$\tilde v = I_h(u^* - \lambda_{u^*}\psi^+) + \lambda_{u^*}\psi^+,$$

where I_h is the standard pointwise interpolator on V_h. Note that the interpolation is well defined since the function $w_{u^*} = u^* - \lambda_{u^*}\psi^+$ belongs to $H^2(\Omega)$ and therefore, w_{u^*} is a continuous function. The function $\tilde v - u^*$ belongs to $H_0^1(D_m)$ and does not satisfy the mortar condition. We next modify $\tilde v$ to v to make $u^* - v$ to satisfy the mortar condition (2.4). This is done by $v = \tilde v + \sum_{m=1}^6 \mathcal{H}_m \Pi_m(u^* - \tilde v)$, where the operator $\mathcal{H}_m$ denote the V_h-discrete harmonic extension function with boundary values given on $\overline{D}_m$ and zero on $\partial\Omega\backslash D_m$. In addition, it is easy to check that $v \in u_0 + V_h^+$. We have

$$\|u^* - v\|_{H^1(\Omega)} = \|w_{u^*} - I_h w_{u^*}\|_{H^1(\Omega)} + \|\sum_{m=1}^6 \mathcal{H}_m Q_m(u^* - \tilde v)\|_{H^1(\Omega)}. \tag{4.4}$$

For the first term of (4.4), we use a standard approximation result on pointwise interpolation and (2.2) to obtain

$$\|w_{u^*} - I_h w_{u^*}\|_{H^1(\Omega)} \le Ch\|w_{u^*}\|_{H^2(\Omega)} \le Ch\left(\|f+\Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right).$$

For the second term of (4.4), we use properties of discrete harmonic extensions and $H_{00}^{1/2}$-norm, and the approximation result (2.5) to obtain

$$\begin{aligned}
\|\sum_{m=1}^6 \mathcal{H}_m\Pi_m(u^*-\tilde v)\|_{H^1(\Omega)} &\le C\sum_{m=1}^6 \|\mathcal{H}_m\Pi_m(u^*-\tilde v)\|_{H^1(\Omega)} \\
\le C\sum_{m=1}^6 \|\Pi_m(u^*-\tilde v)\|_{H_{00}^{1/2}(D_m)} &\le C\sum_{m=1}^6 \|u^*-\tilde v\|_{H_{00}^{1/2}(D_m)} \\
\le Ch\|u_0^*\|_{H^{3/2}(D_m)} &\le Ch\|u_0^*\|_{H^2(\Omega)}.
\end{aligned}$$

■

4.2.2. Consistency Error. We next establish that the consistency error is of optimal order.

Lemma 4.3 *The consistency error is of order h*

$$\sup_{z\in V_h^+} \frac{|\int_{\partial\Omega}\partial_n u^* z ds|}{\|z\|_{H^1(\Omega)}} \le Ch\left(\|f\|_{L^2(\Omega)} + \|f+\Delta u_0^*\|_{L^2(\Omega)}\right). \tag{4.5}$$

Proof. We remark that $z \in V_h^+$ implies that z vanishes on $\overline{D}_1$ and $\overline{D}_6$. Therefore,

$$\int_{\partial\Omega} z\partial_n u^* ds = \sum_{m=2}^5 \int_{D_m} z\partial_n u^* ds.$$

By the definition of V_h^+, we have $\int_{D_m} z\mu_m ds = 0, \quad \mu_m \in M_h(D_m)$. Thus,

$$\sum_{m=2}^{5} \int_{D_m} z\partial_n u^* ds = \sum_{m=2}^{5} \int_{D_m} z(\partial_n u^* - \mu_m) ds, \ \ \forall \mu_m \in M_h(D_m),$$

and using duality arguments we obtain

$$\sum_{m=2}^{5} \left| \int_{D_m} z\partial_n u^* ds \right| \leq C\|z\|_{H^{1/2}(D_m)} \inf_{\mu_m \in M_h(D_m)} \|\partial_n u^* - \mu_m\|_{(H^{1/2})'(D_m)}.$$

Let us denote $\Omega_{1/4} = \Omega \cap \{r^2 = x^2 + y^2 \leq 1/16\}$, and $\Omega_{1/4}^c = \Omega \backslash \Omega_{1/4}$. Since $\psi^+ \in H^2(\Omega_{1/4}^c)$, we have $u^* \in H^2(\Omega_{1/4}^c)$, and therefore we can use a trace theorem to obtain $\partial_n u^* \in H^{1/2}(D_m), m = 2, \cdots, 5$. We then use approximation property (2.6), a trace result, and the regularity estimates (2.2) and (2.3) to obtain

$$\inf_{\mu_m \in M_h(D_m)} \|\partial_n u^* - \mu_m\|_{(H^{1/2})'(D_m)} \leq Ch\|\partial_n u^*\|_{H^{1/2}(D_m)} \leq Ch\|u^*\|_{H^2(\Omega_{1/4}^c)}$$

$$\leq Ch(|\lambda_{u^*}|\|\psi^+\|_{H^2(\Omega_{1/4}^c)} + \|w_{u^*}\|_{H^2(\Omega)}) \leq Ch\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right).$$

We finally use that $\|z\|_{H^{1/2}(D_m)} \leq C\|z\|_{H^1(\Omega)}$ to obtain (4.5).

■

4.3. Error in the L^2-norm. We also obtain an optimal error estimates in $L^2(\Omega)$-norm for the problem (1.1).

Lemma 4.4 *The L_2 discrete error is of order h^2*

$$\|u^* - u\|_{L^2(\Omega)} \leq Ch^2\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right). \tag{4.6}$$

Proof. The proof follows easily from an Aubin-Nitche trick argument and by the fact that the enhanced space V_h^+ is used both as the solution space as well as the test function space for (3.1). ■

4.4. Stress Intensive Factor Error. The apriori error estimate for stress intensive factor errors $|\lambda_{u^*} - \lambda_u^h|$ with λ_u^h defined on (3.3), and $|\lambda_{u^*} - \hat{\lambda}_u^h|$ with $\hat{\lambda}_u^h$ defined on (3.7) for the case $u_0^* \equiv 0$, will follow easily from the L_2-error estimates.

Lemma 4.5 *If $f \in L^2(\Omega)$, then the recovering formula (3.3) gives h^2 accuracy*

$$|\lambda_{u^*} - \lambda_u| \leq Ch^2\left(\|f + \Delta u_0^*\|_{L^2(\Omega)} + \|u_0^*\|_{H^2(\Omega)}\right).$$

Proof. We subtract (3.3) from (3.2) and we obtain

$$|\lambda_{u^*} - \lambda_u| = \left|\frac{\int_\Omega f^-(u - u^*)}{\pi}\right| \leq \|f^-\|_{L^2(\Omega)}\|u - u^*\|_{L^2(\Omega)}.$$

The lemma follows from the Lemma 4.4 and the smoothing properties of the smoothed cut-off function ρ. ■

Using similar arguments we obtain:

Lemma 4.6 *If $f \in L^2(\Omega)$ and $u_0^* \equiv 0$, then the recovering formula (3.7) gives h^2 accuracy*

$$|\lambda_{u^*} - \lambda_{u^*}^h| \leq Ch^2\|f\|_{L^2(\Omega)}.$$

Table 5.1: Results with $f = -\triangle s^+ - \triangle s_2^+ + 6x(y^2 - y^4) + (x - x^3)(12y^2 - 2)$

k	$\lambda^k - 1$	σ^k	$1 - \hat{\lambda}^k$	$\hat{\sigma}^k$	e_2^k	ϵ_2^k	e_1^k	ϵ_1^k
2	2.967e-1	–	2.698e-3	–	7.512e-2	–	9.032e-1	–
3	9.457e-2	1.6497	6.914e-4	1.9642	2.415e-2	1.6380	5.027e-1	0.8454
4	2.651e-2	1.8349	1.673e-4	2.0474	6.805e-3	1.8275	2.673e-1	0.9115
5	6.862e-3	1.9497	4.083e-5	2.0152	1.764e-3	1.9475	1.361e-1	0.9736
6	1.730e-3	1.9873	1.006e-5	2.0216	4.454e-4	1.9858	6.839e-2	0.9928
7	4.341e-4	1.9952	2.550e-6	1.9832	1.116e-4	1.9958	3.424e-2	0.9980
8	1.085e-5	1.9996	6.290e-7	2.0154	2.794e-5	1.9991	1.713e-2	0.9994

5. Numerical Experiments. An advantage of the proposed methods is in the construction of the stiffness matrix of (3.1). Its construction requires few work on numerical integrations since we do integrations by parts on $a(\psi^+, \varphi_i)$ or $a(\psi^+, \psi^+)$. Here the function φ_i stands for a nodal basis function of V_h. The only integrations that cannot be done exact are on $D_m, m = 2, \cdots, 5$. There, the singular function is very smooth and therefore easy in in numerical integrations.

In the set of experiments, we solve the discrete Poisson equation (3.1) with $f = -\triangle s^+ - \triangle s_2^+ + 6x(y^2 - y^4) + (x - x^3)(12y^2 - 2)$. Hence, the exact solution is $u = s^+ + s_2^+ + (x - x^3)(y^2 - y^4)$. Here, $s^+ = \rho(r)\psi^+$ and $s_2^+ = \rho(r)\psi_2^+$, where ψ_2^+ is the next singular function associated to the problem (1.1); i.e. $\psi_2^+ = r^{4/3}\sin(4/3\theta)$. The integer k is the level of refinement of the mesh; $k = 0$ is a mesh with 2 triangles per quadrant. The L^2 norm (H^1 semi-norm) discretization error on the kth level mesh is given by $e_2^k = \|u - u^*\|_{L^2(\Omega)}$ ($e_1^k = |u - u^*|_{H^1(\Omega)}$). The discrete stress intensity factor are given by $\lambda^k = \lambda_u^h$ and $\hat{\lambda}^k = \hat{\lambda}_u^h$. In our example, $\lambda_{u^*} = 1$. We also measure the rate of convergences for the four discrete errors given by

$$\sigma^k = \log_2\left(\frac{|\lambda^{k-1} - 1|}{|\lambda^k - 1|}\right),\ \hat{\sigma}^k = \log_2\left(\frac{|\hat{\lambda}^{k-1} - 1|}{|\hat{\lambda}^k - 1|}\right)\ \epsilon_2^k = \log_2\left(\frac{e_2^{k-1}}{e_2^k}\right),\ \text{and}\ \epsilon_1^k = \log_2\left(\frac{e_1^{k-1}}{e_1^k}\right).$$

The numerical experiments confirm the theory showing optimality of the proposed algorithms and show that the recovering formula (3.7) is very accurate.

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