
Mortar Finite Elements with Dual Lagrange Multipliers: Some Applications^{*}

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Summary. Domain decomposition techniques provide a powerful tool for the numerical approximation of partial differential equations. We consider mortar techniques with dual Lagrange multiplier spaces to couple different discretization schemes. It is well known that the discretization error for linear mortar finite elements in the energy norm is of order h . Here, we apply these techniques to curvilinear boundaries, nonlinear problems and the coupling of different model equations and discretizations.

1 Introduction

The numerical approximation of partial differential equations is often a challenging task. When different physical models should be used in different subregions, a suitable discretization scheme has to be used in each region. Mortar methods yield efficient and flexible coupling techniques for different discretization schemes. The central idea of mortar methods is to decompose the domain of interest into non-overlapping subdomains and impose a weak continuity condition across the interface by requiring that the jump of the solution is orthogonal to a suitable Lagrange multiplier space, see Bernardi et al. [1993, 1994]. Here, we work with mortar techniques and dual Lagrange multiplier spaces. These non-standard Lagrange multipliers show the same qualitative a priori estimates and quantitative numerical results as the standard ones and yield locally supported basis functions for the constrained space leading to a cheaper numerical realization, see Wohlmuth [2001]. This paper is concerned with applications of mortar methods to couple different physical models in different simulation regions. In the next section, we apply mortar methods to couple compressible and nearly incompressible materials in linear elasticity. In Section 3, the linear Laplace operator is coupled with the non-linear p -Laplace operator. Finally in Section 4, we show an application to an elasto-acoustic problem, and a generalized eigenvalue problem has to be solved. For

^{*} This work was supported in part by the Deutsche Forschungsgemeinschaft, SFB 404, C12.

all our models, we provide numerical results. The weak coupling in terms of dual Lagrange multipliers results in a diagonal matrix on the slave side. As a consequence, the Lagrange multiplier can be eliminated locally, and optimal multigrid methods can be applied to the resulting positive definite system.

2 Compressible and Nearly Incompressible Materials

In this section, we consider a problem in linear elasticity with two different materials in two subdomains, one of them being nearly incompressible. We assume that the domain $\Omega \subset \mathbb{R}^2$ is decomposed into two non-overlapping subdomains Ω_1 and Ω_2 with a common interface $\bar{\Gamma} = \bar{\Omega}_1 \cap \bar{\Omega}_2$, and the subdomain Ω_1 is occupied with a nearly incompressible material having a very large Lamé parameter λ_1 . It is well-known that standard low order finite elements for nearly incompressible materials suffer from locking, see Babuška and Suri [1992], and various approaches have been introduced to improve the numerical results. Working with a mixed formulation on Ω_1 , see, e.g., Braess [2001], and standard finite element approach on Ω_2 , we use mortar techniques with dual Lagrange multipliers to realize the coupling between the two formulations. On each subdomain, we define the space

$$\mathbf{H}_*^1(\Omega_k) := \{\mathbf{v} \in H^1(\Omega_k)^2, \mathbf{v}|_{\partial\Omega \cap \partial\Omega_k} = 0\}, \quad k = 1, 2,$$

and consider the constrained product space

$$\mathbf{V} := \{\mathbf{v} \in \prod_{k=1}^2 \mathbf{H}_*^1(\Omega_k) \mid \int_{\Gamma} [\mathbf{v}] \cdot \boldsymbol{\psi} \, d\sigma = 0, \boldsymbol{\psi} \in \mathbf{M}\},$$

where $\mathbf{M} := \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ is the Lagrange multiplier space, and $[\mathbf{v}]$ is the jump of \mathbf{v} across Γ . Introducing an additional unknown $p := \lambda_1 \operatorname{div} \mathbf{u}$ in Ω_1 , the variational problem is given by: find $[\mathbf{u}, p] \in \mathbf{V} \times L^2(\Omega_1)$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= l(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) - \frac{1}{\lambda_1} c(p, q) &= 0, \quad q \in L^2(\Omega_1), \end{aligned}$$

where $l \in \mathbf{V}'$ and

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= \sum_{i=1}^2 2\mu_i \int_{\Omega_i} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \lambda_2 \int_{\Omega_2} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, dx, \\ b(\mathbf{v}, q) &:= \int_{\Omega_1} \operatorname{div} \mathbf{v} \, q \, dx, \quad c(p, q) := \int_{\Omega_1} p \, q \, dx, \quad \text{and} \quad l(\mathbf{v}) := \int_{\Omega} f \cdot \mathbf{v} \, dx. \end{aligned}$$

Here, $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linear strain tensor. For our example, the domain $\Omega := \operatorname{conv}\{(0, 0), (48, 44), (48, 60), (0, 44)\}$ is decomposed into two subdomains Ω_1

and Ω_2 with $\Omega_1 := \text{conv}\{(12, 20.25), (36, 38.75), (36, 50.25), (12, 38.75)\}$, and $\Omega_2 := \Omega \setminus \bar{\Omega}_1$. Here, $\text{conv}\xi$ is the convex hull of the set ξ . The decomposition of the domain and the initial triangulation are shown in the left picture of Figure 1. Here, the left boundary of Ω is fixed and the right boundary is subjected to an in-plane shearing load of 100N along the positive y -direction. The lower and upper boundaries are set free, and we do not apply any volume force. The material parameters are taken to be $E_1 = 250\text{Pa}$, $E_2 = 80\text{Pa}$, $\nu_1 = 0.4999$, and $\nu_2 = 0.35$ to get a nearly incompressible response in Ω_1 , where E_i and ν_i are the Young's modulus and the Poisson ratio on Ω_i , $i = 1, 2$, respectively. The displacement field is discretized with bilinear finite elements, and the pressure in Ω_1 is discretized with piecewise constant functions. The right picture of Figure 1 shows the vertical displacement at $(48, 60)$ versus the number of elements. We compare three different numerical schemes. Using standard conforming finite elements (standard) in Ω does not give satisfying numerical results, whereas the more expensive mixed formulation (mixed) in Ω provides good results. Our numerical results show that the mortar approach (coupled) is almost as good as the mixed formulation and significantly better than the standard one.

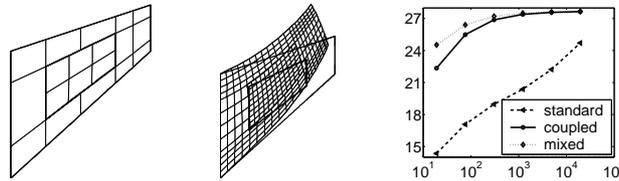


Fig. 1. Decomposition of the domain and initial triangulation (left), distorted grid on level 2 (middle), and vertical tip displacement versus number of elements (right)

3 The Laplace and the p -Laplace Operator

In this section, we consider the coupling of a linear and a non-linear model. The linear model is described by a Poisson equation, and we use the p -Laplacian for the non-linear model. Here, we decompose the domain $\Omega := (-1, 1) \times (-1, 1)$ into four non-overlapping subdomains defined by $\Omega_1 := (-1, 0) \times (-1, 0)$, $\Omega_2 := (0, 1) \times (-1, 0)$, $\Omega_3 := (-1, 0) \times (0, 1)$ and $\Omega_4 := (0, 1) \times (0, 1)$. We have given the decomposition of the domain and the initial triangulation in the left picture of Figure 2. We consider the Poisson equation $-\nabla \cdot (\alpha \nabla) u = f$ in Ω_1 and Ω_4 and the p -Laplacian $-\nabla \cdot (\alpha |\nabla u|^{p-2} \nabla u) = f$ in Ω_2 and Ω_3 . The p -Laplace equation occurs in the theory of two-dimensional plasticity under longitudinal shear or in the diffusion problem with non-linear diffusivity, see Atkinson and Champion [1984], and we are considering here different material models in different subdomains. For the regularity of the solutions and error estimates of the p -Laplacian, we refer to Liu and Barret [1993] and Liu and Yan

[2001]. Let \mathcal{T}_{h_k} be a shape regular simplicial triangulation on Ω_k with mesh-

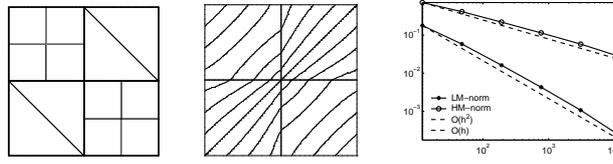


Fig. 2. Decomposition of the domain and initial triangulation (left), isolines of the solution (middle) and discretization errors versus number of elements (right)

sizes bounded by h_k , and $\mathcal{S}(\Omega_k, \mathcal{T}_{h_k})$ stands for the space of linear conforming finite elements in the subdomain Ω_k associated with the triangulation \mathcal{T}_{h_k} satisfying the Dirichlet boundary conditions on $\partial\Omega_k \cap \partial\Omega$, $k = 1, \dots, 4$. Then, the unconstrained finite element space X_h is given by $X_h := \prod_{k=1}^4 \mathcal{S}(\Omega_k, \mathcal{T}_{h_k})$. The interface $\Gamma := \{(0, y), -1 < y < 1\} \cup \{(x, 0), -1 < x < 1\}$ inherits its one-dimensional triangulation \mathcal{S}_Γ from the mesh on Ω_2 and Ω_3 . We recall that $(0, 0)$ is a crosspoint, and M_h does not have any degree of freedom at this point. Now, the Lagrange multiplier space M_h is defined on Γ and is associated with the triangulation \mathcal{S}_Γ . Assuming $q_1 := 2$, $q_2 := p$, $q_3 := p$, and $q_4 := 2$, we can write the weak formulation of the problem as: find $(u_h, \lambda_h) \in X_h \times M_h$ such that

$$\begin{aligned} a(u_h, v) + b(v, \lambda_h) &= l(v), & v \in X_h, \\ b(u_h, \mu) &= 0, & \mu \in M_h, \end{aligned} \tag{1}$$

where $a(u, v) := \sum_{i=1}^4 \int_{\Omega_i} \alpha |\nabla u|^{q_i-2} \nabla u \cdot \nabla v \, dx$, $b(v, \mu) := \int_\Gamma [v] \mu \, d\sigma$, and $l(v) := \int_\Omega f v \, dx$. If $\alpha > 0$, and the right hand side function f is sufficiently smooth, we can show by monotonicity techniques that the problem (1) has a unique solution, see Liu [1999]. However, the regularity of the solution is not known. Let $u_h := \sum_{k=1}^n u_k \phi_k$ and $\lambda_h := \sum_{k=1}^{n_s} \lambda_k \mu_k$, where n and n_s are the dimensions of X_h and M_h , respectively. Suppose $w = (u_1, \dots, u_n, \lambda_1, \dots, \lambda_{n_s})$ be a vector. Now, we define $F(w) := (F_1(w), F_2(w))^T$ with

$$F_1(w) := \begin{pmatrix} a(u_h, \phi_1) + b(\phi_1, \lambda_h) - l(\phi_1) \\ \vdots \\ a(u_h, \phi_n) + b(\phi_n, \lambda_h) - l(\phi_n) \end{pmatrix}, \quad F_2(w) := \begin{pmatrix} b(u_h, \mu_1) \\ \vdots \\ b(u_h, \mu_{n_s}) \end{pmatrix}.$$

The system $F(w) = 0$ is a non-linear system if $p \neq 2$, and we apply the Newton method to solve this system. First, we initialize the solution vector w_0 satisfying the given Dirichlet boundary conditions. Then, we iterate until convergence with

$$J_k \Delta w_k = F(w_k),$$

where $\Delta w_k := w_k - w_{k+1}$, and J_k is the Jacobian of F evaluated at w_k . Working with a dual Lagrange multiplier space has the advantage that the

Jacobian matrix J_k has exactly the same form as the saddle point matrix arising from the mortar finite elements with a dual Lagrange multiplier space for the Laplace operator, see Wohlmuth and Krause [2001]. Hence, we can apply the multigrid approach introduced in Wohlmuth and Krause [2001] to solve the linear system on each level. Suppose that $\tilde{\Omega}_l := \Omega_1 \cup \Omega_4$, and $\tilde{\Omega}_p := \Omega_2 \cup \Omega_3$. In our numerical example, we choose $\alpha = 0.5$ in $\tilde{\Omega}_l$, and $\alpha = 1$ in $\tilde{\Omega}_p$, $p = 1.5$, and $f = 0$. For boundary conditions, we set $u(-1, -1) = u(1, 1) = 0$, $u(-1, 1) = 1$ and $u(1, -1) = -1$, and the Dirichlet boundary condition on $\partial\Omega$ is imposed by taking the linear combination of them in between. Here, we do not have the exact solution. To get the approximation of the discretization errors, we compute a reference solution u_{ref} at a fine level and compare it with the solution u_h at each level until $h_{ref} \leq 2h$. We have given the discretization errors in the LM - and HM - norms defined by

$$\|v\|_{LM} := \|v\|_{L^2(\tilde{\Omega}_l)} + \|v\|_{L^p(\tilde{\Omega}_p)}, \quad \text{and} \quad \|v\|_{HM} := |v|_{W^{1,2}(\tilde{\Omega}_l)} + |v|_{W^{1,p}(\tilde{\Omega}_p)}$$

in the right picture of Figure 2, and the isolines of the solution are given in the middle. Although the regularity of the solution is not known, we get convergence of order h^2 in the LM -norm and of order h in the HM -norm.

4 Application to an Elasto-Acoustic Problem

In this section, we show the application of mortar finite element methods for an elasto-acoustic problem. We consider the situation that the fluid is completely surrounded by the structure. The problem is described by a linear elastic structure occupying a subdomain $\Omega_S \subset \mathbb{R}^2$ coupled with an irrotational fluid in $\Omega_F \subset \mathbb{R}^2$. The interface $\Gamma (= \partial\Omega_S \cap \partial\Omega_F)$ separates the fluid and solid regions. Given the fluid-density ρ_F , the solid-density ρ_S , and the acoustic speed c , we seek the frequency ω , the velocity-field \mathbf{u} , and the pressure p such that

$$\begin{aligned} \nabla p - \omega^2 \rho_F \mathbf{u}_F &= \mathbf{0} && \text{in } \Omega_F, \\ p + c^2 \rho_F \nabla \cdot \mathbf{u}_F &= 0 && \text{in } \Omega_F, \\ \nabla \cdot \sigma(\mathbf{u}_S) + \omega^2 \rho_S \mathbf{u}_S &= \mathbf{0} && \text{in } \Omega_S, \\ \mathbf{u}_S &= \mathbf{0} && \text{on } \Gamma_D, \\ \sigma(\mathbf{u}_S) \cdot \mathbf{n}_S &= \mathbf{0} && \text{on } \Gamma_N, \\ \sigma_n(\mathbf{u}_S) + p &= 0, \quad \sigma_t(\mathbf{u}_S) = \mathbf{0}, && \text{and } (\mathbf{u}_F - \mathbf{u}_S) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \end{aligned}$$

Here, σ is the usual stress tensor from linear elasticity, $\sigma_n = \mathbf{n} \cdot (\sigma \cdot \mathbf{n})$ is the normal stress on Γ , and $\sigma_t = \sigma \cdot \mathbf{n} - \sigma_n \mathbf{n}$ is the tangential traction vector on Γ , where \mathbf{n} is the outward normal to Ω_F on Γ . This problem has become a subject of different papers, see, e.g., Hansbo and Hermansson [2003], Bermúdez and Rodríguez [1994], Alonso et al. [2001]. We introduce the following function spaces to formulate our problem in the weak form

$X := \mathbf{H}(\operatorname{div}, \Omega_F) \times \mathbf{H}_{\Gamma_D}^1(\Omega_S)$, and $\mathbf{V} := \{(\mathbf{u}_F, \mathbf{u}_S) \in X, [\mathbf{u}] \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$,

where

$$\begin{aligned} \mathbf{H}(\operatorname{div}, \Omega_F) &:= \{\mathbf{v} \in L^2(\Omega_F)^2, \|\nabla \cdot \mathbf{v}\|_{L^2(\Omega_F)} < \infty\}, \\ \mathbf{H}_{\Gamma_D}^1(\Omega_S) &:= \{\mathbf{v} \in H^1(\Omega_S)^2, \mathbf{v}|_{\Gamma_D} = 0\}, \quad \text{and} \quad [\mathbf{u}] := (\mathbf{u}_F - \mathbf{u}_S). \end{aligned}$$

The weak form of the continuous problem is: find $\mathbf{u} \in \mathbf{V}$ and $\omega \in \mathbb{R}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \omega^2 m(\mathbf{u}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}, \quad \text{where} \\ a(\mathbf{u}, \mathbf{v}) &:= (\rho_F c^2 \nabla \cdot \mathbf{u}_F, \nabla \cdot \mathbf{v}_F)_{\Omega_F} + (\sigma(\mathbf{u}_S), \epsilon(\mathbf{v}_S))_{\Omega_S}, \quad \text{and} \\ m(\mathbf{u}, \mathbf{v}) &:= (\omega^2 \rho_F \mathbf{u}_F, \mathbf{v}_F)_{\Omega_F} + (\omega^2 \rho_S \mathbf{u}_S, \mathbf{v}_S)_{\Omega_S}. \end{aligned}$$

Here, $\epsilon(\mathbf{v}_S)$ is the linear strain tensor and is related to the stress tensor by Hooke's law, i.e., $\sigma_{ij}(\mathbf{v}_S) = 2\mu\epsilon_{ij}(\mathbf{v}_S) + \lambda \sum_{k=1}^2 \epsilon_{kk}(\mathbf{v}_S)\delta_{ij}$, $i, j = 1, 2$. Let \mathcal{T}_{h_s} and \mathcal{T}_{h_f} be shape regular simplicial triangulations on Ω_S and Ω_F , respectively, and Γ inherits its triangulation \mathcal{S}_Γ from the side of Ω_F . It is a well-known fact that if standard Lagrangian finite elements are used to discretize the fluid, it will give rise to spurious eigensolutions with positive eigenvalues interspersed among the 'real' ones, and a possible remedy of this problem is to use Raviart-Thomas elements in the fluid domain, see Bermúdez et al. [1995]. Therefore, we discretize the fluid domain with Raviart-Thomas elements of lowest order:

$$RT_0 := \{\mathbf{u} \in \mathbf{H}(\operatorname{div}, \Omega_F) : \mathbf{u}|_K = (a + bx, c + by), K \in \mathcal{T}_{h_f}, a, b, c \in \mathbb{R}\},$$

and the solid domain with Lagrangian finite elements of lowest order:

$$W_h^D := S_D(\Omega_S, \mathcal{T}_{h_s}) \times S_D(\Omega_S, \mathcal{T}_{h_s}),$$

where $S_D(\Omega_S, \mathcal{T}_{h_s})$ is the finite element space on Ω_S satisfying the Dirichlet boundary condition on Γ_D . The kinematic constraint can be imposed by piecewise constant Lagrange multipliers yielding a uniform inf-sup condition. Suppose $X_h := RT_0 \times W_h^D$, and $M_h := \{\mu_h \in L^2(\Gamma) : \mu_h|_e \in \mathcal{P}_0(e), e \in \mathcal{S}_\Gamma\}$. Now the finite element space can be written as

$$\mathbf{V}_h := \{(\mathbf{u}_{hF}, \mathbf{u}_{hS}) \in X_h, \int_\Gamma [\mathbf{u}_h] \cdot \mathbf{n} \mu_h \, d\sigma = 0, \mu_h \in M_h\}.$$

The discrete problem reads: find $\mathbf{u}_h \in \mathbf{V}_h$, and $\omega_h \in \mathbb{R}$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = \omega_h^2 m(\mathbf{u}_h, \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{V}_h.$$

Remark 1. We remark that the Lagrange multiplier λ_h approximates the pressure on the interface Γ . The Lagrange multipliers are associated with the one-dimensional mesh inherited from the triangulation on the fluid domain. Due to the special structure of the support of the nodal basis functions of RT_0 and M_h , the degree of freedom corresponding to the Lagrange multiplier can locally be eliminated by inverting a diagonal mass matrix.

In Alonso et al. [2001], an adaptive finite element scheme is analyzed to solve the fluid-structure vibration problem, where the kinematic constraint is imposed by means of piecewise constant Lagrange multiplier. Following this technique, we arrive at the same mortar setting as we discuss here. Now, we consider the domain $\Omega := \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 1\}$ decomposed into two subdomains Ω_S and Ω_F with $\Omega_F := \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 0.6\}$, and $\Omega_S := \Omega \setminus \bar{\Omega}_F$. Here, $\Gamma_D = \{(\cos \theta, \sin \theta), \frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4}\}$. We have used the following parameters in our numerical example: $\rho_F = 1000\text{kg/m}^3$, $c = 1430\text{m/s}$, $\rho_S = 7700\text{kg/m}^3$, $E = 144\text{GPa}$, and $\nu = 0.35$. The first three consecutive eigenmodes along with the pressure in the fluid domain and the distorted grids in the solid domain are shown in Figure 3. We note that Γ defines a curvilinear interface. To evaluate the weak coupling, we commit an additional variational crime by projecting the mesh of the structure side to the mesh on the fluid side.

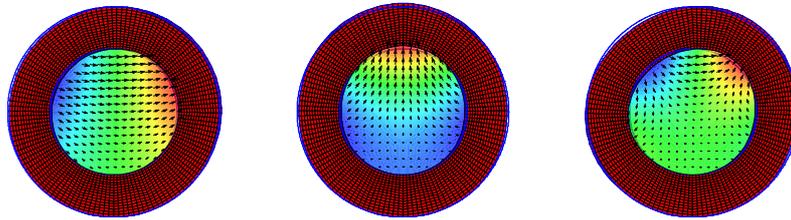


Fig. 3. The first, second and the third eigenmodes corresponding to the eigenvalues 809.1481, 1980.7519 and 3606.3907 (rad/s)

The second numerical example is taken from Bermúdez and Rodríguez [1994]. The domain $\Omega := (0, 1.5) \times (0, 1.5)$ is decomposed into two subdomains Ω_S and Ω_F with $\Omega_F := (0.25, 1.25) \times (0.25, 1.25)$, $\Omega_S := \Omega \setminus \bar{\Omega}_F$, and $\Gamma_D = \{(x, 0) \in \mathbb{R}^2, 0 \leq x \leq 1.5\}$. We have used the same physical parameters as in the previous example. The computed eigenfrequencies (in rad/s) along with the extrapolated ones referred to as ‘Exact’ in Bermúdez and Rodríguez [1994] are given in Table 1.

Table 1. The computed eigenfrequencies using mortar techniques compared with the extrapolated eigenfrequencies (‘Exact’) in Bermúdez and Rodríguez [1994]

Eigenmodes	Computed Eigenfrequencies	‘Exact’
1	648.1847	641.837
2	2147.3593	2116.398
3	3419.5020	3201.475
4	3885.9022	3804.124
5	4214.0865	4211.620
6	4699.6782	4687.927

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