A FETI-DP Method for the Mortar Discretization of Elliptic Problems with Discontinuous Coefficients

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Summary. Second order elliptic problems with discontinuous coefficients are considered. The problem is discretized by the finite element method on geometrically conforming non-matching triangulations across the interface using the mortar technique. The resulting discrete problem is solved by a FETI-DP method. We prove that the method is convergent and its rate of convergence is almost optimal and independent of the jumps of coefficients. Numerical experiments for the case of four subregions are reported. They confirm the theoretical results.

1 Introduction

In this paper we discuss a second order elliptic problem with discontinuous coefficients defined on a polygonal region $\Omega \subset R^2$ which is a union of many polygons Ω_i. The problem is discretized by the finite element method on geometrically conforming non-matching triangulations across $\Gamma = \cup_i \partial\Omega_i \backslash \partial\Omega$ using the mortar technique, see Bernardi et al. [1994]. The resulting discrete problem is solved by a FETI-DP method, see Farhat et al. [2001], Klawonn et al. [2002], Mandel and Tezaur [2001] for the matching triangulation and Dryja and Widlund [2002], Dryja and Widlund [2003] for the non-matching one. The method is discussed under the assumption of continuity of the solution at vertices of Ω_i. We prove that the method is convergent and its rate of convergence is almost optimal and independent of the jumps of coefficients.

The presented results are a generalization of results obtained in Dryja and Widlund [2002], Dryja and Widlund [2003] for continuous coefficients and many subregions, and in Dryja and Proskurowski [2003] for discontinuous coefficients and two subregions Ω_i. In the first two papers two different preconditioners, a standard one and a generalized one, are analyzed for the mortar discretization which is not standard. The mortar condition there is modified at the vertices of Ω_i using the continuity of the solution at these vertices. In the present paper we consider a standard mortar discretization and a standard preconditioner. Numerical experiments for the case of four subregions are reported. They confirm the theoretical results.

The paper is organized as follows. In Section 2, the differential and discrete problems are formulated. In Section 3, a matrix form of the discrete problem is given. The preconditioner is described and analyzed in Section 4. Numerical experiments are presented in Section 5.

2 Differential and discrete problem

We consider the following differential problems. Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \quad v \in H_0^1(\Omega), \tag{1}$$

where $a(u,v) = (\rho(x) \bigtriangledown u, \bigtriangledown u)_{L^2(\Omega)}, \quad f(v) = (f,v)_{L^2(\Omega)}$.

We assume that Ω is a polygonal region and $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$, Ω_i are disjoint polygonal subregions of diameter H_i, $\rho(x) = \rho_i$ is a positive constant on Ω_i and $f \in L^2(\Omega)$. We solve (1) by the FEM on non–matching triangulation across $\partial\Omega_i$. To describe a discrete problem the mortar technique is used, see Bernardi et al. [1994].

We impose on Ω_i a triangulation with triangular elements and parameter h_i. The resulting triangulation in Ω is non-matching across $\partial\Omega_i$. We assume that the triangulation on each Ω_i is quasiuniform and additionally that the parameters h_i and h_j on a common edge of Ω_i and Ω_j are proportional. Let $X_i(\Omega_i)$ be a finite element space of piecewise linear continuous functions defined on the introduced triangulation. We assume that functions of $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. Let

$$X^h(\Omega) = X_1(\Omega_1) \times \ldots \times X_N(\Omega_N). \tag{2}$$

Note that $X^h(\Omega) \subset L^2(\Omega)$ but $X^h(\Omega) \not\subset H_0^1(\Omega)$. To formulate a discrete problem for (1) we use the mortar technique for geometrically conforming case. For that the following notation is used. Let Γ_{ij} be a common edge of two substructures Ω_i and Ω_j, $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. Let $\Gamma = (\cup_i \partial\Omega_i) \backslash \partial\Omega$. We now select open edges $\gamma_m \subset \Gamma$, called *mortar* such that $\overline{\Gamma} = \cup\overline{\gamma}_m$ and $\gamma_m \cap \gamma_n = 0$ for $m \neq n$. Let Γ_{ij} as an edge of Ω_i be denoted by $\gamma_{m(i)}$ and called *mortar* (master), and let Γ_{ij} as an edge of Ω_j be denoted by $\delta_{m(j)}$ and called *non-mortar* (slave). The criteria for choosing $\gamma_{m(i)}$ as the mortar side is that $\rho_i \geq \rho_j$, the coefficients on Ω_i and Ω_j, respectively.

Let $M(\delta_{m(j)})$ be a subspace of $W_j(\delta_{m(j)})$, the restriction of $X_j(\Omega_j)$ to $\delta_{m(j)}$, $\delta_{m(j)} \subset \partial\Omega_j$. Functions of $M(\delta_{m(j)})$ are constants on elements of the triangulation on $\delta_{m(j)}$ which touch $\partial\delta_{m(j)}$. We say that $u_i \in X_i(\Omega_i)$ and $u_j \in X_j(\Omega_j)$ on $\delta_m \equiv \delta_{m(j)} = \gamma_{m(i)} = \Gamma_{ij}$, an edge common to Ω_i and Ω_j, satisfy the mortar condition if

$$\int_{\delta_m} (u_i - u_j)\psi ds = 0, \quad \psi \in M(\delta_m). \tag{3}$$

We are now in a position to introduce V^h, the space for discretization of (1). Let $V^h(\Omega)$ be a subspace of $X^h(\Omega)$ of functions which satisfy the mortar condition (3) for each $\delta_m \subset \Gamma$ and which are continuous at common vertices of the substructures. The discrete problem for (1) in V^h is defined as follows.

Find $u_h^* \in V^h$ such that

$$a_H(u_h^*, v_h) = f(v_h), \quad v_h \in V^h \tag{4}$$

where $a_H(u,v) = \sum_{i=1}^{N} a_i(u,v), \quad a_i(u,v) = \rho_i(\bigtriangledown u, \bigtriangledown v)_{L^2(\Omega_i)}$. The problem has a unique solution and the error bound is known, see Bernardi et al. [1994].

3 FETI-DP equation

To derive FETI-DP method we first rewrite the problem (4) as a saddle-point problem using Lagrange multipliers. For $u = \{u_i\}_{i=1}^N \in X^h(\Omega)$ and $\psi = \{\psi_p\}_{p=1}^P \in M(\Gamma) = \prod_m M(\delta_m)$, the mortar condition (3) can be rewritten as

$$b(u,\psi) \equiv \sum_{i=1}^{N} \sum_{\delta_{m(i)} \subset \partial\Omega_i} \int_{\delta_{m(i)}} (u_i - u_j)\psi_k ds = 0, \tag{5}$$

where $\delta_{m(i)} = \gamma_{m(j)} = \Gamma_{ij}, \psi_k \in M(\delta_{m(i)})$. Let $\tilde{X}^h(\Omega)$ denote a subspace of $X^h(\Omega)$ of functions which are continuous at common vertices of substructures.

The problem now consists of finding $(u_h^*, \lambda_h^*) \in \tilde{X}^h(\Omega) \times M(\Gamma)$ such that

$$a(u_h^*, v_h) + b(v_h, \lambda_h^*) = f(v_h), \quad v_h \in \tilde{X}^h(\Omega), \tag{6}$$

$$b(u_h^*, \psi_h) = 0, \quad \psi_h \in M(\Gamma). \tag{7}$$

It can be proved that u_h^*, the solution of (6) - (7) is the solution of (4) and vice versa. Therefore the problem (6) - (7) has a unique solution.

To derive a matrix form of (6) - (7) we first need a matrix formulation of (7). Using the nodal basis functions $\varphi^{(l)}_{\delta_{m(i)}} \in W_i(\delta_{m(i)})$, $\varphi^{(k)}_{\gamma_{m(j)}} \in W_j(\gamma_{m(j)})$ and $\psi^{(p)}_{\delta_{m(i)}} \in M_m(\delta_{m(i)})$ ($\delta_{m(i)} = \gamma_{m(j)} = \Gamma_{ij}$) the equation (7) can be rewritten on $\bar{\delta}_{m(i)}$ as

$$B_{\delta_{m(i)}} u_{i\delta_{m(i)}} - B_{\gamma_{m(j)}} u_{j\gamma_{m(j)}} = 0, \tag{8}$$

where $u_{i\delta_{m(i)}}$ and $u_{j\gamma_{m(j)}}$ are vectors which represent $u_i|_{\delta_{m(i)}} \in W_i(\delta_{m(i)})$ and $u_j|_{\gamma_m(j)} \in W_j\ (\gamma_{m(j)})$, and ($n_{\delta_{(i)}} \equiv n_{\delta_{m(i)}}$ and $n_{\gamma_{(j)}} \equiv n_{\gamma_{m(j)}}$)

$$\begin{aligned} B_{\delta_{m(i)}} &= \{(\psi^{(p)}_{\delta_{m(i)}}, \varphi^{(k)}_{\delta_{m(i)}})_{L^2(\delta_{m(i)})}\},\ p = 1,..,n_{\delta(i)},\ k = 0,..,n_{\delta(i)}+1, \\ B_{\gamma_{m(j)}} &= \{(\psi^{(p)}_{\delta_{m(i)}}, \varphi^{(l)}_{\gamma_{m(j)}})_{L^2(\gamma_{m(j)})}\},\ p = 1,..,n_{\delta(i)},\ l = 0,..,n_{\gamma_{(j)}}+1. \end{aligned} \tag{9}$$

Here $n_{\delta(i)}, n_{\delta(i)}+2$ and $n_{\gamma(j)}+2$ are the dimensions of $M_m(\delta_{m(i)}),\ W_i(\delta_{m(i)})$ and $W_j(\gamma_{m(j)})$, respectively. Note that $B_{\delta_{m(i)}}$ and $B_{\gamma_{m(j)}}$ are rectangular matrices. We split the vectors $u_{i\delta_{m(i)}}$ and $u_{j\gamma_{m(j)}}$ into vectors $u^{(r)}_{i\delta_{m(i)}},\ u^{(c)}_{i\delta_{m(i)}}$ and $u^{(r)}_{j\gamma_{m(j)}},\ u^{(c)}_{j\gamma_{m(j)}}$, respectively, where $u^{(c)}_{i\delta_{m(i)}}$ and $u^{(c)}_{j\gamma_{m(j)}}$ represent values of functions u_i and u_j at the end points of $\delta_{m(i)}$ and $\gamma_{m(j)}$, and $u^{(r)}_{i\delta_{m(i)}}$ and $u^{(r)}_{j\gamma_{m(j)}}$ represent values of u_i and u_j at the interior nodal points of $\delta_{m(i)}$ and $\gamma_{m(j)}$. Using this notation one can rewrite (8) as

$$(B^{(r)}_{\delta_{m(i)}} u^{(r)}_{i\delta_{m(i)}} + B^{(c)}_{\delta_{m(i)}} u^{(c)}_{i\delta_{m(i)}}) - (B^{(r)}_{\gamma_{m(j)}} u^{(r)}_{j\gamma_{m(j)}} + B^{(c)}_{\gamma_{m(j)}} u^{(c)}_{j\gamma_{m(j)}}) = 0. \tag{10}$$

Note that

$$B^{(r)}_{\delta_{m(i)}} = \{(\psi^{(p)}_{\delta_{m(i)}}, \varphi^{(k)}_{\delta_{m(i)}})_{L^2(\delta_{m(i)})}\}, \quad p,\ k = 1, \ldots, n_{\delta(i)} \tag{11}$$

is a square tridiagonal matrix $n_{\delta(i)} \times n_{\delta(i)}$, symmetric and positive definite and $cond(B^{(r)}_{\delta_{m(i)}}) \sim 1$, while the remaining matrices $B^{(c)}_{\delta_{m(i)}},\ B^{(c)}_{\gamma_{m(j)}},\ B^{(r)}_{\gamma_{m(j)}}$ are rectangular with dimensions $n_{\delta(i)} \times 2,\ n_{\delta(i)} \times 2,\ n_{\delta(i)} \times n_{\gamma(j)}$, respectively.

Let $K^{(l)}$ be the stiffness matrix of $a_l(.\,,.\,)$. It is represented as

$$K^{(l)} = \begin{pmatrix} K^{(l)}_{ii} & K^{(l)}_{ir} & K^{(l)}_{ic} \\ K^{(l)}_{ri} & K^{(l)}_{rr} & K^{(l)}_{rc} \\ K^{(l)}_{ci} & K^{(l)}_{cr} & K^{(l)}_{cc} \end{pmatrix}, \tag{12}$$

where the rows correspond to the interior unknowns $u^{(i)}_l$ of Ω_l, $u^{(r)}_l$ to its edges, and $u^{(l)}_c$ to its vertices. Let $S^{(l)}$ denote the Schur complement of $K^{(l)}$ with respect to the second and third rows, i.e. to the unknowns $u^{(r)}_l$ and $u^{(c)}_l$. This matrix is represented as

$$S^{(l)} = \begin{pmatrix} S^{(l)}_{rr} & S^{(l)}_{rc} \\ S^{(l)}_{cr} & S^{(l)}_{cc} \end{pmatrix}, \tag{13}$$

where the first row corresponds to the unknowns $u^{(r)}_l$ while the second one to $u^{(c)}_l$. Let

$$S = \text{ diag } \{S^{(l)}\}^N_{l=1}, \quad S_{rr} = \text{ diag } \{S^{(l)}_{rr}\}^N_{l=1}, \quad S_{cr} = (S^{(1)}_{cr}, \ldots, S^{(N)}_{cr}), \tag{14}$$

and the solution u^*_h of (6) - (7) be represented as $(u^{(i)}, u^{(r)}, u^{(c)})$ where these global sub-vectors correspond to the local unknowns $u^{(i)}_l$, $u^{(r)}_l$, $u^{(c)}_l$, respectively. We have taken into account that the values of $u^{(c)}_l$ at the common vertices of substructures are equal.

We set $\tilde{\lambda}^* = \{B^{(r)}_{\delta_{m(i)}} \lambda^*_{\delta_{m(i)}}\},\ \delta_{m(i)} \subset \Gamma$, where $\lambda^* = \{\lambda^*_{\delta_{m(i)}}\}$ is the solution of (6) - (7). The mortar condition is represented by $B = (B_r, B_c)$, where these global diagonal matrices are represented by the local ones $(I^{(r)}_{\delta_{m(i)}},\ -(B^{(r)}_{\delta_{m(i)}})^{-1}B^{(r)}_{\gamma_{m(j)}})$ and $((B^{(r)}_{\delta_{m(i)}})^{-1}B^{(c)}_{\delta_{m(i)}},\ -(B^{(r)}_{\delta_{m(i)}})^{-1}B^{(c)}_{\gamma_{m(j)}})$, respectively and $I^{(r)}_{\delta_{m(i)}}$ is an identity matrix of $n_{\delta(i)} \times n_{\delta(i)}$. The form of these matrices follows from (10) after multiplying it by $(B^{(r)}_{\delta_{m(i)}})^{-1}$.

To represent (6) - (7) in the matrix form we first eliminate unknowns corresponding to the interior nodal points of Ω_l, then use the assumption that the unknowns corresponding to the common vertices of Ω_l are the same (the continuity at the vertices) and finally setting $\tilde{\lambda}^* = \{B^{(r)}_{\delta_{m(i)}}\lambda^*_{\delta_{m(i)}}\}$ we get

$$S_{rr}u^{(r)} + S_{rc}u^{(c)} + B_r^T\tilde{\lambda}^* = g_r, \tag{15}$$

$$S_{cr}u^{(r)} + \tilde{S}_{cc}u^{(c)} + B_c^T\tilde{\lambda}^* = g_c, \tag{16}$$

$$B_r u^{(r)} + B_c u^{(c)} = 0. \tag{17}$$

Here S_{rr} and S_{cr} ($S_{rc} = S_{cr}^T$) are defined in (14) while $\tilde{S}_{cc}$ is defined by $S^{(l)}_{cc}$, see (13), taking into account that $u_l^{(c)}$ at common vertices of substructures are the same.

Eliminating $u^{(r)}$ and $u^{(c)}$ from (15) - (17) we get

$$F\tilde{\lambda}^* = d, \tag{18}$$

where $F = B\tilde{S}^{-1}B^T,\ d = B\tilde{S}^{-1}g,\ B = (B_r, B_c),\ g = (g_r, g_c)^T$ and

$$\tilde{S} = \begin{pmatrix} S_{rr} & S_{rc} \\ S_{cr} & \tilde{S}_{cc} \end{pmatrix}. \tag{19}$$

We check straightforwardly that F and d can be represented as follows:

$$F = F_{rr} - F_{rc}F_{cc}^{-1}F_{cr}, \quad F_{rc}^T = F_{cr}, \tag{20}$$

where
$F_{rr} = B_r S_{rr}^{-1} B_r^T, \quad F_{rc} = B_c - B_r S_{rr}^{-1} S_{rc}, \quad F_{cc} = \tilde{S}_{cc} - S_{cr} S_{rr}^{-1} S_{rc},$
$d = d_r - F_{rc}F_{cc}^{-1}d_c, \quad d_r = B_r S_{rr}^{-1} g_r, \quad d_c = g_c - S_{cr}S_{rr}^{-1}g_r.$

In the next section we analyze the preconditioner for F.

4 FETI-DP preconditioner

The preconditioner M for (18) is defined as

$$M^{-1} = B_r S_{rr} B_r^T. \tag{21}$$

An ordering of substructures Ω_l is called Neumann-Dirichlet (N-D) ordering (a check board coloring) if all sides of a fixed Ω_l are mortar while all sides of the neighboring substructures of Ω_l are non-mortar.

Theorem 1. *Let the mortar side be chosen where the coefficient ρ_i is larger. Then for $\lambda \in M(\Gamma)$ the following holds*

$$c_0 \left(1 + \log \frac{H}{h}\right)^{\alpha} \langle M\lambda, \lambda\rangle \leq \langle F\lambda, \lambda\rangle \leq c_1 \left(1 + \log \frac{H}{h}\right)^{2} \langle M\lambda, \lambda\rangle, \qquad (22)$$

where $\alpha = 0$ for N-D ordering of substructures and $\alpha = -2$ in the general case; c_0 and c_1 are positive constants independent of h_i, H_i, and the jumps of ρ_i; $h = \min_i h_i, H = \max_i H_i$.

In the proof of Theorem 1 we will need the following lemmas.

Lemma 1. *For $w \in X_1(\partial\Omega_1) \times \ldots \times X_N(\partial\Omega_N)$ with the same values at the vertices of Ω_i the following holds*

$$||B_r^T B_r z_r||^2_{S_{rr}} \leq C(1 + \log \frac{H}{h})^2 ||w||^2_S, \qquad (23)$$

where $z_r = w - I_H w$ and $I_H w$ is a linear interpolant of w on edges of $\partial\Omega_i$ with values w at the end points of the edges.

Lemma 2. *For $\lambda \in M(\Gamma)$*

$$C(1 + \log \frac{H}{h})^{\alpha} \langle M\lambda, \lambda\rangle \leq \langle F_{rr}\lambda, \lambda\rangle, \qquad (24)$$

where $\alpha = 0$ for the N-D ordering of substructures Ω_l and $\alpha = -2$ in the general case, C is independent of h, H and the jumps of ρ_i.

Proofs of these estimates are slight modifications of the proofs of statements in Dryja and Widlund [2002]. The only item one needs to take into account is that the coefficients ρ_i are larger on the mortar sides. Therefore the proofs of these lemmas are omitted.

Proof. To prove the RHS of Theorem 1 we proceed as follows. For $-\lambda \in M(\Gamma)$ we compute $w = (w^{(r)}, w^{(c)})$ by solving (15) - (16) with $g_r = 0$ and $g_c = 0$. Note that this problem has a unique solution under the assumption that $u^{(c)}$ is continuous at the cross points. Using this, after some manipulations we obtain

$$\langle F\lambda, \lambda\rangle = \langle B_r w^{(r)} + B_c w^{(c)}, \lambda\rangle = \langle Bw, \lambda\rangle. \qquad (25)$$

Let $I_H w$ be a linear interpolant of w on edges with values w at the end points of each edge. Note that $Bw = B(w - I_H w) = B_r z_r$ since $z_r \equiv w - I_H w = 0$ at the end points of the edges. Using that in (25), we get

$$\langle F\lambda, \lambda\rangle = \langle Bw, \lambda\rangle = \langle B_r z_r, \lambda\rangle. \qquad (26)$$

On the other hand, using that $\tilde{S}w = B^T\lambda$ and $\langle \tilde{S}w, w\rangle = \langle Sw, w\rangle$, see (15) - (17), we have

$$\langle Bw, \lambda \rangle = \frac{\langle Bw, \lambda \rangle^2}{\langle Bw, \lambda \rangle} = \frac{\langle B_r z_r, \lambda \rangle^2}{\langle Sw, w \rangle} \leq \frac{||M^{1/2}\lambda||^2 ||M^{-1/2} B_r z_r||^2}{||w||_S^2}. \qquad (27)$$

By Lemma 1

$$||M^{-1/2} B_r z_r||^2 = ||B_r^T B_r z_r||_{S_{rr}}^2 \leq C(1 + \log \tfrac{H}{h})^2 ||w||_S^2. \qquad (28)$$

Substituting this into (27) we have

$$\langle Bw, \lambda \rangle \leq C(1 + \log \frac{H}{h})^2 ||M^{1/2}\lambda||^2. \qquad (29)$$

Using this in (26) we get the RHS estimate of Theorem 1.

To prove the LHS of Theorem 1 we first note that, $F \leq F_{rr}$, see (20), and then use Lemma 2.

5 Numerical results

The test example for all our experiments is the weak formulation, see (1), of

$$-div(\rho(x)\nabla u) = f(x) \text{ in } \Omega, \qquad (30)$$

with the Dirichlet boundary conditions on $\partial\Omega$, where Ω is a union of four disjoint square subregions Ω_i, $i = 1, \ldots, 4$, of a diameter one, and $\rho(x) = \rho_i$ is a positive constant in each Ω_i. The mortar and non-mortar sides are chosen such that $\rho_\gamma \geq \rho_\delta$, see Theorem 1. The region Ω is cut into 4 subregions in a checkerboard coloring way: two equidistant grids (with the ratios 1:1, 2:1, 4:1, etc.) are imposed, one on the black, the other on the white squares. A random right hand side to f of (30) is chosen. Numerical experiments have been carried out with different scaling of the coefficients in the preconditioner. The best results were obtained for the preconditioner with $\rho_\delta = \rho_\gamma = 1$. They are reported in Table 1 and Table 2, and they confirm the theory.

References

C. Bernardi, Y. Maday, and A. T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In H. Brezis and J.-L. Lions, editors, *Collège de France Seminar*. Pitman, 1994. This paper appeared as a technical report about five years earlier.

M. Dryja and W. Proskurowski. On preconditioners for mortar discretization of elliptic problems. *Numerical Linear Algebra with Applications*, 10:65–82, 2003.

M. Dryja and O. Widlund. A FETI-DP method for a mortar discretization of elliptic problems. In P. L. and T. A., editors, *Recent developments in domain decomposition methods*, pages 41–52. Springer Verlag, 2002.

Table 1. Iteration count without (denoted NO) and with the FETI-DP preconditioner (denoted DP) for different grid ratios $\frac{h_\delta}{h_\gamma}$; continuous coefficients.

	$\frac{h_\delta}{h_\gamma}=1:1$		$\frac{h_\delta}{h_\gamma}=2:1$		$\frac{h_\delta}{h_\gamma}=4:1$		$\frac{h_\delta}{h_\gamma}=8:1$		$\frac{h_\delta}{h_\gamma}=16:1$	
h_δ	NO	DP	NO	DP	NO	DP	NO	DP	NO	DP
1/8	12	3	12	8	11	10	12	10	11	10
1/16	17	3	16	9	15	11	15	11	15	11
1/32	23	4	22	9	20	11	20	12	19	11
1/64	28	4	28	9	27	11	25	12	-	-
1/128	38	4	36	9	33	11	-	-	-	-
1/256	49	4	47	9	-	-	-	-	-	-
1/512	63	5	-	-	-	-	-	-	-	-

Table 2. Iteration count without (denoted NO) and with the FETI-DP preconditioner (denoted DP) for different grid ratios $\frac{h_\delta}{h_\gamma}$ and $\frac{\rho_\gamma}{\rho_\delta}=1000$.

	$\frac{h_\delta}{h_\gamma}=1:1$		$\frac{h_\delta}{h_\gamma}=2:1$		$\frac{h_\delta}{h_\gamma}=4:1$		$\frac{h_\delta}{h_\gamma}=8:1$		$\frac{h_\delta}{h_\gamma}=16:1$	
h_δ	NO	DP	NO	DP	NO	DP	NO	DP	NO	DP
1/8	13	5	10	6	9	5	9	5	9	5
1/16	17	6	14	7	12	6	12	5	12	5
1/32	22	6	18	7	16	6	15	5	16	5
1/64	29	7	25	7	21	6	21	5	-	-
1/128	37	7	30	7	27	6	-	-	-	-
1/256	49	7	42	7	-	-	-	-	-	-
1/512	63	7	-	-	-	-	-	-	-	-

M. Dryja and O. Widlund. A generalized FETI-DP method for the mortar discretization of elliptic problems. In I. H. et al., editor, *Fourteenth International Conference in Domain Decomposition methods*, pages 27–38. UNAM, Mexico City, Mexico, 2003.

C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen. FETI-DP: A dual-primal unified FETI method - part i: A faster alternative to the two-level FETI method. *Int. J. Numer. Meth. Engrg.*, 50:1523–1544, 2001.

A. Klawonn, O. B. Widlund, and M. Dryja. Dual-Primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients. *SIAM J.Numer.Anal.*, 40, 159-179 2002.

J. Mandel and R. Tezaur. On the convergence of a dual-primal substructuring method. *Numer. Math.*, 88:543–558, 2001.