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14. Wave Propagation Analysis of Multigrid Methods for Convection Dominated Problems

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1. Introduction. The basic multigrid principle is that the smoother damps the oscillatory high frequency errors whereas the coarse grid correction reduces the smooth low frequency errors. However, this principle may not hold for convection dominated problems since the success of the standard techniques often rely on the intrinsic properties of elliptic PDEs, for instance, symmetry and positive definiteness, which are not generally true for convection dominated problems.

Several smoothing techniques have been proposed for convection dominated problems. One approach is to apply Gauss-Seidel with the so-called downwind ordering [3, 1, 6, 11, 16]. The idea is that the linear system given by the upwind discretization can be well-approximated by the lower triangular part if the unknowns are ordered according to the flow direction. Another approach is to use time-stepping methods as smoothers [7, 8, 10, 13]. The idea is that this class of smoothers do not just reduce the high frequency errors, but more importantly, also propagate the errors along the flow directions. Thus, the multigrid process can be interpreted as speeding up the error propagation by taking larger time step sizes on the coarse grids.

To analyze the efficiency of multigrid methods, one must then take into account the wave propagation property. In the classical Fourier-based analysis of multigrid methods [17], only the magnitude of the Fourier error components are considered, thus ignoring completely the phase angles which account for the wave propagation [15]. Gustafsson and Lötstedt [4, 12] first analyze the phase speed of this multigrid approach, and prove that a speedup of $2^K - 1$ is obtained using K grids for smooth errors. In this paper, we present a more refined phase velocity analysis which is able to explain the dispersive behavior of multigrid process that turns out to have significant influence on the convergence rate.

Phase velocity analysis is not just useful for analyzing the wave propagation multigrid approach but also applicable to explain the efficiency of other coarse grid correction methods as well. One common coarse grid correction approach is to use the discretization matrices as the coarse grid operators together with an exact coarse grid solve. It has been shown by Brandt and Yavneh [3] that the resulting coarse grid correction is not accurate for the Fourier components in the characteristic direction. Our phase velocity analysis not only recovers the same result, but also proves that coarse grid correction is only first order accurate for components in the cross-characteristic direction due to the phase shift error caused by the discretization coarse grid operators. Another approach is to use Galerkin coarsening [14, 19]. It turns out that its phase error is minimal, resulting in more accurate coarse grid correction.

In Section 3, explicit analytic formulae for the asymptotic expansion of the phase velocity of the different coarse grid correction approaches are established in one dimension. In Section 4, similar results in two dimensions are presented with the emphasis on Fourier components in the characteristic and cross characteristic directions. Numerical results are given in Section 5 to compare how these coarse grid correction approaches affect the actual multigrid convergence. Finally, concluding remarks are given in Section 6.

2. Model problem. The model problem we are interested in is the steady state solution of the convection-diffusion equation:

$$u_t - \epsilon \Delta u + w \cdot \nabla u = f \qquad x \in \Omega,$$

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subject to appropriate boundary conditions, and Ω is a d-dimensional unit cube. Here, we assume $\epsilon \ll 1$ and hence the equation is convection dominated. Discretizing the equation by finite difference methods on a standard uniform fine grid Ω^h with mesh size h results in a linear system

$$L^h u^h = f^h.$$

We consider solving the discrete problem using K grids, $\{\Omega^l\}_{l=0}^{K-1}$.

For pure hyperbolic equations, it is well-known that dissipation and dispersion are two fundamental quantities for analyzing numerical methods. Consider the 1D wave equation

$$u_t + a u_x = 0. \tag{2.1}$$

Given a finite difference scheme, suppose the Fourier transform of the numerical solution at time step $n+1$ can be written as

$$\hat{u}^{n+1}(\mu) = g(\mu)\hat{u}^n(\mu),$$

where $g(\mu)$ is the amplification factor. The scheme is dissipative if $|g(\mu)| < 1$, and it is dispersive if the phase speed [15], $\kappa(\mu)$, defined as,

$$\kappa(\mu) \equiv -\frac{\arg(g(\mu))}{\mu\pi\Delta t},$$

is different for different Fourier modes μ.

Thus, the classical multigrid analysis using Fourier analysis is deemed to be inadequate since it only considers the dissipation property. To give a more precise account of the wave propagation property of multigrid V-cycles, Gustafsson and Lötstedt [4, 12] analyzed the phase velocity of a two-grid iteration matrix M. Let $\hat{M}$ be its Fourier transform. It is well-known [5, 17] that $\hat{M}$ is block diagonal with 2×2 subblocks $\hat{M}_\mu$ where $\mu = 0, \ldots, N-1$.

Theorem 2.1 *Let λ_1 be the first eigenvalue of $\hat{M}_\mu$. For frequency $\mu \approx 0$,*

$$\lambda_1(\mu) = 1 - (\Delta t_h + \Delta t_H) i\mu\pi + O(\mu^2).$$

Consequently, the phase velocity of a two-grid method is

$$\kappa(\mu) = -\frac{\arg(\lambda_1(\mu))}{\mu\pi\Delta t_h} = 1 + \frac{\Delta t_H}{\Delta t_h} = 3.$$

The result can be generalized to K-level multigrid, in which case, $\kappa(\mu) = 2^K - 1$.

We remark that their analysis focuses primarily on the leading order terms of the asymptotic expansion of λ_1. If the initial wave consists of nonnegligible higher frequency modes, the effective speed of wave propagation is much slower than the analysis predicts. Figure 2.1 shows the propagation of a square wave by a three-level multigrid V-cycle on a grid with 128 grid points. It should have converged in $128/(2^3 - 1) \approx 36$ iterations; but instead, it takes more than 100 iterations due to numerical oscillations generated.

In the next section, we give a more detail analysis to explain the oscillation phenomenon. Furthermore, we consider two other coarse grid correction approaches and study their phase error behaviors for convection dominated problems in one and two dimensions. For all these approaches, we show that the convergence behavior of multigrid can be precisely described by the phase velocity analysis of the coarse grid correction matrix.

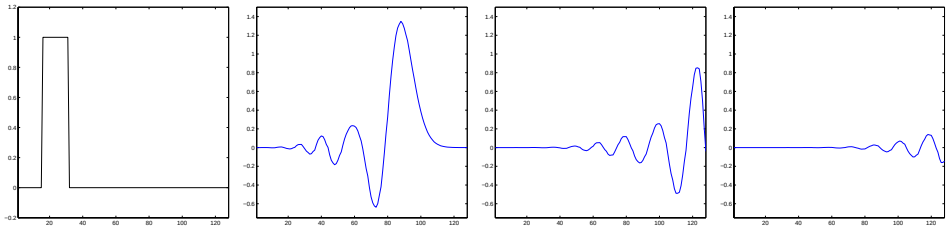

Figure 2.1: The numerical solutions given by a 3-level multigrid V-cycle at (a) iteration = 0, (b) iteration = 20, (c) iteration = 40, (d) iteration = 60.

3. One dimension. We start with 1D in which explicit formulae for the asymptotic expansion of the phase velocity can be established. The model problem becomes (assume ϵ is negligibly small):

$$u_x = f(x) \qquad -1 < x < 1,$$

with periodic boundary condition: $u(-1) = u(1)$. Without loss of generosity, we assume that $f^h \equiv 0$. Thus, we are interested in how the iteration error converges to zero. We shall consider three types of coarse grid correction approaches commonly used in the literature.

3.1. Inexact coarse grid correction. The coarse grid problem is solved inexactly by a few smoothing steps and the coarse grid operator is obtained by direct discretization. This is also the same approach considered by Gustafsson and Lötstedt [4], and others [7, 13]. Here, we extend the phase velocity analysis of Gustafsson and Lötstedt to include also the first correction term in the asymptotic expansions.

In a two-grid method consists of one pre-smoothing (one step of m-stage Runge-Kutta on the fine grid) followed by the coarse grid correction (one step of m-stage of Runge-Kutta on the coarse grid), the iteration matrix M of the two-grid method can be written as $M = CS$, where the coarse grid correction matrix C and smoothing matrix S are

$$\begin{aligned} C &= I + \sum_{j=1}^{m} \Delta t_H^j \, p \, (L^H)^{j-1} \prod_{k=m-j+1}^{m} (-\alpha_k) r L^h \\ S &= I + \sum_{j=1}^{m} \Delta t_h^j \, (L^h)^j \prod_{k=m-j+1}^{m} (-\alpha_k), \end{aligned}$$

where p is the linear interpolation and $r = \frac{1}{2} p^T$ its transpose.

Let $\hat{M} = \hat{C}\hat{S}$ be the Fourier transform of M. In two-grid analysis, it is customary to reorder the rows and columns of $\hat{M}$ ($\hat{C}$, $\hat{S}$) such that the low and high frequency modes are paired up; see [5, 17] for details. As a result, $\hat{M}$ ($\hat{C}$, $\hat{S}$) is block diagonal with 2×2 subblocks, $\hat{M}_\mu$, indexed by the wave numbers: $\mu = -N/2, \ldots, N/2 - 1$ corresponding to smooth or less oscillatory waves.

For easy exposition, we assume $m = 1$, and the coarse grid time step size, $\Delta t_H = \lambda H$. Then the coarse grid correction matrix can be simplified as:

$$C = I - \lambda H p r L^h,$$

Hence, the 2×2 subblocks of the Fourier transform of C are given by

$$\begin{aligned} \hat{C}_\mu &= I - \lambda H \hat{p}_\mu \hat{r}_\mu \hat{L}_\mu^h \\ &= I - \lambda H \begin{bmatrix} c_\mu^2 \\ -s_\mu^2 \end{bmatrix} \begin{bmatrix} c_\mu^2 & -s_\mu^2 \end{bmatrix} \frac{1}{h} \begin{bmatrix} 1 - e^{-\mu\pi h i} & 0 \\ 0 & 1 + e^{-\mu\pi h i} \end{bmatrix}. \end{aligned} \tag{3.1}$$

Since the high frequency errors have been reduced by the smoothing process, we are more interested in the "low-low" interaction, i.e. how the smooth waves are changed by the coarse grid correction. Hence, we focus just on the (1,1) entry of $\hat{C}_\mu$. By (3.1),

$$\hat{C}_\mu(1,1) = 1 - 2\lambda c_\mu^4(1 - e^{-\mu\pi hi}) \equiv |\hat{C}_\mu(1,1)|e^{-i\kappa(\mu)\mu\pi h\lambda}.$$

Here is our result on the dispersion of $\hat{C}_\mu(1,1)$ which is not considered explicitly by Gustafsson and Lötstedt [4, 12].

Theorem 3.1 *The dissipation and phase velocity of $\hat{C}_\mu$ are given, respectively, by*

$$\begin{aligned} |\hat{C}_\mu(1,1)| &\leq 1 \qquad \text{if and only if} \qquad 0 < \lambda \leq \frac{1}{2}, \\ \kappa(\mu) &= 2 + \frac{8\lambda - 15}{12}(\mu\pi h)^2 + O(\mu\pi h)^4. \end{aligned}$$

Proof. Express $\hat{C}_\mu(1,1)$ in terms of $s_\mu \equiv \sin(\mu\pi h)$ and $c_\mu \equiv \cos(\mu\pi h)$, and use the Taylor expansions of s_μ, c_μ and arctan. ∎

Remarks: (1) The coarse grid correction is dissipative. Moreover, the CFL condition on λ is more restrictive than the standard upwinding. (2) While the leading order term indicates propagation speed of 2 on the coarse grid, the negative second term shows that it is dispersive, which accounts for the oscillations observed in Figure 2.1.

Based on our more refined phase velocity analysis, convergence will be slowed down by oscillations unless the the smoother is extremely effective in damping most of the high frequency modes, for instance, by the use of artificial viscosity [13], or modified Runge-Kutta methods [7]. Otherwise, a fundamental change in the algorithm is needed to obtain a nonoscillatory multigrid method [9].

3.2. Exact coarse grid correction. In the second coarse grid correction approach, we consider exact coarse grid solve instead. Thus the coarse grid correction matrix becomes:

$$C = I - p(L^H)^{-1} r L^h.$$

As in the previous approach, we are interested in the low-low interaction, i.e. the (1,1) entry of $\hat{C}_\mu$.

Theorem 3.2 *The coarse grid correction of smooth waves given by the exact coarse grid solve together with linear interpolation is only first order accurate, i.e.*

$$|\hat{C}_\mu(1,1)| = \frac{\mu\pi h}{2} + O(\mu\pi h)^2.$$

Proof. By direct calculation,

$$\hat{C}_\mu(1,1) = 1 - c_\mu^3 e^{\mu\pi hi/2},$$

and the result follows by Taylor expansion. ∎

In the expression of $\hat{C}_\mu(1,1)$, the second term shows that after the coarse grid solve, the error is damped by c_μ^3, and more importantly, it has a phase error of $\mu\pi h/2$, implying that the coarse grid error is shifted precisely by 1/2 grid point (to the left). This shift arises from the discretization of the first order PDE on two different mesh sizes and consequently, leads to only first order accuracy in the coarse grid correction.

3.3. Galerkin coarse grid correction. Thirdly, we consider the use of Galerkin approach to form the coarse grid correction operator

$$G^H = rL^h p,$$

together with exact coarse grid solve. Then, the Fourier transform of C is given by

$$\hat{C}_\mu = I - \hat{p}(\hat{G}_\mu^H)^{-1}\hat{r}\hat{L}^h.$$

Theorem 3.3 *The coarse grid correction of smooth waves given by the exact coarse grid solve together with Galerkin coarse grid operator is third order accurate, i.e.*

$$|\hat{C}_\mu(1,1)| = \frac{1}{8}(\mu\pi h)^3 + O(\mu\pi h)^5.$$

Proof. Using the above formula, we obtain $\hat{C}_\mu(1,1) = (s_\mu^3 - ic_\mu^3)s_\mu^3/(c_\mu^6 + s_\mu^6)$. By Taylor expansion, we have

$$|\hat{C}_\mu(1,1)| = \frac{s_\mu^3}{\sqrt{c_\mu^6 + s_\mu^6}} = \frac{1}{8}(\mu\pi h)^3 + O(\mu\pi h)^5.$$

■

By a similar argument as before, it is easy to see from the expression of $\hat{C}_\mu(1,1)$ that the phase error $= O(\mu\pi h)^3$, which is negligibly small, and hence the coarse grid correction is much more accurate.

4. Two dimensions. The phase velocity analysis of Section 3.1 can be extended to 2D. Consider the convection dominated problem on a unit square:

$$-\epsilon\Delta u + a(x,y)u_x + b(x,y)u_y = f \qquad x \in \Omega = (-1,1)\times(-1,1),$$

with periodic boundary condition. In particular, we focus on two model problems:

(1) Entering flow (constant coefficient):

$$a(x,y) \equiv a, \quad b(x,y) \equiv b.$$

(2) Recirculating flow (variable coefficient):

$$a(x,y) = 4x(x-1)(1-2y), \quad b(x,y) = -4y(y-1)(1-2x).$$

We discretize the equation using the first order upwind scheme for the convection terms and center differencing for the Laplacian [3]. Our primary focus is on the limit $\epsilon \to 0$. We remark that some of the proves of the results in this section are similar to the 1D case and hence they are omitted.

4.1. Inexact coarse grid correction. The Fourier transform of the coarse grid correction matrix C is given by

$$\hat{C}_{\mu,\nu} \quad = \quad I - \lambda H \hat{p}_{\mu,\nu}\hat{r}_{\mu,\nu}\hat{L}^h_{\mu,\nu}.$$

where p is bilinear interpolation and r is full-weighting restriction. The (1,1) entry of $\hat{C}_{\mu,\nu}$ is then given by

$$\hat{C}_{\mu,\nu}(1,1) = 1 - 2\lambda h c_\mu^4 c_\nu^4 [\frac{a}{h}(1 - e^{-\mu\pi h i}) + \frac{b}{h}(1 - e^{-\nu\pi h i})].$$

To get more insight into the formula of $\hat{C}_{\mu,\nu}(1,1)$, we consider the special case where $a = b = 1$ and frequencies in the characteristic direction, i.e. $\nu = \mu$.

Theorem 4.1 *Assume $a = b = 1$. In the characteristic direction, i.e. $\nu = \mu$, the coarse grid correction is dissipative for $0 < \lambda \leq 1/4$, and dispersive, i.e.*

$$\begin{aligned} |\hat{C}_\mu(1,1)| &\leq 1 \qquad \text{if and only if} \qquad 0 < \lambda \leq \frac{1}{4}, \\ \kappa(\mu) &= 2 + (2\lambda - \frac{9}{4})(\mu\pi h)^2 + O(\mu\pi h)^4. \end{aligned}$$

Remark: the 2D approach is also dispersive, consistent with the 1D result.

As an example, we solve the model entering flow problem by multigrid, and snap shots of the errors in the first 15 V-cycles are shown in Figure 4.1. The mesh size is $h = 1/32$, and $\lambda = 0.25$. We observe that oscillations are generated at the tail as the square wave propagates from (-1,-1) to (1,1), which is justified by our phase velocity analysis. For the recirculating flow problem, Fourier analysis is not feasible, and yet we still observe a similar wave propagation phenomenon as in the entering flow problem.

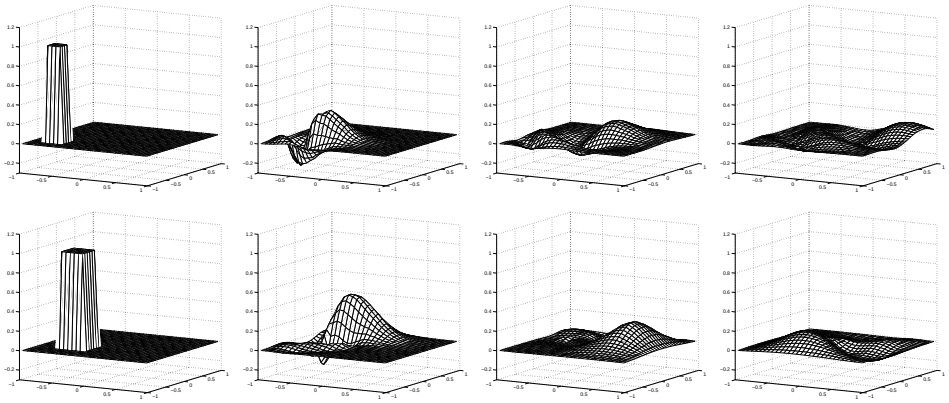

Figure 4.1: Numerical solutions given by a 3-level multigrid for (top row) the entering flow problem, and (bottom row) recirculating flow problem at iteration = 0, 5, 10, 15.

4.2. Exact coarse grid correction. With exact coarse grid correction and direct discretization for the coarse grid operator, the Fourier transform of the coarse grid correction matrix is

$$\hat{C}_{\mu,\nu} = I - \hat{p}_{\mu,\nu}(\hat{L}^H_{\mu,\nu})^{-1}\hat{r}_{\mu,\nu}\hat{L}^h_{\mu,\nu}.$$

Therefore,

$$\hat{C}_{\mu,\nu}(1,1) = 1 - c_\mu^4 c_\nu^4 \frac{\frac{a}{h}(1 - e^{-\mu\pi hi}) + \frac{b}{h}(1 - e^{-\nu\pi hi})}{\frac{a}{2h}(1 - e^{-\mu\pi 2hi}) + \frac{b}{2h}(1 - e^{-\nu\pi 2hi})}.$$

To facilitate understanding, we consider two special and yet important cases: frequency components in the characteristic direction, i.e. (μ, ν) such that

$$b\mu - a\nu = 0,$$

and, cross-characteristic direction [2, 3, 18], i.e. (μ, ν) such that

$$a\mu + b\nu = 0.$$

Theorem 4.2 *For the components in the characteristic direction and assuming $a = b$,*

$$|\hat{C}_{\mu,\nu}(1,1)| = \frac{\mu\pi h}{2} + O(\mu\pi h)^2.$$

For the components in the cross-characteristic direction and general a, b,

$$\lim_{\mu\to 0} \hat{C}_{\mu,\nu}(1,1) = \frac{1}{2}.$$

In particular, for $a = b$, then $\hat{C}_{\mu,\nu}(1,1) = 1 - c_\mu^6/2$.

Proof. In the characteristic direction, and $a = b$, then

$$\hat{C}_{\mu,\nu}(1,1) = 1 - c_\mu^7 e^{\frac{\mu\pi h i}{2}},$$

and hence

$$|\hat{C}_{\mu,\nu}(1,1)| = \frac{\mu\pi h}{2} + O(\mu\pi h)^2.$$

In the cross-characteristic direction, results follows from l'Hospital's rule. ■

We note that our analysis for the cross-characteristic direction is consistent with the result of Brandt and Yavneh [3] in which they consider the special case $b = 0$, and they point out that the coarse grid error is not a good approximation to the fine grid error for components in the cross-characteristic directions.

However, in both [3, 18], phase errors are not discussed. In the characteristic direction, the magnitude of the coarse grid error is in fact accurate: $|c_\mu^7 e^{\frac{\mu\pi h i}{2}}| = c_\mu^7$, but it has a phase error of $\mu\pi h/2$, just like the 1D case. Qualitatively speaking, the coarse grid error is shifted by $h/2$ in the characteristic direction, leading to the first order accuracy of $\hat{C}_{\mu,\nu}(1,1)$.

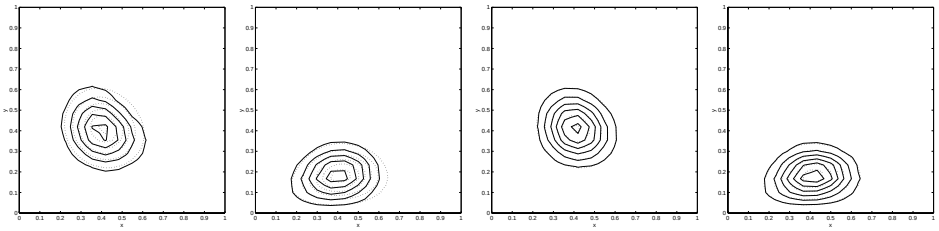

Figure 4.2: Contour plots of the fine grid error (dashed line) and the interpolated coarse grid error (solid line) for (a) the entering flow (exact coarse grid solve), (b) the recirculating flow (exact coarse grid solve), (c) the entering flow (Galerkin), (d) the recirculating flow (Galerkin).

Figure 4.2(a) and (b) show the contour plots of the fine grid error (dashed line) and the interpolated coarse grid error (solid line) for the entering flow and recirculating flow, respectively. Both results agree with the phase analysis that the interpolated coarse grid errors are shifted behind the directions of the flow.

4.3. Galerkin coarse grid correction. The Fourier transform is given by

$$\hat{C}_{\mu,\nu} = I - \hat{p}_{\mu,\nu}(\hat{r}_{\mu,\nu}\hat{L}^h_{\mu,\nu}\hat{p}_{\mu,nu})^{-1}\hat{r}_{\mu,\nu}\hat{L}^h_{\mu,\nu}.$$

We again consider the characteristic and cross-characteristic components.

Theorem 4.3 *For the components in the characteristic direction and assuming $a = b$,*

$$|\hat{C}_{\mu,\nu}(1,1)| = \frac{(\mu\pi h)^3}{8} + O(\mu\pi h)^5.$$

For the components in the cross-characteristic direction and general a, b,

$$\lim_{\mu\to 0} \hat{C}_{\mu,\nu}(1,1) = 0.$$

In particular, if $a = b$, then

$$|\hat{C}_{\mu,\nu}(1,1)| = \frac{(\mu\pi h)^2}{4} + O(\mu\pi h)^4.$$

In the Galerkin approach, the phase error in both directions is negligibly small as opposed to the exact coarse grid correction approach; see Figure 4.2(c) and (d). As a result, the coarse grid correction is second and third order accurate in the characteristic and cross-characteristic components, respectively.

5. Numerical results. In practice, the inexact coarse grid approach is appealing since it is simple and the same smoothing method can be used on all the coarse grids. However, such coarse grid correction is dispersive and oscillations generated delay multigrid convergence. In the exact coarse grid correction approach, the same smoothing method can also be used on all the coarse grids. Moreover, with exact coarse grid solve, the dispersive effect is much improved. However, the coarse grid correction is only first order accurate due to phase error, resulting in slower convergence. For the Galerkin approach, the coarse grid correction is more accurate, and hence the resulting multigrid convergence should be like the elliptic case.

We note that although our analysis suggests that the Galerkin approach has the least phase error, in practice, however, there are several drawbacks. It has been observed that the Galerkin coarse grid operator on the coarse grids become more and more like the central finite difference discretization. Operator-dependent interpolations may be needed to remedy the problems [19]. Another issue is extra storage for the coarse grid operators.

In the following, we compare the effects on the convergence of multigrid V-cycle by the inexact, nonGalerkin and Galerkin coarse grid correction approaches. The first example is the steady state solution of the one-dimensional linear wave equation:

$$u_t + u_x = 0.$$

First order Runge-Kutta method is used as the smoother for all the approaches with CFL number $\lambda = 0.5$. Linear interpolation and full weighting restriction are used between grids. The multigrid V-cycle iterations stop when the relative residual norm is less than 10^{-6}.

The number of multigrid V-cycles are shown in Table 5.1. To verify the results of the previous sections, we use two multigrid levels and consider a smooth initial guess and a square wave initial guess (in parenthesis). The results show that the number of multigrid V-cycles taken by the inexact coarse grid correction increases as mesh size decreases; thus we do not observe the classical mesh-independent convergence. Moreover, the convergence is slow due to the dispersion. Both exact and Galerkin coarse grid correction approaches, which use

h	Inexact	Exact	Galerkin
1/32	35 (31)	14 (13)	11 (8)
1/64	52 (44)	14 (9)	12 (5)
1/128	83 (73)	14 (6)	12 (3)
1/256	144 (141)	14 (5)	12 (3)

Table 5.1: Number of two-grid V-cycles for the 1D linear wave equation using inexact, exact, and Galerkin coarse grid corrections.

	Inexact					Exact				
h	2	3	4	5	6	2	3	4	5	6
1/32	31	35	39			13	25	34		
1/64	44	43	45	51		9	22	37	46	
1/128	73	52	58	61	61	6	13	31	49	55
1/256	141	83	64	72	72	5	9	19	40	59

Table 5.2: Number of multigrid V-cycles for the 1D linear wave equation using inexact and exact coarse grid corrections.

exact coarse grid solve, show much better convergence. Because of the shifting of the coarse grid error, the exact approach is not as efficient as the Galerkin approach.

In Table 5.2, it shows the multilevel results of the inexact and nonGalerkin coarse grid correction approach. The Galerkin approach requires different smoothing parameters on the coarse grids and hence it is not tested in this case. For the inexact coarse grid correction approach, the convergence should, in principle, have been improved by using more coarse grids based on the result of Gustafsson and Lötstedt (cf. Theorem 2.1). It is true when the mesh size is very small ($h = 1/256$) and hence the small wave number components are more dominant in the initial guess. But when the coarse grid gets smaller, the convergence starts to deteriorate. For the exact coarse grid correction approach, the multigrid convergence also starts to deteriorate on the coarser grids due to the phase shift of the coarse grid errors which is more serious with larger mesh size.

h	Inexact	Exact	Galerkin
1/32	28 (29)	13 (14)	7 (7)
1/64	41 (45)	13 (14)	5 (8)
1/128	70 (77)	11 (14)	5 (9)

Table 5.3: Number of two-grid V-cycles for the 2D entering flow problem using inexact, exact, and Galerkin coarse grid corrections.

We next consider the model entering flow and recirculating flow problems in two dimensions (cf. Section 4). Similar multigrid setting as in the one-dimensional case: Euler's smoothing, linear interpolation and full weighting restriction. Since the CFL number $\lambda = 0.25$ in two dimension, we use 2 presmoothing and postsmoothing steps instead.

The two-grid results are shown in Table 5.3 with a smooth initial guess and a square wave initial guess (in parenthesis). As in the 1D case, the multigrid convergence of the inexact coarse grid solve depends on the mesh size whereas the other two do not. Also, the Galerkin approach is more efficient than the exact coarse grid correction approach.

6. Conclusions. We have demonstrated that phase velocity analysis is a useful tool to analyze multigrid methods for convection dominated problems, and brings more insight into the efficiency of different coarse grid correction approaches.

For inexact coarse grid correction, the propagation of smooth waves is accelerated by using coarse grids. However, dispersion occurs in the coarse grid correction process which slows down substantially the multigrid convergence. The exact coarse grid correction approach does not rely on wave propagation and hence dispersion is not an issue. However, due to the use of the discretization matrix as the coarse grid operator, there is a phase error in the coarse grid solve which deteriorates the multigrid convergence. The Galerkin approach has the advantage of maintaining small phase shift error in the coarse grid correction. However, one needs to form the coarse grid operators on every grids, and hence to determine new sets of parameters, e.g. time-step size, for the smoother to obtain good smoothing efficiency.

We have addressed the issue of phase velocity analysis of multigrid methods for convection dominated problems. However, the design of new multigrid methods which possess good phase velocity property requires further investigation.

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