Mixed Finite Element Methods for Diffusion Equations on Nonmatching Grids

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Summary. The hybridization technique is applied to replace the macro-hybrid mixed finite element problem for the diffusion equation by the equivalent cell-based formulation. The underlying algebraic system is condensed by eliminating the degrees of freedom which represent the interface flux and cell pressure variables to the system containing the Lagrange multipliers variables. An approach to the numerical solution of the condensed system is briefly discussed.

1 Introduction

In this paper, we consider macro-hybrid mixed finite element method for the diffusion equation on nonmatching grids. The paper is organized as follows. The four-field macro-hybrid mixed formulation for the diffusion equation is given in Sect. 2.

In Sect. 3, we apply the hybridization technique to replace the macro-hybrid formulation by the cell-based formulation and describe the condensation procedure to reduce the underlying algebraic system to the system for the Lagrange multipliers only. In Sect. 4, we briefly discuss an algebraic solution method for the condensed system.

2 Problem formulation

We consider the diffusion problem in the form of a system of the first order differential equations

$$
\begin{aligned}
K^{-1}\,\bar{u} + \operatorname{grad} p &= 0 \\
\operatorname{div} \bar{u} + \quad cp &= f
\end{aligned}
\tag{1}
$$

in a bounded connected polygonal (polyhedral) domain Ω in $\mathbb{R}^d$, $d = 2$ $(d = 3)$ with homogeneous boundary conditions

$$p = 0 \quad \text{on} \quad \Gamma_D,$$
$$\bar{u} \cdot \bar{n} = 0 \quad \text{on} \quad \Gamma_N. \tag{2}$$

Here Γ_D and Γ_N are the Dirichlet and the Neumann parts of the boundary $\partial\Omega$, $\bar{n}$ is the outward unit normal to $\partial\Omega$, $K = K(x)$ is the diffusion tensor, $K = K^T > 0$, $c = c(x)$ is a nonnegative function, and $f \in L_2(\Omega)$. We assume that Γ_D is a closed subset of $\partial\Omega$ consisting of a finite number of segments (polygons) in the case $d = 2$ $(d = 3)$.

For the sake of simplicity, in the paper, we consider only the case $d = 2$. The extension to the three dimensional problem is basically straightforward.

The weak formulation of (1), (2) reads as follows: find

$$\bar{u} \in V \equiv \left\{ \bar{v} : \bar{v} \in H_{\mathrm{div}}(\Omega), \int_{\partial\Omega} |\bar{v} \cdot \bar{n}|^2 \, \mathrm{ds} < +\infty, \ \bar{v} \cdot \bar{n} = 0 \text{ on } \Gamma_N \right\},$$

$p \in Q \equiv L_2(\Omega)$ such that

$$\int_\Omega \left(K^{-1}\bar{u} \right) \cdot \bar{v} \, \mathrm{dx} - \int_\Omega p(\nabla \cdot \bar{v}) \, \mathrm{dx} = 0$$
$$\int_\Omega (\nabla \cdot \bar{u})q \, \mathrm{dx} \quad + \int_\Omega cpq \, \mathrm{dx} \quad = \int_\Omega fq \, \mathrm{dx} \tag{3}$$

for all $(\bar{v}, \ q) \in V \times Q$.

Let Ω_h be a partitioning of Ω into m nonoverlapping polygonal cells e_k:

$$\Omega_h = \bigcup_{k=1}^{m} e_k, \tag{4}$$

and V_h and Q_h be finite element subspaces of V and Q, respectively. We assume that the partitioning Ω_h is conforming, i.e. the interface Γ_{st} between any adjacent cells e_s and e_t is always a common edge for both cells and the set $\Gamma_N \cap \Gamma_D$ belongs to the set of vertices in Ω_h. If all the cells e_k are triangles then V_h can be chosen as the proper subspace of the lowest Raviart-Thomas finite element space $\mathrm{RT}_0(\Omega_h)$ (see, Brezzi and Fortin [1991]). Otherwise, we can use the new method for the construction of V_h recently invented in Kuznetsov and Repin [2003]. The normal components $\bar{u} \cdot \bar{n}_{st}$ of the flux $\bar{u}$ at the interfaces Γ_{st} between cells e_s and e_t are constants in both choices of V_h. Here $\bar{n}_{st}$ denotes the unit normal to Γ_{st} directed from e_s to e_t.

The mixed finite element approximation to (1), (2) reads as follows: find $(\bar{u}_h, \ p_h) \in V_h \times Q_h$ such that

$$\int_\Omega \left(K^{-1}\bar{u}_h \right) \cdot \bar{v} \, \mathrm{dx} - \int_\Omega p_h(\nabla \cdot \bar{v}) \, \mathrm{dx} = 0$$
$$\int_\Omega (\nabla \cdot \bar{u}_h) \, q \, \mathrm{dx} \quad + \int_\Omega cp_h q \, \mathrm{dx} \quad = \int_\Omega fq \, \mathrm{dx} \tag{5}$$

for all $(\bar{v}, \ q) \in V_h \times Q_h$.

Let Ω be splitted into two nonoverlapping subdomains Ω_1 and Ω_2 with a piece-wise linear simply connected interface boundary Γ. Then, the four-field macro-hybrid mixed formulation of (1), (2) originally proposed in Kuznetsov and Wheeler [1995] reads as follows: find $(\bar{u}_k,\ p_k,\ \lambda_k) \in V_k \times Q_k \times \Lambda_k$, $k = 1, 2$, $\phi \in \Phi$ such that

$$a_1(\bar{u}_1,\ \bar{v}_1) + b_1(p_1,\ \bar{v}_1) + c_1(\lambda_1,\ \bar{v}_1) = 0$$
$$a_2(\bar{u}_2,\ \bar{v}_2) + b_2(p_2,\ \bar{v}_2) + c_2(\lambda_2,\ \bar{v}_2) = 0$$

$$b_1(q_1,\ \bar{u}_1) - \sigma_1(p_1,\ q_1) \qquad\qquad = l_1(q_1)$$
$$b_2(q_2,\ \bar{u}_2) - \sigma_2(p_2,\ q_2) \qquad\qquad = l_2(q_2)$$

$$c_1(\mu_1,\ \bar{u}_1) \qquad\qquad + d_1(\phi,\ \mu_1) = 0$$
$$c_2(\mu_2,\ \bar{u}_2) \qquad\qquad + d_2(\phi,\ \mu_1) = 0$$

$$d_1(\psi,\ \lambda_1) \quad + \quad d_2(\psi,\ \lambda_2) \qquad = 0$$

$$(6)$$

for all $(\bar{v}_k,\ p_k,\ \mu_k) \in V_k \times Q_k \times \Lambda_k$, $k = 1, 2$, $\psi \in \Psi$.
Here

$$V_k = \Big\{ \bar{v} : \bar{v} \in H_{\mathrm{div}}(\Omega_k), \int\limits_{\partial\Omega_k} (\bar{v} \cdot \bar{n}_k)^2\, \mathrm{ds} < +\infty,\ \bar{v} \cdot \bar{n} = 0 \text{ on } \partial\Omega_k \cap \Gamma_N \Big\},$$
$$Q_k = L_2(\Omega_k), \quad \Lambda_k = L_2(\Gamma), \quad k = 1, 2,$$
$$\Phi = L_2(\Gamma),$$

$$(7)$$

and

$$a_k(\bar{u}, \bar{v}) = \int\limits_{\Omega_k} \big(K^{-1}\bar{u}\big) \cdot \bar{v}\, \mathrm{dx}, \quad \sigma_k(p,\ q) = \int\limits_{\Omega_k} cpq\, \mathrm{dx},$$

$$b_k(p,\ \bar{v}) = -\int\limits_{\Omega_k} p\,(\nabla \cdot \bar{v})\, \mathrm{dx}, \quad c_k(\lambda,\ \bar{v}) = (-1)^{k-1}\int\limits_{\Gamma} \lambda\,(\bar{v} \cdot \bar{n}_\Gamma)\, \mathrm{ds}, \quad (8)$$

$$d_k(\phi,\ \mu) = (-1)^k \int\limits_{\Gamma} \phi\mu\, \mathrm{ds}, \quad l_k(q) = -\int\limits_{\Omega_k} fq\, \mathrm{dx},$$

$k = 1, 2$, where $\bar{n}_\Gamma$ is the unit normal vector to Γ directed from Ω_1 to Ω_2.

Let $\Omega_{k,h}$ be a partitioning of Ω_k into m_k polygons $e_i^{(k)}$, $k = 1, 2$. We assume that both partitionings are conformal and the set of vertices of Γ belongs to the set of vertices of both partitionings $\Omega_{1,h}$ and $\Omega_{2,h}$. Subspaces $V_{k,h}$ and $Q_{k,h}$ of the spaces V_k and Q_k, respectively, are defined similar to V_h and Q_h in problem (5).

Let $\varGamma_h^{(k)} = \bigcup_{i=1}^{n_k} \gamma_{i,h}^{(k)}$ be the trace of $\Omega_{k,h}$ onto $\varGamma$ where $\gamma_{i,h}^{(k)}$ are the edges of the cells in $\Omega_{k,h}$ adjacent to $\varGamma$, $i = \overline{1,\ n_k}$, $k = 1,2$. Here n_k is the number of cells' edges in Ω_h belonging to $\varGamma$, $k = 1,2$. We define $\varLambda_{k,h}$ by

$$\varLambda_{k,h} = \left\{\lambda : \lambda = \text{const on } \gamma_{i,h}^{(k)},\ i = \overline{1,\ n_k}\right\} \tag{9}$$

$k = 1, 2$, and choose

$$\varPhi_h = \varLambda_{1,h}. \tag{10}$$

The finite element approximation to (6)-(8) reads as follows: find
$(\bar{u}_{k,h},\ p_{k,h},\ \lambda_{k,h}) \in V_{k,h} \times Q_{k,h} \times \varLambda_{k,h}$, $k = 1,2$, $\phi_h \in \varPhi_h$, such that the equations (6) with $\bar{u}_k = \bar{u}_{k,h}$, $p_k = p_{k,h}$, $\lambda_k = \lambda_{k,h}$, $k = 1,2$, $\phi = \phi_h$ are satisfied for all $(\bar{v}_k,\ q_k,\ \mu_k) \in V_{k,h} \times Q_{k,h} \times \varLambda_{k,h}$, $k = 1,2$, $\psi \in \varPhi_h$. This approximation results in the system

$$\mathcal{A} \begin{pmatrix} w_1 \\ w_2 \\ \phi \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} \tag{11}$$

with the matrix

$$\mathcal{A} = \begin{pmatrix} A_1 & 0 & D_1^T \\ 0 & A_2 & D_2^T \\ D_1 & D_2 & 0 \end{pmatrix} \tag{12}$$

where

$$A_k = \begin{pmatrix} M_k & B_k^T & C_k^T \\ B_k & -\Sigma_k & 0 \\ C_k & 0 & 0 \end{pmatrix} \tag{13}$$

are the saddle point matrices, $k = 1,2$, and

$$w_k = \begin{pmatrix} u_k \\ p_k \\ \lambda_k \end{pmatrix}, \quad F_k = \begin{pmatrix} 0 \\ -f_k \\ 0 \end{pmatrix}, \quad k = 1,2. \tag{14}$$

Here M_k is a symmetric positive definite matrix, and Σ_k is a symmetric positive definite (or semidefinite) matrix, $k = 1,2$.

3 Hybridization and condensation

The extension of (6)-(8) to the case of many subdomains is straightforward. We consider the hybrid mixed formulation based on partitionings of $\Omega_{k,h}$ into subdomains/cells $e_i^{(k)}$ used in Sect. 2 for the approximation of the problem (6)-(8).

We introduce new spaces $V_{k,i,h}$ and $Q_{k,i,h}$ to be the restrictions onto $e_i^{(k)}$ of $V_{k,h}$ subject to $\Gamma_N = \emptyset$ and $Q_{k,h}$, respectively, $i = \overline{1, m_k}$, $k = 1, 2$, and define the spaces $\widehat{V}_{k,h}$ and $\widehat{Q}_{k,h}$ as the products of the spaces $V_{k,i,h}$ and $Q_{k,i,h}$, $i = \overline{1, \ m_k}$, respectively, $k = 1, 2$. Then, we introduce spaces $\Lambda_{k,i,h}$ of functions λ defined on $\partial e_i^{(k)}$ which are constants on each interface $\Gamma_{k,i,h}$ between $e_i^{(k)}$ and adjacent cells $e_j^{(k)}$ as well as on the intersections $\partial e_i^{(k)}$ with the linear parts of $\partial\Omega$, $i = \overline{1, \ m_k}$, $k = 1, 2$. The functions in $\Lambda_{k,i,h}$ should vanish on Γ_D. We denote by $\widehat{\Lambda}_{k,h}$ the product of all spaces $\Lambda_{k,i,h}$, $k = 1, 2$. Finally, we preserve the definition for Φ_h from Sect. 2.

The new finite element problem reads as follows: find $(\hat{u}_{k,h}, \ \hat{p}_{k,h}, \ \hat{\lambda}_{k,h}) \in \widehat{V}_{k,h} \times \widehat{Q}_{k,h} \times \widehat{\Lambda}_{k,h}$, $k = 1, 2$, $\hat{\phi}_h \in \Phi_h$, such that

$$\hat{a}_1(\hat{u}_{1,h}, \ \bar{v}_1) + \hat{b}_1(\hat{p}_{1,h}, \ \bar{v}_1) + \hat{c}_1(\hat{\lambda}_{1,h}, \ \bar{v}_1) = 0$$
$$\hat{a}_2(\hat{u}_{2,h}, \ \bar{v}_2) + \hat{b}_2(\hat{p}_{2,h}, \ \bar{v}_2) + \hat{c}_2(\hat{\lambda}_{2,h}, \ \bar{v}_2) = 0$$

$$\hat{b}_1(q_1, \ \hat{u}_{1,h}) - \hat{\sigma}_1(\hat{p}_{1,h}, \ q_1) \qquad\qquad = \hat{l}_1(q_1)$$
$$\hat{b}_2(q_2, \ \hat{u}_{2,h}) - \hat{\sigma}_2(\hat{p}_{2,h}, \ q_2) \qquad\qquad = \hat{l}_2(q_2)$$

$$\hat{c}_1(\mu_1, \ \hat{u}_{1,h}) \qquad\qquad + \hat{d}_1(\hat{\phi}_h, \ \mu_1) = 0 \tag{15}$$
$$\hat{c}_2(\mu_2, \ \hat{u}_{2,h}) \qquad\qquad + \hat{d}_2(\hat{\phi}_h, \ \mu_1) = 0$$

$$\hat{d}_1(\psi, \ \hat{\lambda}_{1,h}) \quad + \quad \hat{d}_2(\psi, \ \hat{\lambda}_{2,h}) \qquad = 0$$

for all $(\bar{v}_k, \ p_k, \ \mu_k) \in \widehat{V}_{k,h} \times \widehat{Q}_{k,h} \times \widehat{\Lambda}_{k,h}$, $k = 1, 2$, $\psi \in \Phi_h$. Here,

$$\hat{a}_k(\bar{u}_k, \ \bar{v}_k) = \sum_{i=1}^{m_k} \int_{e_i^{(k)}} \left(K^{-1}\bar{u}_{k,i}\right) \cdot \bar{v}_{k,i} \, \mathrm{dx}, \quad \hat{b}_k(p_k, \ \bar{v}_k) = -\sum_{i=1}^{m_k} \int_{e_i^{(k)}} p_{k,i} \left(\nabla \cdot \bar{v}_{k,i}\right) \mathrm{dx},$$

$$\hat{c}_k(\lambda_k, \ \bar{v}_k) = \sum_{i=1}^{m_k} \int_{\partial e_i^{(k)} \backslash \Gamma_D} \lambda_{k,i} \left(\bar{v}_{k,i} \cdot \bar{n}_{k,i}\right) \mathrm{ds}, \quad \hat{\sigma}_k(p_k, \ q_k) = \sum_{i=1}^{m_k} \int_{e_i^{(k)}} c p_{k,i} q_{k,i} \, \mathrm{dx},$$

$$\hat{d}_k(\phi, \ \mu_k) = (-1)^k \sum_{i=1}^{m_k} \int_{\Gamma \cap \partial e_i^{(k)}} \phi \mu_{k,i} \, \mathrm{ds}, \quad \hat{l}_k(q_k) = -\sum_{i=1}^{m_k} \int_{e_i^{(k)}} f q_{k,i} \, \mathrm{dx},$$

$$\tag{16}$$

where $\bar{n}_{k,i}$ is the outward unit normal to $\partial e_i^{(k)}$, $i = \overline{1, \ m_k}$, $k = 1, 2$.

The finite element problem (15), (16) is said to be the hybridization of the finite element problem of the previous Section. It can be proved that the problems are equivalent, i.e. the restrictions of $\bar{u}_{k,h}$ and $p_{k,h}$ onto a cell $e_i^{(k)}$ coincide with $\hat{u}_{k,i,h}$ and $\hat{p}_{k,i,h}$, respectively, $\lambda_{k,h}$ coincides with restriction of $\hat{\lambda}_{k,h}$ onto Γ, and ϕ_h coincides with $\hat{\phi}_h$.

Problem (15), (16) results in the system of linear algebraic equations

$$
\mathcal{A}
\begin{pmatrix}
w_1 \\
\lambda_1 \\
w_2 \\
\lambda_2 \\
\phi
\end{pmatrix}
=
\begin{pmatrix}
F_1 \\
0 \\
F_2 \\
0 \\
0
\end{pmatrix}
\tag{17}
$$

with the 5×5 block matrix

$$
\mathcal{A} =
\begin{pmatrix}
A_1 & C_1^T & 0 & 0 & 0 \\
C_1 & 0 & 0 & 0 & D_1^T \\
0 & 0 & A_2 & C_2^T & 0 \\
0 & 0 & C_2 & 0 & D_2^T \\
0 & D_1 & 0 & D_2 & 0
\end{pmatrix}
\tag{18}
$$

where A_k is the block diagonal matrix with the diagonal blocks

$$
A_{k,i} =
\begin{pmatrix}
M_{k,i} & B_{k,i}^T \\
B_{k,i} & -\Sigma_{k,i}
\end{pmatrix},
\tag{19}
$$

$$
C_k = \begin{pmatrix} C_{k,1} & \ldots & C_{k,m_k} \end{pmatrix},
\tag{20}
$$

and

$$
F_k =
\begin{pmatrix}
F_{k,1} \\
\vdots \\
F_{k,m_k}
\end{pmatrix},
\qquad
F_{k,i} =
\begin{pmatrix}
0 \\
-f_{k,i}
\end{pmatrix},
\quad i = \overline{1, \; m_k},
\tag{21}
$$

$k = 1, 2$. The subvectors w_1 and w_2 can be excluded from the system by the block Gauss elimination method. The reduced system is given by

$$
\begin{pmatrix}
S_1 & 0 & -D_1^T \\
0 & S_2 & -D_2^T \\
-D_1 & -D_2 & 0
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\phi
\end{pmatrix}
=
\begin{pmatrix}
g_1 \\
g_2 \\
0
\end{pmatrix}
\tag{22}
$$

where

$$
S_k = \sum_{i=1}^{m_k} C_{k,i} \, A_{k,i}^{-1} \, C_{k,i}^T
\tag{23}
$$

$$
g_k = \sum_{i=1}^{m_k} C_{k,i} \, A_{k,i}^{-1} \, F_{k,i},
\tag{24}
$$

$k = 1, 2$. The system (22)-(24) is said to be the condensation of the system (17)-(21).

4 Algebraic solvers

The saddle point system (22) can be explicitly reduced to a system with a positive definite matrix. With additional block partitioning

$$
S_k = \begin{pmatrix} S_{kk} & S_{k\Gamma} \\ S_{\Gamma k} & S_{\Gamma\Gamma}^{(k)} \end{pmatrix}, \quad D_k = \begin{pmatrix} 0 & D_{k\Gamma} \end{pmatrix},
$$
$$
\lambda_k = \begin{pmatrix} \lambda_{kk} \\ \lambda_{k\Gamma} \end{pmatrix}, \quad g_k = \begin{pmatrix} g_{k1} \\ g_{k\Gamma} \end{pmatrix}
$$

(25)

where the blocks $S_{\Gamma\Gamma}^{(k)}$, $D_{k\Gamma}$, $\lambda_{k\Gamma}$, and $g_{k\Gamma}$ correspond to the degrees of freedom located on the interface Γ, $k = 1, 2$. System (22) can be written in the form of 5×5 block system

$$
\begin{pmatrix} S_{11} & S_{1\Gamma} & 0 & 0 & 0 \\ S_{\Gamma 1} & S_{\Gamma\Gamma}^{(1)} & 0 & 0 & -D_{1\Gamma}^T \\ 0 & 0 & S_{22} & S_{2\Gamma} & 0 \\ 0 & 0 & S_{\Gamma 2} & S_{\Gamma\Gamma}^{(2)} & -D_{2\Gamma}^T \\ 0 & -D_{1\Gamma} & 0 & -D_{2\Gamma} & 0 \end{pmatrix} \begin{pmatrix} \lambda_{11} \\ \lambda_{1\Gamma} \\ \lambda_{22} \\ \lambda_{2\Gamma} \\ \phi \end{pmatrix} = \begin{pmatrix} g_{11} \\ g_{1\Gamma} \\ g_{22} \\ g_{2\Gamma} \\ 0 \end{pmatrix}.
$$

(26)

In this system $D_{1\Gamma}$ is the diagonal matrix. Then, excluding the subvectors $\lambda_{1\Gamma}$ and ϕ by the block Gauss elimination method we get the system

$$
R \begin{pmatrix} \lambda_{11} \\ \lambda_{22} \\ \lambda_{2\Gamma} \end{pmatrix} = \begin{pmatrix} g_{11} \\ g_{22} \\ g_{\Gamma} \end{pmatrix}
$$

(27)

with the symmetric positive definite matrix

$$
R = \begin{pmatrix} S_{11} & 0 & R_{1\Gamma}^T \\ 0 & S_{22} & S_{2\Gamma} \\ R_{1\Gamma} & S_{\Gamma 2} & R_{\Gamma\Gamma} \end{pmatrix}
$$

(28)

where

$$
\begin{aligned}
R_{\Gamma\Gamma} &= S_{\Gamma\Gamma}^{(2)} + D_{2\Gamma}^T D_{1\Gamma}^{-1} S_{\Gamma\Gamma}^{(1)} D_{1\Gamma}^{-1} D_{2\Gamma}, \\
R_{1\Gamma} &= -D_{2\Gamma}^T D_{1\Gamma}^{-1} S_{\Gamma 1}, \\
g_{\Gamma} &= g_{2\Gamma} - D_{2\Gamma}^T D_{1\Gamma}^{-1} g_{1\Gamma}.
\end{aligned}
$$

(29)

To solve the system (27) we can use iterative techniques developed for algebraic systems with symmetric positive definite matrices. We recall that for the mortar P_1 finite element methods the above explicit reduction is not applicable.

The preconditioned Lanczos method is a good candidate to solve the saddle point system (22). In Kuznetsov [1995] an efficient preconditioner was

proposed for the P_1 mortar element method. By coupling the ideas from Kuznetsov [1995], Kuznetsov and Wheeler [1995] with the new results from the recent publication by Kuznetsov [2003] we are able to derive efficient preconditioners for the system matrix in (22) as well. This is a topic for another publication.

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