Balancing Neumann-Neumann Methods for Elliptic Optimal Control Problems

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Summary. We present Neumann-Neumann domain decomposition preconditioners for the solution of elliptic linear quadratic optimal control problems. The preconditioner is applied to the optimality system. A Schur complement formulation is derived that reformulates the original optimality system as a system in the state and adjoint variables restricted to the subdomain boundaries. The application of the Schur complement matrix requires the solution of subdomain optimal control problems with Dirichlet boundary conditions on the subdomain interfaces. The application of the inverses of the subdomain Schur complement matrices require the solution of subdomain optimal control problems with Neumann boundary conditions on the subdomain interfaces. Numerical tests show that the dependence of this preconditioner on mesh size and subdomain size is comparable to its counterpart applied to elliptic equations only.

1 Introduction

We are interested in domain decomposition methods for the solution of large-scale linear quadratic problems

$$\text{minimize } \frac{1}{2}\mathbf{y}^T\mathbf{M}\mathbf{y} + \mathbf{c}^T\mathbf{y} + \mathbf{y}^T\mathbf{N}\mathbf{u} + \frac{\alpha}{2}\mathbf{u}^T\mathbf{H}\mathbf{u} + \mathbf{d}^T\mathbf{u}, \tag{1a}$$

$$\text{subject to } \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} + \mathbf{b} = 0, \tag{1b}$$

arising from the finite element discretization of elliptic optimal control problems. In (1) $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{u} \in \mathbb{R}^n$ are called the (discretized) state and the (discretized) control, respectively, and $\mathbf{A}\mathbf{y}+\mathbf{B}\mathbf{u}+\mathbf{b} = 0$ is called the (discretized) state equation. Throughout this paper we assume that

A. $\mathbf{A} \in \mathbb{R}^{m\times m}$ is invertible, $\mathbf{B} \in \mathbb{R}^{m\times n}$, $\mathbf{N} \in \mathbb{R}^{m\times n}$, $\mathbf{M} \in \mathbb{R}^{m\times m}$ is symmetric, $\mathbf{H} \in \mathbb{R}^{n\times n}$ is symmetric and the reduced Hessian $\widehat{\mathbf{H}} = \alpha\mathbf{H} - \mathbf{B}^T\mathbf{A}^{-T}\mathbf{N} - \mathbf{N}^T\mathbf{A}^{-1}\mathbf{B} + \mathbf{B}^T\mathbf{A}^{-T}\mathbf{M}\mathbf{A}^{-1}\mathbf{B}$ is positive definite.

The assumption that $\widehat{\mathbf{H}}$ is positive definite is equivalent to the assumption that the Hessian of (1a) is positive definite on the null-space of the linear constraints (1b). Under the assumption A, the necessary and sufficient optimality conditions for (1) are given by

$$\begin{pmatrix} \mathbf{M} & \mathbf{N} & \mathbf{A}^T \\ \mathbf{N}^T & \alpha\mathbf{H} & \mathbf{B}^T \\ \mathbf{A} & \mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \\ \mathbf{b} \end{pmatrix}. \tag{2}$$

The system matrix in (2) is also called a KKT (Karush-Kuhn-Tucker) matrix. Large-scale linear quadratic problems of the form (1) arise as subproblems in Newton or sequential quadratic programming (SQP) type optimization algorithms for nonlinear PDE constrained optimization problems. The solution of these subproblems is a very time consuming part in Newton or SQP type optimization algorithms and therefore the development of preconditioners for such problems is of great interest. Domain decomposition methods for steady state optimal control problems were considered in Benamou [1996], Bounaim [1998], Dennis and Lewis [1994], Biros and Ghattas [2000], Lions and Pironneau [1998] and other preconditioners for the system matrix in (2) are discussed, e.g., by Ascher and Haber [2003], Battermann and Sachs [2001], Hoppe et al. [2002], Keller et al. [2000]. Although (2) is a saddle point problem, its structure is quite different from the saddle point problems arising, e.g., from the Stokes problem (see, e.g.,Pavarino and Widlund [2002]) or from mixed finite element discretizations of elliptic PDEs.

We present a Neumann-Neumann (NN) domain decomposition preconditioner for the solution of discretized elliptic linear quadratic optimal control problems. The preconditioner is applied to the optimality system (2). A Schur complement formulation is derived that reformulates (2) as a system in the state and adjoint variables restricted to the subdomain boundaries. The application of the Schur complement matrix requires the solution of subdomain optimal control problems with Dirichlet boundary conditions on the subdomain interfaces. The application of the inverses of the subdomain Schur complement matrices require the solution of subdomain optimal control problems with Neumann boundary conditions on the subdomain interfaces. Our numerical tests in Section 4 show that the dependence of this preconditioner on mesh size and subdomain size is comparable to that of its counterpart applied to elliptic PDEs only. Numerical tests also indicate that, unlike several other KKT preconditioners, the proposed NN preconditioner is rather insensitive to the choice of the penalty parameter α. Unlike several other KKT preconditioners, our preconditioner does not require a preconditioner for the reduced Hessian $\widehat{\mathbf{H}}$, which is often difficult to obtain. Due to page limitations, we only present the algebraic view of the preconditioner. For more details we refer to Heinkenschloss and Nguyen [2004].

2 The Example Problem

We are interested in the solution $y \in H^1(\Omega)$, $u \in L^2(\partial\Omega)$ of the optimal control problem

$$\text{minimize} \quad \frac{1}{2}\int_\Omega (y(x) - \hat{y}(x))^2 dx + \frac{\alpha}{2}\int_{\partial\Omega} u^2(x)dx, \tag{3a}$$

$$\text{subject to} \quad a(y,\psi) + b(u,\psi) = \int_\Omega f(x)\psi(x)dx \quad \forall \psi \in H^1(\Omega), \tag{3b}$$

where $a(y,\psi) = \int_\Omega \nabla y(x)\nabla\psi(x) + y(x)\psi(x)dx$ and $b(u,\psi) = -\int_{\partial\Omega} u(x)\psi(x)dx$. The desired state $\hat{y} \in L^2(\Omega)$ and $f \in L^2(\Omega)$ are given functions, and $\alpha > 0$ is a given parameter. It is shown in Lions [1971] that (3) has a unique solution.

We discretize (3) using conforming finite elements. Let $\{T_l\}$ be a triangulation of Ω. We divide Ω into nonoverlapping subdomains Ω_i, $i = 1, \ldots, d$, such that each T_l belongs to exactly one $\overline{\Omega}_i$. We approximate the state y by a function $y_h = \sum_k y_k\psi_k$ which is continuous on Ω and linear on each T_l. Our discretized controls u_h are not chosen to be continuous and piecewise linear on $\partial\Omega$ (see the left plot in Figure 1). A domain decomposition formulation based on such a discretization would introduce 'interface controls' (dotted hat function in the left plot in Figure 1) defined on a 'band' of width $O(h)$ around $\partial\Omega \cap \partial\Omega_i \cap \partial\Omega_j$, $i \neq j$. Since the evaluation of $u \in L^2(\partial\Omega)$ on $\partial\Omega \cap \partial\Omega_i \cap \partial\Omega_j$ does not make sense, we avoid interface controls. We discretize the control u by a function $u_h = \sum_k u_k\mu_k$ which is continuous on each $\partial\Omega_i$, $i = 1, \ldots, d$, and linear on each $\partial\Omega \cap \partial\Omega_i \cap T_l$. The discretized control u_h is not assumed to be continuous at $\partial\Omega \cap \partial\Omega_i \cap \partial\Omega_j$, $i \neq j$. In particular, for each point $x_k \in \partial\Omega \cap \partial\Omega_i \cap \partial\Omega_j$, $i \neq j$, there are two discrete controls u_{k_i}, u_{k_j} belonging to subdomains Ω_i and Ω_j, respectively (see the right plot in Figure 1). Hence, our control discretization depends on the partition $\{\Omega_i\}_{i=1}^d$ of the domain Ω.

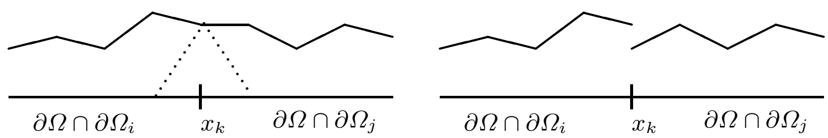

Fig. 1. Sketch of the Control Discretization for the Case $\Omega \subset \mathbb{R}^2$

3 The Domain Decomposition Preconditioners

We define

$$\mathbf{K}^i_{\Gamma\Gamma} = \begin{pmatrix} \mathbf{M}^i_{\Gamma\Gamma} & (\mathbf{A}^i_{\Gamma\Gamma})^T \\ \mathbf{A}^i_{\Gamma\Gamma} & \end{pmatrix}, \ \mathbf{A}^i = \begin{pmatrix} \mathbf{A}^i_{II} & \mathbf{A}^i_{I\Gamma} \\ \mathbf{A}^i_{\Gamma I} & \mathbf{A}^i_{\Gamma\Gamma} \end{pmatrix}, \ \mathbf{M}^i = \begin{pmatrix} \mathbf{M}^i_{II} & \mathbf{M}^i_{I\Gamma} \\ \mathbf{M}^i_{\Gamma I} & \mathbf{M}^i_{\Gamma\Gamma} \end{pmatrix},$$

$i = 1, \ldots, d$, and $\mathbf{K}_{\Gamma\Gamma} = \sum_{i=1}^{d} \mathbf{K}_{\Gamma\Gamma}^{i}$ $\mathbf{x}_{\Gamma} = \begin{pmatrix} \mathbf{y}_{\Gamma} \\ \mathbf{p}_{\Gamma} \end{pmatrix}$, $\mathbf{g}_{\Gamma} = \begin{pmatrix} \mathbf{c}_{\Gamma} \\ \mathbf{b}_{\Gamma} \end{pmatrix}$. Furthermore, for indices i with $\partial\Omega_i \cap \partial\Omega \neq \emptyset$, we define

$$\mathbf{K}_{II}^{i} = \begin{pmatrix} \mathbf{M}_{II}^{i} & \mathbf{N}_{II}^{i} & (\mathbf{A}_{II}^{i})^T \\ (\mathbf{N}_{II}^{i})^T & \mathbf{H}_{II}^{i} & (\mathbf{B}_{II}^{i})^T \\ \mathbf{A}_{II}^{i} & \mathbf{B}_{II}^{i} & \end{pmatrix}, \mathbf{K}_{\Gamma I}^{i} = \begin{pmatrix} \mathbf{M}_{\Gamma I}^{i} & \mathbf{N}_{\Gamma I}^{i} & (\mathbf{A}_{I\Gamma}^{i})^T \\ \mathbf{A}_{\Gamma I}^{i} & \mathbf{B}_{\Gamma I}^{i} & \end{pmatrix},$$

$$\mathbf{B}^{i} = \begin{pmatrix} \mathbf{B}_{II}^{i} \\ \mathbf{B}_{\Gamma I}^{i} \end{pmatrix}, \ \mathbf{N}^{i} = \begin{pmatrix} \mathbf{N}_{II}^{i} \\ \mathbf{N}_{\Gamma I}^{i} \end{pmatrix}, \ \mathbf{x}_{I}^{i} = \begin{pmatrix} \mathbf{y}_{I}^{i} \\ \mathbf{u}_{I}^{i} \\ \mathbf{p}_{I}^{i} \end{pmatrix}, \ \mathbf{g}_{I}^{i} = \begin{pmatrix} \mathbf{c}_{I}^{i} \\ \mathbf{d}_{I}^{i} \\ \mathbf{b}_{I}^{i} \end{pmatrix},$$

and for indices i with $\partial\Omega_i \cap \partial\Omega = \emptyset$, we define

$$\mathbf{K}_{II}^{i} = \begin{pmatrix} \mathbf{M}_{II}^{i} & (\mathbf{A}_{II}^{i})^T \\ \mathbf{A}_{II}^{i} & \end{pmatrix}, \ \mathbf{K}_{\Gamma I}^{i} = \begin{pmatrix} \mathbf{M}_{\Gamma I}^{i} & (\mathbf{A}_{I\Gamma}^{i})^T \\ \mathbf{A}_{\Gamma I}^{i} & \end{pmatrix}, \ \mathbf{x}_{I}^{i} = \begin{pmatrix} \mathbf{y}_{I}^{i} \\ \mathbf{p}_{I}^{i} \end{pmatrix}, \ \mathbf{g}_{I}^{i} = \begin{pmatrix} \mathbf{c}_{I}^{i} \\ \mathbf{b}_{I}^{i} \end{pmatrix}.$$

Most of this notation is a direct adaption of the notation used for domain decomposition of PDEs (see, e.g., Smith et al. [1996]). For example, $\mathbf{y}_I^i$ is the subvector containing the coefficients y_k of the discretized state belonging to nodes x_k in the interior of Ω_i. Note that in our particular control discretization, all basis functions μ_k for the discretised control u_h have support in only one subdomain boundary $\partial\Omega_i$ (see the right plot in Figure 1). Consequently, there is no $\mathbf{u}_\Gamma$.

After a symmetric permutation, (2) can be written as

$$\begin{pmatrix} \mathbf{K}_{II}^{1} & & & (\mathbf{K}_{\Gamma I}^{1})^T \\ & \ddots & & \vdots \\ & & \mathbf{K}_{II}^{d} & (\mathbf{K}_{\Gamma I}^{d})^T \\ \mathbf{K}_{\Gamma I}^{1} & \cdots & \mathbf{K}_{\Gamma I}^{d} & \mathbf{K}_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{x}_I^1 \\ \vdots \\ \mathbf{x}_I^d \\ \mathbf{x}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{g}_I^1 \\ \vdots \\ \mathbf{g}_I^d \\ \mathbf{g}_\Gamma \end{pmatrix}. \tag{4}$$

Frequently, we use the compact notation

$$\begin{pmatrix} \mathbf{K}_{II} & \mathbf{K}_{\Gamma I}^T \\ \mathbf{K}_{\Gamma I} & \mathbf{K}_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{x}_I \\ \mathbf{x}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{g}_I \\ \mathbf{g}_\Gamma \end{pmatrix}, \tag{5}$$

or even $\mathbf{K}\mathbf{x} = \mathbf{g}$ instead of (4). We make the following assumptions.

B. $\mathbf{A}_{II}^{i} \in \mathbb{R}^{m_i^I \times m_i^I}$ is invertible and $\mathbf{M}_{II}^{i} \in \mathbb{R}^{m_i^I \times m_i^I}$ is symmetric, $i = 1, \ldots, d$. For i with $\partial\Omega_i \cap \partial\Omega \neq \emptyset$, $\mathbf{H}_{II}^{i} \in \mathbb{R}^{n_i^I \times n_i^I}$ is symmetric and $\widehat{\mathbf{H}}_{II}^{i} = \alpha\mathbf{H}_{II}^{i} - (\mathbf{B}_{II}^{i})^T(\mathbf{A}_{II}^{i})^{-T}\mathbf{N}_{II}^{i} - (\mathbf{N}_{II}^{i})^T(\mathbf{A}_{II}^{i})^{-1}\mathbf{B}_{II}^{i} + (\mathbf{B}_{II}^{i})^T(\mathbf{A}_{II}^{i})^{-T}\mathbf{M}_{II}^{i}(\mathbf{A}_{II}^{i})^{-1}\mathbf{B}_{II}^{i}$ is positive definite.

C. $\mathbf{A}^{i} \in \mathbb{R}^{m_i \times m_i}$ is invertible and $\mathbf{M}^{i} \in \mathbb{R}^{m_i \times m_i}$ is symmetric, $i = 1, \ldots, d$. For i with $\partial\Omega_i \cap \partial\Omega \neq \emptyset$, $\mathbf{H}_{II}^{i} \in \mathbb{R}^{n_i^I \times n_i^I}$ is symmetric and $\widehat{\mathbf{H}}^{i} = \alpha\mathbf{H}_{II}^{i} - (\mathbf{B}^{i})^T(\mathbf{A}^{i})^{-T}\mathbf{N}^{i} - (\mathbf{N}^{i})^T(\mathbf{A}^{i})^{-1}\mathbf{B}^{i} + (\mathbf{B}^{i})^T(\mathbf{A}^{i})^{-T}\mathbf{M}^{i}(\mathbf{A}^{i})^{-1}\mathbf{B}^{i}$ is positive definite.

Assumptions A, B, C are satisfied for our example problem.

Assumption B guarantees that $\mathbf{K}_{II}$ is invertible. Hence, we can form the Schur complement system

$$\mathbf{S}\mathbf{x}_\Gamma = \mathbf{r}, \tag{6}$$

where $\mathbf{S} = \mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{K}_{\Gamma I}^T$ and $\mathbf{r} = \mathbf{g}_\Gamma - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{g}_I$. The Schur complement matrix $\mathbf{S}$ can be written as a sum of subdomain Schur complement matrices. Let $\tilde{\mathbf{R}}_i^y$, $i = 1, \ldots, d$, be the restriction operator which maps from the vector of coefficient unknowns on the artificial boundary, $\mathbf{y}_\Gamma$, to only those associated with the boundary of Ω_i. Let

$$\tilde{\mathbf{R}}_i = \begin{pmatrix} \tilde{\mathbf{R}}_i^y & \\ & \tilde{\mathbf{R}}_i^p \end{pmatrix}, \quad \tilde{\mathbf{R}}_i^p = \tilde{\mathbf{R}}_i^y \tag{7}$$

The Schur complement can be written as $\mathbf{S} = \sum_i \tilde{\mathbf{R}}_i^T \mathbf{S}_i \tilde{\mathbf{R}}_i$, where $\mathbf{S}_i = \mathbf{K}_{\Gamma\Gamma}^i - \mathbf{K}_{\Gamma I}^i(\mathbf{K}_{II}^i)^{-1}(\mathbf{K}_{\Gamma I}^i)^T$. It is shown in Heinkenschloss and Nguyen [2004] that the application $\mathbf{S}_i$ to a vector $\tilde{\mathbf{R}}_i(\mathbf{y}_\Gamma^T, \mathbf{p}_\Gamma^T)^T$ corresponds to solving a subdomain optimal control problem in Ω_i with Dirichlet boundary conditions for the state on $\partial\Omega_i \setminus \partial\Omega$ and then extracting Neumann data of the optimal state and corresponding adjoint on $\partial\Omega_i \setminus \partial\Omega$.

Theorem 1. *If Assumptions* A *and* B *are valid, then the Schur complement matrix* $\mathbf{S}$ *has* $m - \sum_{i=1}^d m_i^I$ *positive and* $m - \sum_{i=1}^d m_i^I$ *negative eigenvalues. If Assumptions* B *and* C *are valid, then the subdomain Schur complement matrix* $\mathbf{S}_i$, $i = 1, \ldots, d$, *has* $m_i - m_i^I$ *positive and* $m_i - m_i^I$ *negative eigenvalues.*

Proof. Recall that $\mathbf{S} = \mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{K}_{\Gamma I}^T$. It is easy to verify that

$$\begin{pmatrix} \mathbf{K}_{II} & \mathbf{K}_{\Gamma I}^T \\ \mathbf{K}_{\Gamma I} & \mathbf{K}_{\Gamma\Gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{II} & 0 \\ \mathbf{K}_{\Gamma I} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{K}_{II}^{-1} & 0 \\ 0 & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{K}_{II} & \mathbf{K}_{\Gamma I}^T \\ 0 & \mathbf{I} \end{pmatrix}.$$

The matrix $\mathbf{K}$ is a symmetric permutation of the system matrix in (2) and, hence, both matrices have the same eigenvalues. It is well known that the system matrix in (2) and, hence, $\mathbf{K}$ has $m+n$ positive and m negative eigenvalues (see, e.g., Keller et al. [2000]). Similarly, the matrix $\mathbf{K}_{II}$ has $\sum_{i=1}^d m_i^I + n_i^I$ positive and $\sum_{i=1}^d m_i^I$ negative eigenvalues. By Sylvester's law of inertia, the number of positive [negative] eigenvalues of $\mathbf{K}$ is equal to the number of positive [negative] eigenvalues of $\mathbf{K}_{II}^{-1}$ plus the number of positive [negative] eigenvalues of $\mathbf{S}$. Since $n = \sum_{i=1}^d n_i^I$, this implies that $\mathbf{S}$ has $m - \sum_{i=1}^d m_i^I$ positive and $m - \sum_{i=1}^d m_i^I$ negative eigenvalues.

The second assertion can be proven analogously.

If Assumption C is valid, then $\mathbf{S}_i^{-1}$ exists. It is shown in Heinkenschloss and Nguyen [2004] that the application $\mathbf{S}_i^{-1}$ to a vector $\tilde{\mathbf{R}}_i(\mathbf{v}_\Gamma^T, \mathbf{q}_\Gamma^T)^T$ corresponds to solving a subdomain optimal control problem in Ω_i with Neumann boundary conditions for the state on $\partial\Omega_i \setminus \partial\Omega$ and then extracting Dirichlet data of the optimal state and corresponding adjoint on $\partial\Omega_i \setminus \partial\Omega$.

It is now relatively easy to generalize the Neumann-Dirichlet and Neumann-Neumann preconditioners used in the context of elliptic PDEs to the optimal control context. We focus on Neumann-Neumann (NN) preconditioners.

Let $\mathbf{D}_i^y$ be the diagonal matrix, whose entries are computed as follows. If $x_k \in \partial\Omega_i$, then $(\mathbf{D}_i^y)_{kk}^{-1}$ is the number of subdomains that share node x_k. Note that $\sum_i \mathbf{D}_i^y = \mathbf{I}$. Furthermore, let $\tilde{\mathbf{D}}_i^p = \tilde{\mathbf{D}}_i^y$ and

$$\mathbf{D}_i = \begin{pmatrix} \mathbf{D}_i^y & \\ & \mathbf{D}_i^p \end{pmatrix}.$$

If assumptions B, C are valid, then we can form $\mathbf{S}_i$ and $\mathbf{S}_i^{-1}$. In this case the one-level NN preconditioner is given by

$$\mathbf{P} = \sum_i \mathbf{D}_i \tilde{\mathbf{R}}_i^T \mathbf{S}_i^{-1} \tilde{\mathbf{R}}_i \mathbf{D}_i. \tag{8}$$

It is well known that the performance of one-level NN preconditioners for elliptic PDEs deteriorates fast as the number of subdomains increases. The same is observed for the NN preconditioner (8) in the optimal control context (see Section 4). To avoid this, we include a coarse grid. More precisely, we adapt the balanced NN preconditioner due to Mandel [1993] to the optimal control context. Following the description in [Smith et al., 1996, Sec. 4.3.3], the balanced-NN for the optimal control problem is given by

$$\mathbf{P} = \left(\mathbf{I} - \tilde{\mathbf{R}}_0^T \mathbf{S}_0^{-1} \tilde{\mathbf{R}}_0 \mathbf{S}\right) \left(\sum_{i=1}^d \mathbf{D}_i \tilde{\mathbf{R}}_i^T \mathbf{S}_i^{-1} \tilde{\mathbf{R}}_i \mathbf{D}_i\right) \left(\mathbf{I} - \mathbf{S} \tilde{\mathbf{R}}_0^T \mathbf{S}_0^{-1} \tilde{\mathbf{R}}_0\right) + \tilde{\mathbf{R}}_0^T \mathbf{S}_0^{-1} \tilde{\mathbf{R}}_0, \tag{9}$$

where $\mathbf{S}_0 = \tilde{\mathbf{R}}_0 \mathbf{S} \tilde{\mathbf{R}}_0^T$ and $\tilde{\mathbf{R}}_0$ is defined as in (7) with $\tilde{\mathbf{R}}_0^y$ being the restriction operator which returns for each subdomain the weighted sum of the values of all the nodes on the boundary of that subdomain. The weight corresponding to an interface node is one over the number of subdomains the node is contained in.

4 Numerical Results

We consider (3) with $\Omega = (-1,1)^2$, $f(x) = (2\pi^2 + 1)\sin(\pi x_1)\sin(\pi x_2)$, $\hat{y}(x) = \sin(\pi x_1)\sin(\pi x_2)$. Numerical observations show that the condition number for the system matrix in (2) computed for a fixed discretization is proportional to α^{-1}. Hence, (3) becomes more difficult to solve as $\alpha > 0$ approaches zero.

The domain Ω is partitioned into equal-sized square subdomains in a checkerboard pattern. The side length of each subdomain is denoted by H. Regular meshes consisting of triangular elements of various widths, denoted by h, are generated. The preconditioned system $\mathbf{PS}\mathbf{x}_\Gamma = \mathbf{P}(\mathbf{g}_\Gamma - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{g}_I)$ is solved using the symmetric QMR (sQMR) algorithm of Freund and Nachtigal [1995]. The preconditioned sQMR iteration is stopped if the ℓ_2-norm of the residual is less than 10^{-8}. The subdomain problems are solved exactly using a sparse LU decomposition.

Tables 1, 2 show the number of preconditioned sQMR iterations needed to solve for various discretizations h and various subdomain sizes H. As expected, the performance of the NN preconditioner (8) without coarse grid gets worse quickly as the number of subdomain increases while the balanced NN preconditioner (9) remains effective. The number of sQMR iterations for the balanced NN preconditioner

remain nearly constant for a fixed H/h ratio.

The observed performance of the NN preconditioners (8), (9) applied to the optimal control problems is similar to the performance of the NN preconditioners applied to the elliptic PDE (3b) with fixed u. A notable result is that both preconditioners depend only weakly on the regularization parameter α. As α is reduced from 1 to 10^{-8}, the iteration count for the balanced NN preconditioner grows by only a factor of about two.

Table 1. Number of preconditioned sQMR iterations, $\alpha = 1$. Left: NN preconditioner (8). Right: Balanced NN preconditioner (9).

H \ h	1/4	1/8	1/16	1/32	1/64	1/128
1/2	12	15	19	24	26	28
1/4		53	69	94	107	119
1/8			170	226	287	345
1/16				509	679	798
1/32					1578	2233

H \ h	1/4	1/8	1/16	1/32	1/64	1/128
1/2	5	6	8	10	11	12
1/4		5	9	12	14	15
1/8			5	10	13	15
1/16				5	9	13
1/32					4	9

Table 2. Number of preconditioned sQMR iterations, $\alpha = 10^{-8}$. Left: NN preconditioner (8). Right: Balanced NN preconditioner (9).

H \ h	1/4	1/8	1/16	1/32	1/64	1/128
1/2	15	16	16	21	21	23
1/4		58	61	63	74	76
1/8			217	191	202	214
1/16				583	536	584
1/32					1255	1249

H \ h	1/4	1/8	1/16	1/32	1/64	1/128
1/2	9	10	13	15	17	20
1/4		11	17	21	26	30
1/8			13	18	24	30
1/16				12	19	24
1/32					11	16

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References

U. M. Ascher and E. Haber. A multigrid method for distributed parameter estimation problems. *Electron. Trans. Numer. Anal.*, 15:1–17, 2003.

A. Battermann and E. W. Sachs. Block preconditioners for KKT systems in PDE-governed optimal control problems. In K.-H. Hoffmann, R. H. W. Hoppe, and V. Schulz, editors, *Fast solution of discretized optimization problems (Berlin, 2000)*, pages 1–18, Basel, 2001. Birkhäuser.

J.-D. Benamou. A domain decomposition method with coupled transmission conditions for the optimal control of systems governed by elliptic partial differential equations. *SIAM J. Numer. Anal.*, 33:2401–2416, 1996.

G. Biros and O. Ghattas. Parallel Lagrange–Newton–Krylov–Schur methods for PDE–constrained optimization. Part I: The Krylov–Schur solver. Technical report, Carnegie Mellon University, 2000.

A. Bounaim. A Lagrangian approach to a DDM for an optimal control problem. In P. Bjørstad, M. Espedal, and D. Keyes, editors, *DD9 Proceedings*, pages 283–289, Bergen, Norway, 1998. Domain Decomposition Press.

J. E. Dennis and R. M. Lewis. A comparison of nonlinear programming approaches to an elliptic inverse problem and a new domain decomposition approach. Technical Report TR94–33, CAAM Dept., Rice University, 1994.

R. W. Freund and N. M. Nachtigal. Software for simplified Lanczos and QMR algorithms. *Applied Numerical Mathematics*, 19:319–341, 1995.

M. Heinkenschloss and H. Nguyen. Neumann-Neumann domain decomposition preconditioners for linear quadratic elliptic optimal control problems. Technical Report TR04-01, Department of Computational and Applied Mathematics, Rice University, 2004.

R. H. W. Hoppe, S. I. Petrova, and V. Schulz. Primal-dual Newton-type interior-point method for topology optimization. *J. Optim. Theory Appl.*, 114(3):545–571, 2002.

C. Keller, N. I. M. Gould, and A. J. Wathen. Constrained preconditioning for indefinite linear systems. *SIAM J. Matrix Anal. and Appl.*, 21:1300–1317, 2000.

J.-L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations.* Springer Verlag, Berlin, Heidelberg, New York, 1971.

J.-L. Lions and O. Pironneau. Sur le contrôle parallèle des systèmes distribués. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, 327 (12):993–998, 1998.

J. Mandel. Balancing domain decomposition. *Comm. Numer. Meth. Engrg.*, 9:233–241, 1993.

L. F. Pavarino and O. B. Widlund. Balancing Neumann-Neumann methods for incompressible Stokes equations. *Comm. Pure Appl. Math.*, 55(3):302–335, 2002.

B. Smith, P. Bjørstad, and W. Gropp. *Domain Decomposition. Parallel Multilevel Methods for Elliptic Partial Differential Equations.* Cambridge University Press, Cambridge, London, New York, 1996.