Stability of the Parareal Algorithm

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Summary. We discuss the stability of the Parareal algorithm for an autonomous set of differential equations. The stability function for the algorithm is derived, and stability conditions for the case of real eigenvalues are given. The general case of complex eigenvalues has been investigated by computing the stability regions numerically.

1 Introduction

This paper represents one of the contributions at a minisymposium on the Parareal algorithm at this domain decomposition conference. The minisymposium was organized by Professor Yvon Maday, who is also one of the originators of the Parareal algorithm. The main objective is to be able to integrate a set of differential equations using domain decomposition techniques in time. We refer to the review article by Yvon Maday in these proceedings for a more detailed introduction to the ideas and motivation behind this algorithm.

In Section 2, we briefly review the Parareal algorithm and introduce the necessary notation. Our main focus is the stability analysis of this algorithm. In Section 3.1, we briefly review the standard stability analysis of ordinary differential equations, and in Section 3.2, we derive the stability function for the Parareal algorithm. In the remaining part of Section 3, we derive the stability conditions in the case of real and complex eigenvalues.

2 Algorithm

The Parareal algorithm was first presented in Lions et al. [2001]. An improved version of the algorithm was presented in Bal and Maday [2002]. Further improvements and understanding, as well as new applications of the algorithm, were presented in Baffico et al. [2002] and Maday and Turinici [2002]; our point of departure is the version of the Parareal algorithm presented in these papers.

We consider a set of ordinary differential equations that we would like to integrate from an initial time $t_0 = 0$ to a final time T. The time interval is first decomposed as

$$t_0 = T_0 < T_1 < \cdots < T_n = n\Delta T < T_{n+1} < T_N = T.$$

The Parareal algorithm is then given as the predictor-corrector scheme

$$\lambda_n^k = \mathcal{F}_{\Delta T}(\lambda_{n-1}^{k-1}) + \mathcal{G}_{\Delta T}(\lambda_{n-1}^{k}) - \mathcal{G}_{\Delta T}(\lambda_{n-1}^{k-1}), \tag{1}$$

where subscript n refers to the time subdomain number, superscript k refers to the (global) iteration number, and λ_n^k represents an approximation to the solution at time level n at iteration number k. The fine propagator $\mathcal{F}_{\Delta T}$ represents a fine time discretization of the differential equations, with the property that

$$\lambda_n = \mathcal{F}_{\Delta T}(\lambda_{n-1}) \ , \ n = 1, ..., N,$$

while the coarse propagator $\mathcal{G}_{\Delta T}$ represents an approximation to $\mathcal{F}_{\Delta T}$.

Notice that $\mathcal{F}_{\Delta T}$ operates on initial conditions λ_{n-1}^{k-1}, which are known. This implies that $\mathcal{F}_{\Delta T}(\lambda_{n-1}^{k-1})$ can be implemented in parallel. The coarse propagator $\mathcal{G}_{\Delta T}$, on the other hand, operates on initial conditions λ_{n-1}^{k} from the current iteration, and is therefore strictly serial.

3 Stability analysis

In Farhat and Chandesris [2003], an investigation of the stability for an autonomous problem is presented. We will here use more of the tools provided by the ODE theory, and extend the stability analysis a bit further.

The departure of our stability analysis is the predictor-corrector scheme (1). A stability analysis is performed on the autonomous differential equation

$$y' = \mu y, \quad y(0) = y_0, \quad \mu < 0 \ . \tag{2}$$

The exact solution to this problem is $y(t) = e^{\mu t} y_0$. Since $\mu < 0$, this is a decaying function for increasing t. The numerical solution of (2) is an approximation to the exact solution. It is well known that a convergent numerical scheme can be arbitrarily accurate by choosing sufficiently small time-steps. A numerical scheme which results in an non-increasing approximation for the chosen time-step is called stable. For a more precise definition of stability, the reader is referred to Hairer et al. [2000].

3.1 Stability analysis for ordinary ODE schemes

To better understand the derivation of the stability properties of the Parareal algorithm, we start by deriving the stability properties for two well known numerical schemes, namely the explicit and implicit Euler methods. Applied to our differential equation, the two schemes can be written as

$$
\begin{aligned}
y_n &= y_{n-1} + \Delta T \mu \, y_{n-1} = (1 + \Delta T \mu)^n y_0 = R(z)^n \, y_0 && \text{explicit Euler} \\
y_n &= y_{n-1} + \Delta T \mu \, y_n \; = (1 - \Delta T \mu)^{-n} \, y_0 = R(z)^n y_0 && \text{implicit Euler}
\end{aligned}
$$

where ΔT is the time-step, $z = \Delta T \mu$ and $R(z)$ is called the *stability function* of the chosen scheme. Obviously, $|R(z)| \leq 1$ will prevent the numerical schemes from blowing up for increasing n.

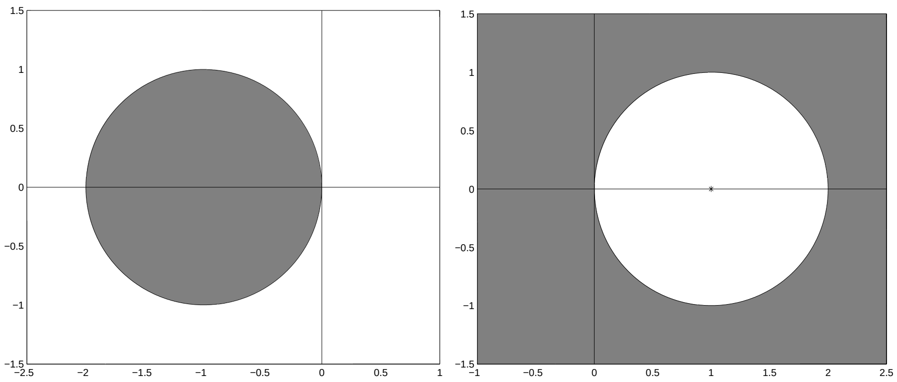

Fig. 1. Stability domain for explicit (left) and implicit(right) Euler. The dark region is the stability domain, i.e., those values of z in the complex plane where $|R(z)| \leq 1$.

From Figure 1 we see that explicit Euler suffers from time-step restrictions, while implicit Euler is stable for all possible choices of the time-step ΔT ($\mu < 0$). In the context of the Parareal algorithm, the coarse propagator $\mathcal{G}_{\Delta T}$ is forced to take large time-steps, which clearly indicates that implicit Euler is a better choice then explicit Euler for the coarse propagator.

Consider now a linear system of M autonomous differential equations

$$y' = Ay, \quad y(0) = y_0. \tag{3}$$

Assuming that a spectral factorization is possible, we may write the system matrix $A \in \mathbb{R}^{M \times M}$ as

$$A = VDV^{-1}$$

where D is a diagonal matrix containing the eigenvalues $\{\mu_1, \ldots, \mu_M\}$ of A, and V is a matrix containing the corresponding eigenvectors of A.

The exact solution of (3) may then be written as

$$y(t) = e^{tA} y_0 = V e^{tD} V^{-1} y_0 \, ,$$

while the approximation of the (3) using implicit Euler can be expressed as

$$y_n = V\,(I - \varDelta T D)^{-n}\, V^{-1}\, y_0\,.$$

Obviously, a method is stable for systems of ODE's if $|R(z_i)| \leq 1$, $i = 1, \ldots, M$, where $z_i = \varDelta T \mu_i$ and μ_i is the i^{th} eigenvalue of A.

3.2 Stability analysis for the Parareal algorithm

In the following analysis, we assume that we may use different integration schemes for the fine and the coarse propagator. Within each coarse time step $\varDelta T$, we will use several fine time steps δt with the fine propagator.

Our first aim is to write the predictor-corrector scheme (1) on the form

$$\lambda_n^k = V\, H(n, k, r(D\delta t), R(D\varDelta T)\, V^{-1}\, \lambda_0,$$

where n is the subdomain number (in time), k is the iteration number, H is the "stability function" for the Parareal scheme, r is the stability function for the fine propagator $\mathcal{F}_{\varDelta T}$, R is the stability function for the coarse propagator $\mathcal{G}_{\varDelta T}$ and D is the diagonal matrix containing all the eigenvalues for the system matrix. To do this we first apply the predictor-corrector scheme (1) to the model problem (2); this gives

$$\lambda_n^k = \bar{r}(\mu\delta t)\lambda_{n-1}^{k-1} + R(\mu\varDelta T)\lambda_{n-1}^k - R(\mu\varDelta T)\lambda_{n-1}^{k-1}, \tag{4}$$

where $\bar{r}(\mu\delta t) = r(\mu\delta t)^s$ is the stability function for the fine operator after $s = \frac{\varDelta T}{\delta t}$ fine time-steps δt, and $R(\mu\varDelta T)$ is the stability function for the coarse operator $\mathcal{G}_{\varDelta T}$. For simplicity we will write $\bar{r} = \bar{r}(\mu\delta t)$ and $R = R(\mu\varDelta T)$.

We rearrange (4) and write

$$\lambda_n^k = R\lambda_{n-1}^k + (\bar{r} - R)\lambda_{n-1}^{k-1} = R\lambda_{n-1}^k + S\lambda_{n-1}^{k-1}. \tag{5}$$

Obviously, the recursion is solved like this:

$$\begin{array}{ccccc}
\lambda_n^k & & & \lambda_n^k & \\
\downarrow\searrow & & & \downarrow R \;\searrow S & \\
\lambda_{n-1}^k & \lambda_{n-1}^{k-1} & & \lambda_{n-1}^k & \lambda_{n-1}^{k-1} \\
\downarrow\searrow & \downarrow\searrow & & & \\
\lambda_{n-2}^k & \lambda_{n-2}^{k-1} & \lambda_{n-2}^{k-2} & & \\
\downarrow\searrow & \downarrow\searrow & \downarrow\searrow & & \\
\lambda_{n-3}^k & \lambda_{n-3}^{k-1} & \lambda_{n-3}^{k-2} & \lambda_{n-3}^{k-3} & \\
\downarrow\searrow & \downarrow\searrow & \downarrow\searrow & \downarrow \quad \searrow & \\
\lambda_{n-4}^k & \lambda_{n-4}^{k-1} & \lambda_{n-4}^{k-2} & \lambda_{n-4}^{k-3} & \lambda_{n-4}^{k-4}
\end{array}$$

We recognize the Pascal tree, and we may write (5) as

$$\lambda_n^k = \left(\sum_{i=0}^{k} \binom{n}{i} (\bar{r} - R)^i R^{n-1} \right) \lambda_0,$$

where we identify the "stability function" H as

$$H(n,k,r,R) = \sum_{i=0}^{k} \binom{n}{i} (\bar{r}-R)^i R^{n-1}.$$

The extension to solve the system (3) is straightforward,

$$\lambda_n^k = V\, H(n,k,r,R)\, V^{-1}\, \lambda_0.$$

Stability is achieved if

$$\sup_{1\le n\le N}\ \sup_{1\le k\le N} |H(n,k,r,R)| \le 1 \qquad \forall \mu_i\ ,\ i=1,...,M. \tag{6}$$

3.3 Special case: μ_i real

In the case of real eigenvalues, the stability condition (6) can be expressed as

$$\begin{aligned} |H| = \left| \sum_{i=0}^{k} \binom{n}{i} (\bar{r}-R)^i R^{n-i} \right| &\le \sum_{i=0}^{k} \binom{n}{i} |(\bar{r}-R)|^i |R|^{n-i} \\ &\le \sum_{i=0}^{n} \binom{n}{i} |(\bar{r}-R)|^i |R|^{n-i} \\ &= (|\bar{r}-R| + |R|)^n \le 1 \qquad \forall \mu_i\ ,\ i=1,...,M, \end{aligned}$$

where $|\bar{r}-R|+|R|$ is either $|(\bar{r}-R)+R|$ or $|(\bar{r}-R)-R|$. [1] The condition $|(\bar{r}-R)+R| = |\bar{r}| \le 1$ is the stability condition for the fine operator, and this should be true independent of the use of the Parareal algorithm.

The condition $|(\bar{r}-R)-R| = |2R-\bar{r}| \le 1$ can be rewritten as

$$\frac{\bar{r}-1}{2} \le R \le \frac{\bar{r}+1}{2}\,. \tag{7}$$

Theorem 1. *Assume we want to solve the autonomous differential equation*

$$y' = \mu y\,, \quad y(0) = y_0\,, \quad 0 > \mu \in \mathbb{R}\,,$$

and that $-1 \le r, R \le 1$ *where* $r = r(\mu\delta t)$ *is the stability function for the fine propagator* $\mathcal{F}_{\Delta T}$ *using time-step* δt *and* $R = R(\mu\Delta T)$ *is the stability function for the coarse propagator* $\mathcal{G}_{\Delta T}$ *using time-step* ΔT*. Then the Parareal algorithm is stable for all possible values of number of subdomains* N *and all number of iterations* $k \le N$ *as long as*

$$\frac{\bar{r}-1}{2} \le R \le \frac{\bar{r}+1}{2}$$

where $\bar{r} = r(\mu\delta t)^s$ *and* $s = \frac{\Delta T}{\delta t}$*.*

[1] This is due to Harald Hanche Olsen, Dept. of Mathematical Sciences, NTNU

It is not obvious from (7) which solvers will fulfil this stability condition. However, Theorem 2 gives some insight by considering a special case.

Theorem 2. *Assume we want to solve the autonomous differential equation*

$$y' = \mu y\,, \quad y(0) = y_0\,, \quad 0 > \mu \in \mathbb{R}\,,$$

using the Parareal algorithm. Assume also that the system is stiff, meaning that $z = \mu \Delta T \ll -1$, and that the fine propagator is close to exact. Then the "stability function" can be written as

$$H(n,k,R) = (-1)^k \binom{n-1}{k} R^n\,,$$

and stability is guaranteed if the following property is fulfilled:

$$R_\infty = \lim_{z \to -\infty} |R(z)| \leq \frac{1}{2}\ . \tag{8}$$

The proofs of Theorem 1 and 2 are not included due to space limitation, but will be included in a future article.

We have tested the condition (8) by solving the one-dimensional unsteady diffusion equation using a spectral Galerkin method in space and a Crank-Nicolson scheme for the fine propagator $\mathcal{F}_{\Delta T}$. We then tested the following schemes for the coarse propagator $\mathcal{G}_{\Delta T}$: the implicit Euler method ($R_\infty = 0$), the Crank-Nicolson scheme ($R_\infty = 1$), and the θ-scheme where we can vary the degree of "implicitness," and hence R_∞; see Hairer et al. [2000] and Hairer and Wanner [2002]. The numerical results demonstrated that rapid convergence of the Parareal scheme is obtained for implicit Euler, while Crank-Nicolson first gives convergence, and then starts to diverge when k increases. However, as k approaches N, the results again start to converge; this is expected since the Parareal algorithm gives precisely the fine solution after N iterations. By varying the degree of "implicitness" in the θ-scheme, we observe that our results are consistent with the "stability condition" (8).

3.4 General case: μ_i complex

Notice that Theorem 1 is true for ODE's and systems of ODE's where the eigenvalues of the system matrix have pure real eigenvalues. For complex eigenvalues, (6) needs to be fulfilled. This is done numerically in Figure 2 and 3 for the two-stage third order Implicit Runge-Kutta-Radau scheme (Radau3); see Hairer and Wanner [2002]. This scheme is chosen because it represents the typical asymptotic behaviour of a scheme which fulfils Theorem 2. The difference in behavior between the various possible schemes lies in the size of the instability regions in the real direction, and in the number of instability regions in the imaginary direction. For example, implicit Euler will have smaller instability regions compared to Radau3.

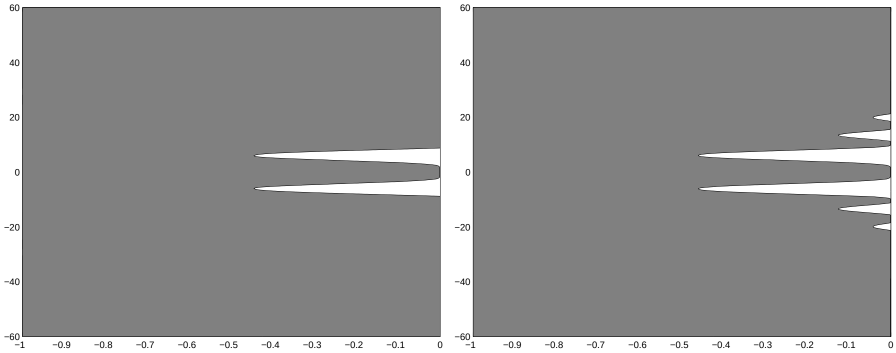

Fig. 2. Stability plots using Radau3 for both $\mathcal{G}_{\Delta T}$ and $\mathcal{F}_{\Delta T}$. The x-axis is $\mathrm{Re}(\mu \Delta T)$ and the y-axis is $\mathrm{Im}(\mu \Delta T)$. The dark regions represent the regions in the complex plane where (6) is satisfied. Here, $N = 10$, and $s = 10$ (left) and $s = 1000$ (right).

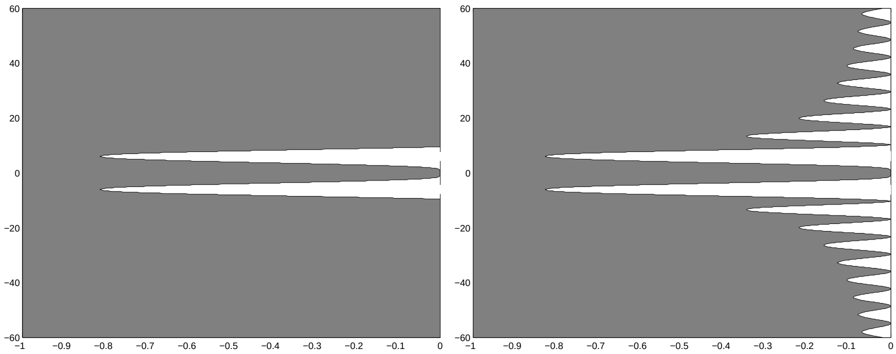

Fig. 3. Stability plots using Radau3 for both $\mathcal{G}_{\Delta T}$ and $\mathcal{F}_{\Delta T}$. The x-axis is $\mathrm{Re}(\mu \Delta T)$ and the y-axis is $\mathrm{Im}(\mu \Delta T)$. The dark regions represent the regions in the complex plane where (6) is satisfied. Here, $N = 1000$, and $s = 10$ (left) and $s = 1000$ (right).

From Figure 2 and 3 we notice that the Parareal algorithm is unstable for pure imaginary eigenvalues, as well as for some complex eigenvalues where the imaginary part is much larger then the real part (notice the difference in scalings along the real and the imaginary axes). No multistage scheme has yet been found that makes the presented formulation of the Parareal algorithm stable for all possible eigenvalues. This means that the numerical solution of some hyperbolic problems, and convection-diffusion problems with highly dominant convection (e.g Navier-Stokes with high Reynolds numbers), are probably unstable using the Parareal algorithm. This is also consistent with the results reported in Farhat and Chandesris [2003].

4 Conclusion and final comments

For an autonomous set of differential equations, we have derived the stability conditions for the Parareal algorithm. The stability conditions corresponding to the case of real eigenvalues are explicitly given, while the general case has been investigated by computing the stability regions numerically. These latter results indicate that the Parareal algorithm is unstable for pure imaginary eigenvalues, which is also consistent with previously reported results.

Numerical results have also been obtained using the Parareal algorithm in the context of solving partial differential equations such as the nonlinear, viscous Burger's equation, and where the coarse propagator incorporates a coarse discretization in space as well as in time. However, a discussion of these results will be reported elsewhere due to space limitation.

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