Weighted Norm-Equivalences for Preconditioning

Karl Scherer

University of Bonn, Institut fuer Angewandte Mathematik

Summary. The theory of *multilevel methods* for solving Ritz-Galerkin equations arising from discretization of elliptic boundary value problems is by now well developed. There exists a variety of survey talks and books in this area (see e.g.Xu [1992],Yserentant [1993],Oswald [1994]). Among them the *additive methods* are based on a suitable decomposition of the underlying projection operator (thus including also domain decomposition methods). In particular there is a close connection with classical concepts in approximation theory via so- called Jackson and Bernstein inequalities. These provide norm equivalences with the bilinear form underlying the Ritz- Galerkin procedure and thus preconditioners for the arising stiffness matrix.

The size of the constants in this equivalence is crucial for the stability of the resulting iteration methods. In this note we establish *robust norm equivalences* with constants which are *independent* of the mesh size and depend only *weakly* on the ellipticity of the problem, including the case of strongly varying coefficients. Extensions to the case of coefficients with discontinuities are possible, see Scherer [2003/4]. In the case of piecewise constant coefficients on the initial coarse grid there exist already estimates of the condition numbers of BPX-type preconditioners independent of the coefficients (see Yserentant [1990], Bramble and J.Xu [1991]) however they depend still on the mesh size (of the finest level).

1 Introduction

Given coefficients $a_{i,k} \in L_\infty(\Omega), \Omega \subset \mathcal{R}^2$ consider the *bilinear form*

$$a(u,v) := \int_\Omega \sum_{i,k=1}^{2} (a_{i,k}(D_i u)(D_k v) \qquad \text{for } u,v \in H^1(\Omega) = W_2^1(\Omega). \tag{1}$$

Here $W_p^r(\Omega)$ denotes the usual *Sobolev space* with norm $(1 \le p < \infty)$

$$\|u\|_{r,p;\Omega} := \|u\|_{p,\Omega} + |u|_{r,p;\Omega}, \qquad |u|_{r,p;\Omega} := \sum_{|\alpha|=r} \|D^\alpha u\|_{p;\Omega}.$$

If $a(u,v)$ is coercive (or L strongly elliptic) the *Lax-Milgram-Theorem* states that the equation $Lu := \sum_{i,k} \partial_i(a_{i,k}\partial_k u) = f$ has a unique generalized solution u satisfying weakly Dirichlet boundary conditions, i.e. $u \in H_0^1(\Omega)$.

Let $\psi_1, \cdots, \psi_N$ be a basis of a finite-dimensional subspace $\mathcal{V}$ of $H_0^1(\Omega)$. The *Ritz-Galerkin-equations* compute an approximate solution $u_N \in \mathcal{V}$ by

$$a(u_N, \psi_k) = (f, \psi_k), \qquad u_N := \sum_{i=1}^{N} \alpha_i \psi_i, \qquad\qquad 1 \le k \le N, .$$

These equations are solved *iteratively* for $\nu = 0, 1, 2, \cdots$:

$$u_N^{(\nu+1)} = u_N^{(\nu)} - \omega\ \mathcal{C} r^{(\nu)}, \qquad r^{(\nu)} := \mathcal{A} u^{(\nu)} - \mathsf{b}, \qquad \mathsf{b} := \{(f, \psi_k)\}.$$

Here ω is a relaxation factor and the matrix $\mathcal{C}$ acts as a preconditioner for the *stiffness matrix* $\mathcal{A} := \Big(a(\psi_i, \psi_k)\Big)_{i,k}$. The speed of convergence of this iteration scheme is governed by the condition number $\kappa(\mathcal{C}\mathcal{A})$.

In the theory of additive multi-level- methods preconditioners for $\mathcal{A}$ have been constructed for which $\kappa(\mathcal{C}\mathcal{A}) = \kappa\ (\mathcal{C}^{1/2}\mathcal{A}\mathcal{C}^{1/2}) = \mathcal{O}(1)$, independent of the mesh-size of the underlying FE-space. Thereby the matrix $\mathcal{C}$ is derived via a norm equivalence with $a(u,u)$ and the size of $\kappa(\mathcal{C}\mathcal{A})$ depends on the equivalence constants.

2 Norm equivalences and Approximation processes

Given a hierarchical sequence of subspaces

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_J := \mathcal{V}\ \subset\ X := L_2(\Omega), \tag{2}$$

assume that there exist of bounded projections $P_j : \mathcal{V} \longrightarrow \mathcal{V}_j$ satisfying

$$\beta_0\ a(u,u) \le \|P_0\|_X^2 + \sum_{j=1}^{J} d_j\ \|P_j u - P_{j-1} u\|_X^2 \le \beta_1\ a(u,u) \tag{3}$$

with suitable coefficients $\{d_j\}$ and constants β_0, β_1 independent of d_j, $u \in \mathcal{V}$ or J. Define via $(u, Bu) := \|P_0\|_X^2 + \sum_{j=1}^{J} d_j\ \|P_j u - P_{j-1} u\|_X^2$ a positive definite operator B for $u \in \mathcal{V}$ and let $\mathcal{C}$ above be the matrix representing the inverse B^{-1}. It is well known that then $\kappa(\mathcal{C}\mathcal{A}) \le \beta_1/\beta_0$, showing that $\mathcal{C}$ is a suitable preconditioner. More generally one can chose the matrix $\mathcal{C}$ as the discrete analogue of an operator C which is spectrally equivalent to B^{-1}. The derivation of the norm equivalence (3) proceeds in a meanwhile standard manner (cf. Dahmen and Kunoth [1992], Bornemann and Yserentant [1993], Oswald [1994]):

1.Step: use the equivalence of $a(u,v)$ with a Sobolev-norm, i.e.

$$A_1\ a(u,u) \leq\ \|u\|^2_{1,2,\Omega}\ \leq A_2\ a(u,u), \qquad \forall u \in H^1_0(\Omega), \tag{4}$$

with positive constants A_1, A_2 (needed also in the Lax-Milgram theorem).

2.Step: describe the Sobolev-norm via the *K-functional* of J.Peetre

$$K(t,f;X,Z) := \inf_{g\in Z}(\|f-g\|_X + t|g|_Z), \qquad t>0, \quad f\in X.$$

for normed linear spaces X, Z with $Z \subset X$ and seminorm $|\cdot|_Z$ such that Z is complete under norm $\|\cdot\|_X + |\cdot|_Z$. In the K-method of interpolation theory (see Bennett-R.Sharpley [1988], chapter 5) one defines for any integer r and $0<\theta<r$:

$$\|f\|_{(L_2(\Omega),W^r_p(\Omega))_{\theta/r,p,q}}\ := \Big(\sum_{n=0}^{\infty}[2^{n\theta}K(2^{-n\theta},f;L_2(\Omega),W^r_p(\Omega))]^q\Big)^{1/q}. \tag{5}$$

There holds the equivalence with Besov seminorms (see Johnen and Scherer)

$$\|f\|_{(L_2(\Omega),W^r_p(\Omega))_{\theta/r,p,q}}\ \approx \|f\|_{\theta,p,q,\Omega} := \Big(\sum_{n=0}^{\infty}[2^{n\theta}\omega_r(2^{-n},f)_p]^q\Big)^{1/q}$$

where $\approx$ denotes equivalence up to constants not depending on f. Further the equivalence of special Besov- norms with (fractional) Sobolev norms is known (see Triebel [1992],p.9):

$$\|f\|_{\theta,2,2,\Omega} \approx \|f\|_{\theta,2;\Omega} \qquad \text{for}\ \ \theta>0. \tag{6}$$

3.Step: describe the *interpolation norms* created by the K-functional via *approximation processes* $\mathbf{V}$, i.e. sequences of linear bounded operators $\{V_j\}$ defined on a Banach space X satisfying $\lim_{n\to\infty} V_n f = f$ for all $f\in X$. Then define *approximation norms* describing certain rates of approximation by

$$\|f\|_{\theta,q;V} := \Big\{\sum_{n=0}^{\infty}[2^{n\theta}\|V_nf-f\|_X]^q\Big\}^{1/q}, \qquad \theta\geq 0,\ 1\leq q\leq\infty$$

and introduce Jackson- and Bernstein- inequalities:

Definition 1. *An approximation process satisfies a* Jackson-inequality *with respect to the pair X, Z and* order $\alpha>0$ *if there exists a constant C_V*

$$\|V_nf-f\|_X \leq C_V\ 2^{-\alpha n}\ |f|_Z, \qquad \forall f\in Z. \tag{7}$$

and a corresponding Bernstein-inequality *if there exists D_V such that*

$$V_nf\in Z, \qquad |V_nf|_Z \leq D_V\ 2^{\alpha n}\ \|f\|_X, \qquad \forall f\in X. \tag{8}$$

Under these assumptions it has been shown (see Butzer and Scherer [1968], Butzer and Scherer [1972]))

Theorem 1. *For the operator sequences V_n defining an approximation process and satisfying Jackson- and Bernstein-inequalities of order α for a pair X, Z there holds for all $\theta > 0$*

$$\Big\{\sum_{n=0}^{\infty}[2^{n\theta}\|V_n f - V_{n-1}f\|_X]^q\Big\}^{1/q} \approx ||f||_{\theta,q,V} \approx \Big\{\sum_{n=0}^{\infty}[2^{n\theta}K(2^{-n\alpha}, f; X, Z)]^q\Big\}^{1/q}$$

with equivalence constants only depending on α, θ, C_V, D_V and $\sup \|V_n\| < \infty$.

The upper bound in the second equivalence follows from the Jackson-type inequality. For the lower one uses the decomposition $f = \sum_{k=n+1}^{\infty} V_k f - V_{k-1} f$,

$$K(2^{-n\alpha}, V_k f - V_{k-1} f; X, Z) \;\leq\; \min\;(1, 2^{(k-n)\alpha})\|V_k f - V_{k-1} f\|_X \qquad (9)$$

and Hardy's inequalities to estimate the arising double sum (cf. below).

We consider now the case of uniformly bounded linear projections $V_j = P_j : \mathcal{V} \longrightarrow \mathcal{V}_j$ in (3). For later use we assume $\mathcal{V}_j \subset Z = H_0^2(\Omega), \alpha = q = 2$ and $\theta = 1$. Then we obtain in combination with (4), (5) and (6)

Corollary 1. *Given the elliptic bilinear form $a(u, u)$ in (1) suppose that the above projections satisfy Jackson- and Bernstein-inequalities of order 2 for the pair $L_2(\Omega), H_0^2(\Omega)$. Then there holds for $u \in \mathcal{V}$ the equivalence ($P_{-1}u := 0$)*

$$\sum_{j=1}^{J} 4^J \;\|P_j u - P_{j-1}u\|_{2,\Omega}^2 \approx \sum_{n=0}^{\infty}[2^n \; K(2^{-2n}, f; L_2(\Omega), H_0^2(\Omega))]^2 \approx a(u, u),$$

and the equivalence constants do not depend on the level J and u.

We apply this to the case of FE- spaces consisting of piecewise polynomial functions of degree k in (2) with respect to the sequence of triangulations

$$\mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_J := \mathcal{T}. \qquad (10)$$

The coarse initial triangulation $\mathcal{T}_0$ is adaptively refined by dividing each triangle either into 4 congruent triangles or halving it such that each triangle in $\mathcal{T}_k$ is geometrically similar to a triangle of $\mathcal{T}_0$.

Jackson- inequalities for projections into such spaces with respect to the pair $X = L_2(\Omega), Z = H_0^{k+1}(\Omega)$ of order $k + 1$ are well known (cf. Ciarlet [1978]) whereas corresponding Bernstein-inequalities are only possible of maximal order k. However the modified inequality (9) can be proved (in case of maximal smoothness of u) with order $\alpha = k + 1/2$ which suffices for a proof of Theorem 1. Such an inequality follows from a corresponding one for the L_p modulus of continuity $\omega_k(t, f)_p$ (see Oswald [1994]) and the equivalence (see Johnen and Scherer):

$$K(t^k, f; L_p(\Omega), W_p^k(\Omega)) \;\approx\; \omega_k(t, f)_p.$$

3 Weighted norm equivalences

The disadvantage of above approach is that the "equivalence constants " in (3) depend on the ellipticity constants A_1, A_2 in (4). In the following we want to study *robust* norm equivalences, i.e. how they depend on these constants. (In the following constants C will only depend on the initial coarse triangulation $\mathcal{T}_0$). Thereby we restrict us to subspaces $\mathcal{V}_j$ consisting of piecewise linear functions. The basic idea is to introduce for the triangulations (10)

Assumption A: There exist weights $\underline{\omega}_i, \overline{\omega}_i$ such that the bilinear form (1) satisfies on the triangles Z_i of $\mathcal{T}_J$ the ellipticity condition

$$\underline{\omega}_i \sum_{\nu=1}^{2} \xi_\nu^2 \leq \sum_{\nu,\mu=1}^{2} a_{\nu,\mu}(x)\xi_\nu\xi_\mu \leq \overline{\omega}_i \sum_{\nu=1}^{2} \xi_\nu^2, \qquad \text{for all} \quad x \in Z_i. \tag{11}$$

Then one wants to establish Jackson- and Bernstein- inequalities for suitable projections P_j in (3) with respect to a "weighted norm" arising from this assumption. This will be described shortly in the following (for more details see Scherer [2003/4]). Ideally one should take for P_j the Ritz projection Q_j^a defined on $\mathcal{V}$ by

$$a(Q_j^a u, v) = a(u, v), \qquad u \in \mathcal{V}, \quad v \in \mathcal{V}_j \tag{12}$$

since then $v_j := Q_j^a u - Q_{j-1}^a u$ with $Q_{-1}^a u := 0$ satisfies $a(u,u) := ||u||_a^2 = \sum_{j=0}^{J} ||v_j||_a^2$. Then the idea is to replace Q_j^a by projections Q_j^ω with respect to a *weighted norm* (from now on we omit subscript and superscript on ω):

$$(Q_j^\omega u, v)_\omega \;=\; (u,v)_\omega := \sum_{Z_i \in \mathcal{T}_J} \omega_i \int_{Z_i} u \cdot v \, dx, \qquad v \in \mathcal{V}_j.$$

Essential for our analysis are also the *average weights* $\omega_T := \frac{1}{\mu(T)} \sum_{Z_i \subset T} \mu(Z_i)\omega_i$ with corresponding *weighted norms*

$$\|v\|_{j,\omega}^2 := \sum_{T \in \mathcal{T}_j} \omega_T \int_T |v|^2$$

At first two *Bernstein-type inequalities of order 1/2* are proved. To this end we assume a continous weight $\omega(x)$ in Assumption A and work with average weights $\omega_i^* := \frac{1}{\mu(Z_i)} \int_{Z_i} \omega$ as well as corresponding ones ω_T^* for $T \in \mathcal{T}_j$.

Lemma 1. *Define the semi-norm*

$$\|u\|_{1/2,\omega,l} := \Big(\sum_{T \in \mathcal{T}_l} \omega_T^* \int_{\partial T} |u|^2 \Big)^{1/2}$$

for $u \in \mathcal{V}_J \subset H_0^1(\Omega)$. Then there holds for $v_j := Q_j^a u - Q_{j-1}^a u$ and any $w \in \mathcal{V}_j$

$$\|Q^a_j u - Q^a_{j-1}u\|_a \le E^j_\omega \; \|u-w\|_a \; + \; C \; 2^{j/2} \; \|u\|_{1/2,\omega,j}, \tag{13}$$

where $E^j_\omega := \max_{T\in\mathcal{T}_j} \max_{x,y\in T} |[\omega(x)-\omega(y)]/\omega(y)|$ *is a modulus of continuity of* $\omega(x)$.

Lemma 2. *There holds for any* $u \in \mathcal{V}_k$ *and* $k \ge j$

$$\|u\|_{1/2,\omega,j} \le C \; 2^{k/2} \; \|u\|^*_{k,j}, \qquad \|u_k\|^*_{k,j} := \Big(\sum_{U\in\mathcal{T}_j} \omega^*_U \int_{S_k(U)} |u_k|^2\Big)^{1/2}.$$

Here $S_k(U)$ *denotes the strip along* ∂U *consisting of triangles* $T \in \mathcal{T}_k$.

We apply the first lemma to each term $\|v_j\|^2_a$ with $j \ge j_0$ for some $j_0 \ge 1$ to be chosen later, take $u - Q^\omega_{j-1}u$ instead of u, $w = Q^a_{j-1}u - Q^\omega_{j-1}u$ and obtain

$$\sum_{j=j_0}^{J} \|v_j\|^2_a \, \Big(1 \; - \; 2 \sum_{j=j_0}^{J} \big(E^j_\omega\big)^2\Big) \le 2C \sum_{j=j_0}^{J} 2^j \Big(\sum_{k=j}^{J} \|u_k\|_{1/2,\omega,j}\Big)^2 \; . \tag{14}$$

Next we use the decomposition of $u - Q^\omega_{j-1}u = \sum_{k=j}^J u_k$, with $u_k := Q^\omega_k u - Q^\omega_{k-1}u$ and apply the second Bernstein-type inequality. After inserting this into the right hand side of (14) we obtain a double sum which is estimated with a refined version of Hardy's inequality giving

$$\sum_{j=0}^{J} 2^j \, \Big(\sum_{k=j}^{J} 2^{k/2} \; \|u_k\|^*_{k,j}\Big)^2 \Big[1 \; - \; 2E^j_\omega\Big] \le 4 \sum_{j=j_0}^{J} \; 4^j \; \|u_j\|^2_{j,\omega} \tag{15}$$

In addition, by an other application of Lemma 1, one can obtain a bound for the remaining sum $\sum_{j=0}^{j_0-1} \|v_j\|^2_a = \|Q^a_{j_0}\|^2_a$. The final estimate is then

$$a(u,u) \; \le \; C \left\{ \frac{1 + 3(E^{j_0}_\omega)^2}{[1-2E^{j_0}_\omega]\Big[1 - 2\sum_{j=j_0}^{J} \big(E^j_\omega\big)^2\Big]} \sum_{j=j_0}^{J} 4^j \|u_j\|^2_{j,\omega} \right\} + \|Q^\omega_{j_0-1}u\|^2_a.$$

Theorem 2. *If there exists* $j_0 \ge 1$ *such that* $\sum_{j=j_0}^{J} \big(E^j_\omega\big)^2 \le 1/4$ *there holds*

$$a(u,u) \le C \sum_{j=j_0}^{J} 4^j \; \|Q^\omega_j u - Q^\omega_{j-1}u\|^2_{j,\omega} + a(Q^\omega_{j_0-1}u, Q^\omega_{j_0-1}u).$$

Remarks: The assumption of the theorem can be fulfilled for continuous ω. In case $\omega \in C^1$ or $\omega \in C^\alpha$ a more quantitative description can be given, e.g. for $\omega(x) := \exp\{q(x)\}$ we have $E^j_\omega \le c2^{-j}\|\nabla q\|_\infty \; \exp\{c2^{-j}\|\nabla q\|_\infty\}$. The continuity of ω also justifies the choice $\underline{\omega}_i = \overline{\omega}_i = \omega^*_i$ in (11) since then $a(v,v)$ is norm-equivalent to $\tilde{a}(v,v) := \sum_{Z_i\in\mathcal{T}_J} \omega^*_i \int_{Z_i} \|\nabla v\|^2$. Finally we remark that the above argument can be extended also to the case of a weight function $\omega(x)$ which is continous on Ω up to a (smooth) curve. If this curve coincides with the edges of the initial coarse grid the above argument can be applied to each of the two subregions separately with corresponding moduli of continuity. But also the more general case can be treated (see Scherer [2003/4]).

We turn now to the problem of a lower bound for $a(u,u)$ (more details can be found in Scherer [2003/4]). At first observe that by Hardy's inequality

$$\sum_{j=1}^{J}\Big[2^j\|u_j\|_\omega\Big]^2 \le \sum_{j=0}^{J}\Big[2^j\|Q_j^a u-u\|_\omega\Big]^2 \le 4\sum_{j=1}^{J}\Big(2^j\|Q_j^a u-Q_{j-1}^a u\|_\omega\Big)^2.$$

In order to pass from $\|\cdot\|_\omega$ -norm to $\|\cdot\|_{j,\omega}$ - norm use

Theorem 3. *Suppose that there exists a constant* $\gamma\in(0,1]$ *such that for any* $T\in\mathcal{T}_j$ *the weights* ω_i *satisfy*

$$\omega_E\ /\ \omega_T \le \mu(T)\ /\ 2\mu(E), \qquad \forall\ E\subset T \ \textit{with}\ \mu(E)\le\gamma\ \mu(T)\ . \tag{16}$$

Then there holds for $v\in\mathcal{V}_j$

$$[1+\ C\ \gamma^{-1}]^{-1}\ \|v\|_{j,\omega;T}\le\ \|v\|_{\omega;T}\le\ [1+\ \sqrt{6}\ C]\ \|v\|_{j,\omega;T}. \tag{17}$$

Application of this theorem gives

$$\sum_{j=1}^{J}\Big[2^j\|Q_j^\omega u-Q_{j-1}^\omega u\|_\omega\Big]^2 \le\ C\ \sum_{j=1}^{J}\Big(2^j\|Q_j^a u-Q_{j-1}^a u\|_{j,\omega}\Big)^2\ . \tag{18}$$

The crucial step is the following local estimate

Theorem 4. *Let be* U *be the support of a nodal function in* $\mathcal{V}_{j-1}$, $\psi_l^{(j-1)}$ *say. Then there holds (* $S,S'\in\mathcal{T}_j$ *)*

$$\|Q_j^a u-Q_{j-1}^a u\|_{j,\omega,U}\le C\left(\max_{S',S\subset U}\sqrt{\frac{\omega_S}{\omega_{S'}}}\right)\ 2^{-j}\ \|\nabla(Q_j^a u-Q_{j-1}^a u)\|_{j,\omega,U}.$$

Sketch of the proof: Using the duality technique of Aubin-Nitsche one has

$$\|v_j\|_{j,\omega,U}=\sup_{g\in L_\omega(U)}\frac{|(g,\omega\cdot v_j)_U|}{\|g\|_{j,\omega,U}}=\sup_{g\in L_\omega(U)}\frac{|(-\Delta\varphi_g,\omega\cdot v_j)_U|}{\|g\|_{j,\omega,U}}.$$

where $-\Delta\varphi_g=\tilde g$ on $\tilde U\ge U$, $\varphi_g|_{\partial\tilde U}=0$ and $\|\tilde g\|_{\tilde U}\le C\|g\|_U$ with some absolute constant $\tilde C$. Partial integration on each $S\subset U$ gives

$$|(-\Delta\varphi_g,\omega\cdot v_j)_U|\le|\sum_{S\subset U}\omega_S\int_S(\nabla\varphi_g,\nabla v_j)|+|\sum_{S\subset U}\omega_S\int_{\partial S}v_j(\nabla(\varphi_g-v),n_{\partial S})|$$

The bound for the first term uses $\|\nabla\varphi_g\|_{\tilde U}\le C\ \sqrt{\mu(\tilde U)}\|g\|_U$. In the second supremum one chooses $v=v^*\in\mathcal{V}_{j-1}$ with supp $v^*\subset U$ as interpolant of φ_g. Then (cf. Ciarlet [1978])

$$\|\nabla(\varphi_g - v^*)\|_{\infty,T} \leq \ C \text{ diam } T \sum_{|\alpha|=3} \|D^\alpha \varphi_g\|_T, \qquad T \in \mathcal{T}_{j-1},$$

and by the special choice $g = g^* := v_j$ the estimate of the theorem follows.

Summing with respect to U and inserting the result into (18) yields

Theorem 5. *Under assumption A and for uniformly refined triangulations there holds for a(u,u) the lower bound*

$$\sum_{j=1}^{J} \Big[2^j \|Q_j^\omega u - Q_{j-1}^\omega u\|_{j,\omega}\Big]^2 \leq C \left(\max_{1\leq j\leq J} \max_{U\in\mathcal{T}_{j-1}} \max_{S',S\subset U} \sqrt{\frac{\omega_S}{\omega_{S'}}} \right) a(u,u)$$

with C depending only on the shape of the triangles of the initial triangulation.

4 Application to preconditioning

Theorems 2 and 5 can be combined via Theorem 3 to the following

Theorem 6. *Under assumption A assume further that $\omega(x)$ satisfies the assumptions of Theorems 2 and 3. Then for uniformly refined triangulations there holds for $a(u,u)$ and j_0 depending on $\omega(x)$*

$$C_1 \ a(u,u) \ \leq \ a(u_{j_0}, u_{j_0}) + \sum_{j=1}^{J} \Big[2^j \|Q_j^\omega u - Q_{j-1}^\omega u\|_\omega\Big]^2 \leq C_2 \ C_\omega \ a(u,u)$$

where $C_\omega := \max_{1\leq j\leq J} \max_{U\in\mathcal{T}_{j-1}} \max_{S',S\subset U} \sqrt{\omega_S}/\sqrt{\omega_{S'}}$ and C_1, C_2 independent of J and $\omega(x)$.

It seems impossible to dispense with any condition starting from Assumption A. The most restrictive conditions appear in Theorem 2 where the choice of j_0 depends on the decrease of the moduli of continuity E_ω^j in j. This has been discussed in the remarks following it. A discussion of the further condition (16) in Theorem 3 is given in Scherer [2003/4]. The weakest one is probably that of Theorem 5 which requires the boundedness of the constant there. The case of non-uniformly refined meshes can be reduced to that one of uniformly refined meshes by the technique in Bornemann and Yserentant [1993].

For preconditioning we can proceed as in the BPX approach (cf.Bramble et al. [1990]) taking

$$B^{-1} = (Q_{j_0}^\omega)^{-1} + \sum_{j=j_0+1}^{J} 4^{-j} \ (Q_j^\omega \ - \ Q_{j-1}^\omega) \approx (Q_{j_0}^\omega)^{-1} \ + \sum_{j=1}^{J} 4^{-j} \ Q_j^\omega .$$

According to Yserentant [1990] this operator can be replaced by the cheaper one

$$\mathsf{C}\ r := (Q^{\omega}_{j_0})^{-1}\ r + \sum_{j=1}^{J} 4^{-j}\ M_j r, \qquad M_j v := \sum_{i\in\mathcal{N}_j} \frac{(v,\psi_i^{(j)})_\omega}{(1,\psi_i^{(j)})_\omega}\psi_i^{(j)}$$

provided the 'quasi-interpolant' $M_j v$ is spectrally equivalent to Q_j^ω uniformly in j. This property holds if the $\psi_l^{(j)}$ form a *Riesz-basis* with respect to the weighted norm, i.e.

$$\|\sum_{l\in\mathcal{N}_j} \alpha_l \psi_l^{(j)}\|_\omega^2 \approx \sum_{l\in\mathcal{N}_j} |\alpha_l|^2 (1,\psi_l^{(j)})_\omega$$

The proof follows from the norm equivalence $\|\cdot\|_\omega \approx \|\cdot\|_{\omega,j}$ stated in (17). Then the $\mathcal{C}$ can be taken as a discretized version of the operator C above.

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