Iterative Methods for Stokes/Darcy Coupling

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Summary. We present iterative subdomain methods based on a domain decomposition approach to solve the coupled Stokes/Darcy problem using finite elements. The dependence of the convergence rate on the grid parameter h and on the physical data is discussed; some difficulties encountered when applying the algorithms are indicated together with possible improvement strategies.

1 Introduction and problem setting

The simulation of incompressible flows in heterogenous media is an interesting topic with many applications: considering the particular case of free fluids which can filtrate through porous media, we recall for example the hydrological environmental applications and mass transfer in biomechanics.

The Stokes/Darcy coupled system provides a linear model to describe such phenomena. We consider a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ formed by two non-overlapping subdomains Ω_f and Ω_p separated by a surface $\Gamma = \overline{\Omega}_f \cap \overline{\Omega}_p$. Ω_f is the region occupied by the fluid whose motion is described by the Stokes equations which can be written in adimensional form as:

$$\begin{aligned} -Re_f^{-1}\triangle \mathbf{u}_f + \nabla p_f &= \mathbf{f} \\ \nabla \cdot \mathbf{u}_f &= 0 \end{aligned} \quad \text{in } \Omega_f, \tag{1}$$

where $\mathbf{u}_f$ and p_f are the adimensional velocity and pressure, respectively, while Re_f is the Reynolds number defined as $Re_f = L_f U_f/\nu$, $\nu > 0$ being the fluid kinematic viscosity and L_f, U_f a characteristic length and velocity, respectively.

The filtration through the porous region Ω_p is modeled using Darcy's equations, whose adimensional form reads:

$$\begin{aligned} \mathbf{u}_p &= -\varepsilon Re_p \nabla p_p \\ \nabla \cdot \mathbf{u}_p &= 0 \end{aligned} \quad \text{in } \Omega_p, \tag{2}$$

where $\mathbf{u}_p$ and p_p are the adimensional fluid velocity and pressure, respectively, Re_p is the Reynolds number $Re_p = \delta_p U_p/\nu$, U_p being a characteristic velocity through the porous medium and δ_p a characteristic pore size. Finally, $\varepsilon = \delta_p/L_p$ is the adimensional ratio between the micro and the macro scales in Ω_p.

Across the interface Γ the continuity of normal stresses and fluxes is required; precisely, we impose:

$$\begin{aligned} \mathbf{u}_f \cdot \mathbf{n} &= \mathbf{u}_p \cdot \mathbf{n} \\ -\mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} &= p_p \qquad \text{on } \Gamma \\ -\boldsymbol{\tau}_j \cdot \mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} &= \alpha\, \mathbf{u}_f \cdot \boldsymbol{\tau}_j \end{aligned} \tag{3}$$

where $\mathsf{T}(\mathbf{u}_f, p_f)$ is the stress tensor, $\mathbf{n}$ is the unit normal outward vector to $\partial\Omega_f$ and $\boldsymbol{\tau}_j \cdot \mathbf{n} = 0$ $(j = 1, \ldots, d-1)$; α is a dimensionless coefficient depending essentially on ν and the hydraulic conductivity of the porous medium. For an extensive discussion about these coupling conditions we refer to Discacciati et al. [2002], Jäger and Mikelić [1996], Layton et al. [2003], Payne and Straughan [1998].

The mathematical analysis of the coupled problem has been addressed in previous works concerning both the continuous case and the finite element approximation (see Discacciati and Quarteroni [2003], Layton et al. [2003]).

In order to solve the coupled problem, an iterative substructuring method was proposed and analyzed in Discacciati and Quarteroni [2004]. Here, we test it on a model problem in order to investigate its effectiveness and robustness, with particular emphasis on the role that physical and grid parameters play on the convergence properties. We consider the computational domain $\Omega \subset \mathbb{R}^2$, with $\Omega_f = (0,1)\times(1,2)$, $\Omega_p = (0,1)^2$ and interface $\Gamma = (0,1)\times\{1\}$. We rewrite Darcy's equation as $-\nabla \cdot (\varepsilon Re_p \nabla p_p) = 0$ in Ω_p, and consider the following analytic solution: $\mathbf{u}_f = (y^2-2y+1, x^2-x)^T$, $p_f = 2(x+y-1)/Re_f + 1/(3\varepsilon Re_p)$ and $p_p = (x(1-x)(y-1) + y^3/3 - y^2 + y)/(\varepsilon Re_p) + 2x/Re_f$.

Concerning the finite element discretization, $\mathbb{P}_2-\mathbb{P}_1$ Taylor-Hood elements have been used for Stokes equations, while $\mathbb{P}_2$ Lagrangian elements have been adopted for Darcy's problem. All the computational meshes are conforming on Γ.

2 Dirichlet-Neumann (DN) methods

Considering the interface conditions $(3)_1$ and $(3)_2$, we can choose as scalar interface variable λ either $\lambda = \mathbf{u}_f \cdot \mathbf{n}$ on Γ or $\lambda = p_p$ on Γ. These two choices define two different DN-type methods, which can be outlined as follows, respectively:

Algorithm DN$_1$

`0. choose` $\lambda = \mathbf{u}_f \cdot \mathbf{n}$ `on` Γ `and an initial guess` $\lambda^{(0)}$ `on` Γ `;`

For $k = 0, 1, \ldots$ until convergence, Do

1. solve Darcy's equation with b.c. $-\varepsilon Re_p \nabla p_p^{(k+1)} \cdot \mathbf{n} = \lambda^{(k)}$ on Γ ;
2. solve Stokes problem with b.c. $-\mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f^{(k+1)}, p_f^{(k+1)}) \cdot \mathbf{n} = p_p^{(k+1)}$ and $(3)_3$ on Γ;
3. $\lambda^{(k+1)} = \theta \mathbf{u}_f^{(k+1)} \cdot \mathbf{n} + (1-\theta)\lambda^{(k)}$ on Γ , $\theta \in (0,1)$;

End For

Algorithm DN$_2$

0. choose $\lambda = p_p$ on Γ and an initial guess $\lambda^{(0)}$ on Γ ;

   For $k = 0, 1, \ldots$ until convergence, Do

1. solve Stokes problem with b.c. $-\mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f^{(k+1)}, p_f^{(k+1)}) \cdot \mathbf{n} = \lambda^{(k)}$ and $(3)_3$ on Γ;
2. solve Darcy's equation with b.c. $-\varepsilon Re_p \nabla p_p^{(k+1)} \cdot \mathbf{n} = \mathbf{u}_f^{(k+1)} \cdot \mathbf{n}$ on Γ ;
3. $\lambda^{(k+1)} = \theta p_p^{(k+1)} + (1-\theta)\lambda^{(k)}$ on Γ , $\theta \in (0,1)$;

End For

The two DN methods are equivalent to preconditioned Richardson methods to solve the symmetric Steklov-Poincaré equations associated to the coupled problem, and they allow to characterize optimal preconditioners for Krylov type methods (e.g. the Conjugate Gradient) for the corresponding interface problems (see Discacciati and Quarteroni [2004]).

2.1 Numerical results

We consider $Re_f = 1$, $\varepsilon Re_p = 1$ and $tol = 10^{-5}$; in Table 1 we report the number of iterations for both the Richardson and the Preconditioned Conjugate Gradient (PCG) method. These convergence results are satisfactory as they show the optimality of the preconditioners with respect to the grid parameter h.

Number of mesh elements	DN$_1$ ($\theta = 0.7$)	PCG $\lambda = \mathbf{u}_f \cdot \mathbf{n}$	DN$_2$ ($\theta = 0.7$)	PCG $\lambda = p_p$
172	9	4	10	4
688	9	4	10	4
2752	9	4	10	4
11008	9	4	10	4

Table 1. Number of iterations on different grids with $Re_f = 1$ and $\varepsilon Re_p = 1$

However, if the fluid viscosity and the hydraulic conductivity decrease, small relaxation parameters θ must be adopted to guarantee convergence, in

accordance with the theoretical estimate of the upper bound θ_{max} given in Discacciati and Quarteroni [2004]. Unfortunately, in some cases θ should be so small that in practice it prevents the numerical scheme from converging. To quote an example, if $Re_f = 10^3$ and $\varepsilon Re_p = 10^{-2}$, then θ should be unreasonably small (smaller than 10^{-4} !) in DN_1 to prevent divergence.

This difficulty should not be ascribed to the non-optimal choice of the relaxation parameter θ. In fact, if we apply the PCG method which embeds the choice of the optimal acceleration parameter (see, e.g., Quarteroni et al. [2000] p. 150), the iterative algorithm converges, but the optimal properties of the preconditioners are lost, since the number of iterations depends on the mesh parameter h, as reported in Table 2.

Mesh elements	PCG iterations ($\lambda = \mathbf{u}_f \cdot \mathbf{n}$)
688	82
2752	102
11008	148

Table 2. Number of iterations on different grids with $Re_f = 10^3$ and $\varepsilon Re_p = 10^{-2}$

On the basis of the numerical results we have obtained we can conclude that DN methods are effective only when the ratio $Re_f/(\varepsilon Re_p)$ is sufficiently small, while dealing with large values causes some difficulties. However, the latter are the very values of interest in real-life applications and, therefore, a robust numerical method is required.

3 Dirichlet-Neumann for a time-dependent problem

We introduce a formal argument to better understand the results obtained in Sect. 2.1 and to set up a more effective numerical scheme. This approach will be treated from a precise mathematical viewpoint in a forthcoming work Discacciati [2004].

The underlying idea is that our difficulties in solving the Stokes/Darcy problem may come from the different structure of equations $(1)_1$ and $(2)_1$, which become even more dissimilar when $Re_f \gg 1$ and $\varepsilon Re_p \ll 1$. In fact, in that case, under the physically reasonable hypothesis that $\triangle \mathbf{u}_f$ and ∇p_p are sufficiently small, $(1)_1$ reduces almost to $C_f \mathbf{I} + \nabla p_f = \mathbf{f}$, while $(2)_1$ becomes $\mathbf{u}_p + C_p \mathbf{I} = \mathbf{0}$, where C_f and C_p denote two positive constants $\ll 1$. We rewrite $(2)_1$ as

$$(\varepsilon Re_p)^{-1} \mathbf{u}_p + \nabla p_p = \mathbf{0} \quad \text{in } \Omega_p \,, \tag{4}$$

and formally comparing (4) to $(1)_1$, we are led to modify the latter by adding a mass term like $(\varepsilon Re_p)^{-1} \mathbf{u}_p$ as follows:

$$\beta(\varepsilon Re_p)^{-1}\mathbf{u}_f - Re_f^{-1}\triangle\mathbf{u}_f + \nabla p_f = \tilde{\mathbf{f}}\,, \quad \beta \in \mathbb{R}^+\,, \tag{5}$$

possibly with a consequent modification of the right hand side (see Remark 1) that we have denoted by $\tilde{\mathbf{f}}$. In this way we obtain a generalized Stokes momentum equation, and note that now (5) has the same behaviour of (4) in the cases of our interest, that is when $Re_f \gg 1$ and $\varepsilon Re_p \ll 1$.

We expect that the mass term $\beta(\varepsilon Re_p)^{-1}\mathbf{u}_f$ would help improving the positivity of the discrete Steklov-Poincaré operator which acts as a preconditioner in the DN_1 method. With this aim, we have carried out some numerical tests using the PCG algorithm with $\lambda = \mathbf{u}_f \cdot \mathbf{n}$ as interface variable to solve the modified problem (2), (5). The convergence results reported in Table 3 show that the numerical scheme has really improved.

Re_f	εRe_p	β	Iterations on the mesh with		
			688 el.	2752 el.	11008 el.
10^3	10^{-2}	0.1	17	14	13
		1	10	9	7
		10	5	5	4
10^6	10^{-4}	0.1	19	21	19
		1	11	10	10
		10	5	5	4

Table 3. Number of iterations to solve problem (2), (5) for different values of Re_f, εRe_p and β

Remark 1. Equation (5) can be regarded as a discretization in time of the time-dependent Stokes momentum equation $\partial_t\mathbf{u}_f - Re_f^{-1}\triangle\mathbf{u}_f + \nabla p_f = \mathbf{f}$ in Ω_f. Precisely, if we consider

$$\beta(\varepsilon Re_p)^{-1}\mathbf{u}_{f,n+1} - Re_f^{-1}\triangle\mathbf{u}_{f,n+1} + \nabla p_{f,n+1} = \tilde{\mathbf{f}}_{n+1} \qquad n \geq 0$$

with $\tilde{\mathbf{f}}_{n+1} = \mathbf{f}(\mathbf{x}, t_{n+1}) + \beta(\varepsilon Re_p)^{-1}\mathbf{u}_{f,n}$, we have a backward Euler discretization in time with $\beta(\varepsilon Re_p)^{-1}$ playing the role of the inverse of a time step.

From the physical viewpoint, since the fluid velocities in Ω_f are much higher than the ones through the porous medium (see Ene and Sanchez-Palencia [1975]), a time-dependent model better represents the phenomena occurring during the filtration process.

3.1 The tDN algorithm

Let $[0, T]$ be a characteristic time interval; using for the sake of simplicity the first-order backward Euler scheme, denoting by $\Delta t > 0$ the time step and $N = T/\Delta t$, the iterative method that we propose to solve the time-dependent coupled problem reads (the subscript n refers to the nth time level):

Algorithm tDN

For $n = 0, \dots, N-1$ Do

0. choose an initial guess $\lambda_{n+1}^{(0)}$ for the normal velocity on Γ at the $(n+1)$th time level;

For $k = 0, 1, \dots$ until convergence, Do

1. solve Darcy's equation with b.c. $-\varepsilon Re_p \nabla p_{p,n+1}^{(k+1)} \cdot \mathbf{n} = \lambda_{n+1}^{(k)}$ on Γ ;
2. solve Stokes problem

$$\left.\begin{aligned} (\Delta t)^{-1}\mathbf{u}_{f,n+1}^{(k+1)} - Re_f^{-1}\triangle \mathbf{u}_{f,n+1}^{(k+1)} + \nabla p_{f,n+1}^{(k+1)} &= (\Delta t)^{-1}\mathbf{u}_{f,n} + \mathbf{f}_{n+1} \\ \nabla \cdot \mathbf{u}_{f,n+1}^{(k+1)} &= 0 \end{aligned}\right. \quad \text{in } \Omega_f$$

   with b.c. $-\mathbf{n} \cdot \mathsf{T}(\mathbf{u}_{f,n+1}^{(k+1)}, p_{f,n+1}^{(k+1)}) \cdot \mathbf{n} = p_{p,n+1}^{(k+1)}$ and $(3)_3$ on Γ ;
3. $\lambda_{n+1}^{(k+1)} = \theta \mathbf{u}_{f,n+1}^{(k+1)} \cdot \mathbf{n} + (1-\theta)\lambda_{n+1}^{(k)}$ on Γ , $\theta \in (0,1)$;

End For
End For

3.2 Numerical tests

We consider the horizontal section of a channel 12 m long and 8 m wide which is partially occupied by a porous medium with discontinuous conductivity, as represented in Fig. 1 (left). A parabolic inflow profile is imposed on the left hand side boundary with maximal velocity equal to 0.1m/s. On the right an outflow condition is imposed. The time interval is $t \in [0, 0.5]$ and the time step $\Delta t = 10^{-3}$ s; for space discretization three different computational meshes have been adopted.

In a first case we have considered $Re_f = 8 \cdot 10^5$ and a discontinuous coefficient $\varepsilon Re_p = 10^{-3}$ in $\Omega_p^{(1)}$, $\varepsilon Re_p = 10^{-7}$ in $\Omega_p^{(2)}$.

In Fig. 1 (right) we have represented the computed solution at time $t = 0.05$ s, while in Fig. 2 a zoom of the velocity field through the porous medium is shown; it can be seen that the velocity is almost null in the less permeable areas of the porous medium. Finally, Table 4 (left) reports the number of iterations obtained for three computational grids at different time levels, showing that the number of iterations is low and independent of h.

The same test has been performed considering different values of the parameters: $Re_f = 8 \cdot 10^2$, $\varepsilon Re_p = 10^{-1}$ in $\Omega_p^{(1)}$ and $\varepsilon Re_p = 10^{-5}$ in the less permeable part of the porous medium $\Omega_p^{(2)}$. The convergence results show that the number of iterations is essentially independent of these parameters, as it can be seen comparing the previous convergence results with those reported in Table 4 (right).

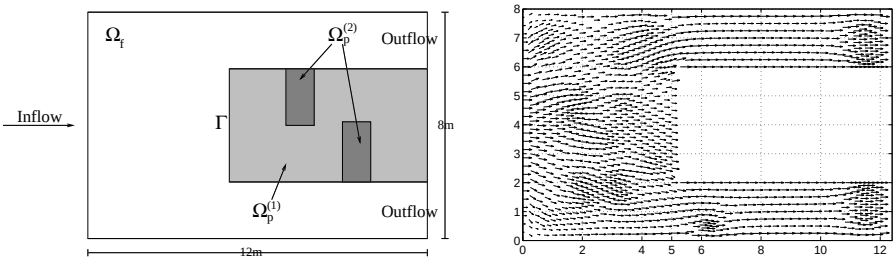

Fig. 1. Computational domain (left) and computed velocity field at $t = 0.05$ s (right)

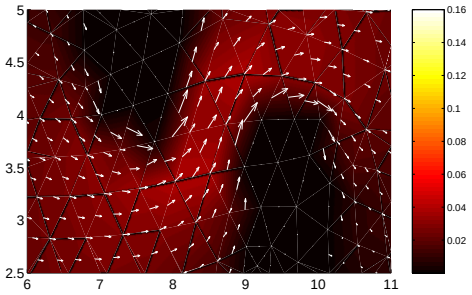

Fig. 2. Zoom of the velocity field through the porous medium

Time level	Iterations on the mesh with 232 el.	928 el.	3712 el.
0.001	21	21	21
0.003	20	19	19
0.006	12	11	11
0.009	10	10	10
0.01	10	10	10

Time level	Iterations on the mesh with 232 el.	928 el.	3712 el.
0.001	22	22	22
0.003	20	20	20
0.006	15	15	15
0.009	15	15	15
0.01	15	15	15

Table 4. Number of iterations on different grids with $Re_f = 8 \cdot 10^5$, $\varepsilon Re_p = 10^{-3}$ and 10^{-7} (left); with $Re_f = 8 \cdot 10^2$, $\varepsilon Re_p = 10^{-1}$ and 10^{-5} (right)

4 Conclusions and perspectives

Numerical results show that considering a time-dependent problem allows to set up a far more efficient DN algorithm for problems with parameters in a range of physical interest. However, as we have shown, the value of Δt generally depends on εRe_p and Re_f, and in some cases we could be forced to consider very small time steps $\Delta t \ll 1$. This could be quite annoying since one might be interested in considering long time scales, for example in modeling the filtration of pollutants in groundwater.

This limitation on Δt drives us to reconsider the steady coupled model. In fact, should we find an algorithm whose behaviour were as much as possible independent of the physical parameters, then not only we would be able to solve the steady problem itself, but we could also use it in the framework of the time-dependent model where Δt would be chosen under the sole requirements of stability and accuracy. A possible approach we are currently considering is a Robin-Robin type method following the ideas presented in Lube et al. [2001] for Oseen equations; its analysis and numerical results will be presented in a future work.

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