

# Physics-based Preconditioners and Optimization

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# Recall Newton methods

- Given  $F(u) = 0$ ,  $F : \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  and iterate  $u^0$  we wish to pick  $u^{k+1}$  such that

$$F(u^{k+1}) \approx F(u^k) + F'(u^k) \mathbf{d}u^k = 0$$

where  $\mathbf{d}u^k = u^{k+1} - u^k$ ,  $k = 0, 1, 2, \dots$

- Neglecting higher-order terms, we get

$$\mathbf{d}u^k = -[J(u^k)]^{-1} F(u^k)$$

where  $J = F'(u^k)$  is the Jacobian matrix, generally large, sparse, and ill-conditioned for PDEs

- In practice, require  $\| F(u^k) + J(u^k) \mathbf{d}u^k \| < \mathbf{e}$
- In practice, set  $u^{k+1} = u^k + \mathbf{l} \mathbf{d}u^k$  where  $\mathbf{l}$  is selected to minimize  $\| F(u^k + \mathbf{l} \mathbf{d}u^k) \|$



# Jacobian-free Newton-Krylov

- In the Jacobian-Free Newton-Krylov (JFNK) method, a Krylov method solves the linear Newton correction equation, requiring Jacobian-vector products
- These are approximated by the Fréchet derivatives

$$J(u)v \approx \frac{1}{\mathbf{e}} [F(u + \mathbf{e}v) - F(u)]$$

(where  $\mathbf{e}$  is chosen with a fine balance between approximation and floating point rounding error) or automatic differentiation, so that the actual Jacobian elements are *never explicitly needed*

- One builds the Krylov space on a true  $F'(u)$  (to within numerical approximation)



# Recall idea of preconditioning

- Krylov iteration is expensive in memory and in function evaluations, so  $k$  must be kept small in practice, through preconditioning the Jacobian with an approximate inverse, so that the product matrix has low condition number in

$$(B^{-1}A)x = B^{-1}b$$

- Given the ability to apply the action of  $B^{-1}$  to a vector, preconditioning can be done on either the left, as above, or the right, as in, e.g., for matrix-free:

$$JB^{-1}v \approx \frac{1}{e} [F(u + eB^{-1}v) - F(u)]$$



# Philosophy of Jacobian-free NK

- To *evaluate* the linear residual, we use the true  $F'(u)$ , giving a true Newton step and asymptotic quadratic Newton convergence
- To *precondition* the linear residual, we do anything convenient that uses understanding of the dominant physics/mathematics in the system and respects the limitations of the parallel computer architecture and the cost of various operations:
  - **Jacobian of lower-order discretization**
  - **Jacobian with “lagged” values for expensive terms**
  - **Jacobian stored in lower precision**
  - **Jacobian blocks decomposed for parallelism**
  - **Jacobian of related discretization**
  - **operator-split Jacobians**
  - **physics-based preconditioning**



# Using Jacobian of related discretization

- To precondition a variable coefficient operator, such as  $\nabla \cdot (\mathbf{a} \nabla \bullet)$ , use  $\overline{\mathbf{a}} \nabla^2$ , based on a constant coefficient average
- Brown & Saad (1980) showed that, because of the availability of fast solvers, it may even be acceptable to use  $-\nabla^2$  to precondition something like

$$-\nabla^2(\bullet) + u \frac{\partial(\bullet)}{\partial x} + v \frac{\partial(\bullet)}{\partial y}$$



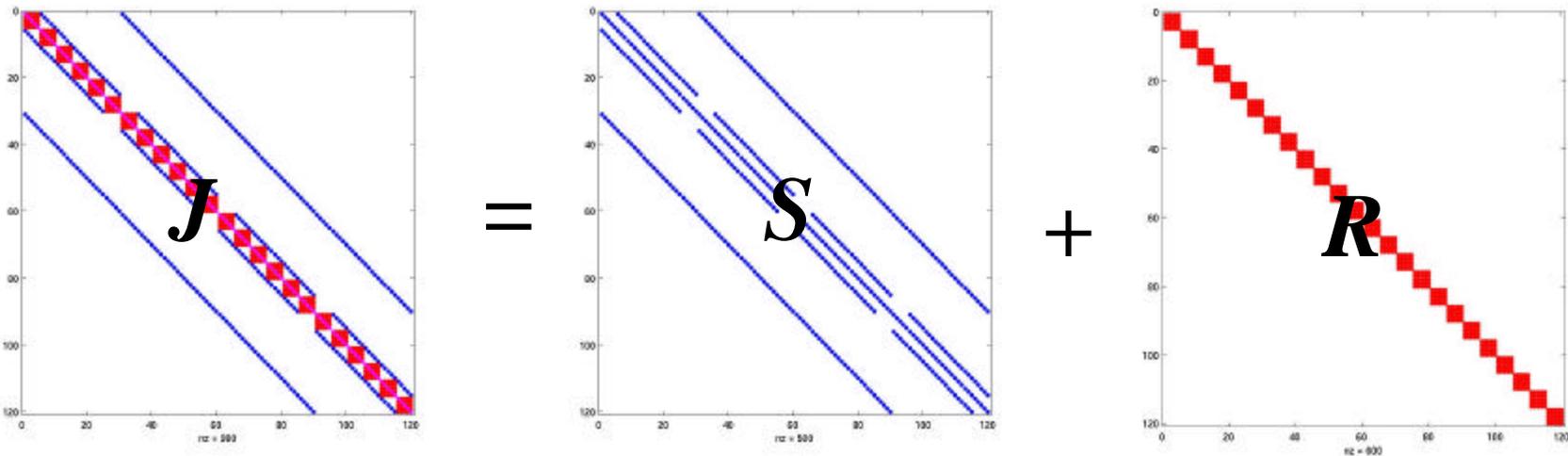
# Operator-split preconditioning

- Subcomponents of a PDE operator often have special structure that can be exploited if they are treated separately
- Algebraically, this is just a generalization of Schwarz, by term instead of by subdomain
- Suppose  $J = \boxed{t^{-1}I + S + R}$  and a preconditioner is to be constructed, where  $I + tS$  and  $I + tR$  are each “easy” to invert
- Form a preconditioned vector from  $u$  as follows:
$$(t^{-1}I + R)^{-1} (I + tS)^{-1} u$$
- Equivalent to replacing  $J$  with  $\boxed{t^{-1}I + S + R} + tSR$
- First-order splitting error, yet *often used as a solver!*



# Operator-split preconditioning, cont.

- Suppose  $S$  is convection-diffusion and  $R$  is reaction, among a collection of fields stored as gridfunctions
- On a small regular 2D grid with a five-point stencil:



- $R$  is trivially invertible in block diagonal form
- $S$  is invertible with one multilevel solve per field



# Operator-split preconditioning, cont.

- **Preconditioners assembled from just the “strong” elements of the Jacobian, alternating the source term and the diffusion term operators, are competitive in convergence rates with block-ILU on the Jacobian**
  - **particularly, since the decoupled scalar diffusion systems are amenable to simple multigrid treatment – not as trivial for the coupled system**
- **The decoupled preconditioners store many fewer elements and significantly reduce memory bandwidth requirements and are expected to be much faster per iteration when carefully implemented**
- **See “alternative block factorization” by Bank et al.; incorporated into SciDAC TSI solver by D’Azevedo**



# Physics-based preconditioning

- In Newton iteration, one seeks to obtain a correction (“delta”) to solution, by inverting the Jacobian matrix on (the negative of) the nonlinear residual:

$$\mathbf{d}u^k = -[J(u^k)]^{-1} F(u^k)$$

- A typical operator-split code also derives a “delta” to the solution, by some implicitly defined means, through a series of implicit and explicit substeps

$$F(u^k) \mapsto \mathbf{d}u^k$$

- This implicitly defined mapping from residual to “delta” is a *natural* preconditioner
- Software must accommodate this!



# Physics-based Preconditioning

- We consider a standard “dynamical core,” the shallow-water wave splitting algorithm, *as a solver*
- Leaves a first-order in time splitting error
- In the Jacobian-free Newton-Krylov framework, this solver, which maps a residual into a correction, can be regarded *as a preconditioner*
- The true Jacobian is never formed yet the time-implicit nonlinear residual at each time step can be made as small as needed for nonlinear consistency in long time integrations



# Example: shallow water equations

- **Continuity (\*)** 
$$\frac{\partial}{\partial t} \times \left( \frac{\partial \mathbf{f}}{\partial t} + \frac{\partial (u \mathbf{f})}{\partial x} \right) = 0$$

- **Momentum (\*\*)** 
$$\frac{\partial}{\partial x} \times \left( \frac{\partial (u \mathbf{f})}{\partial t} + \frac{\partial (u^2 \mathbf{f})}{\partial x} + g \mathbf{f} \frac{\partial \mathbf{f}}{\partial x} \right) = 0$$

- **These equations admit a *fast gravity wave*, as can be seen by cross differentiating, e.g., (\*) by  $t$  and (\*\*) by  $x$ , and subtracting:**

$$\frac{\partial^2 \mathbf{f}}{\partial t^2} - g \mathbf{f} \frac{\partial^2 \mathbf{f}}{\partial x^2} = \text{other terms}$$



# 1D shallow water equations, cont.

- **Wave equation for geopotential:**

$$\frac{\partial^2 \mathbf{f}}{\partial t^2} - g \mathbf{f} \frac{\partial^2 \mathbf{f}}{\partial x^2} = \text{other terms}$$

- **Gravity wave speed  $\sqrt{g\mathbf{f}}$**
- **Typically  $\sqrt{g\mathbf{f}} \gg u$ , but stability restrictions would require timesteps based on the Courant-Friedrichs-Levy (CFL) criterion for the fastest wave, for an explicit method**
- **One can solve *fully implicitly*, or one can filter out the gravity wave by solving *semi-implicitly***



# 1D shallow water equations, cont.

- Continuity (\*)

$$\frac{\mathbf{f}^{n+1} - \mathbf{f}^n}{t} + \frac{\partial (u \mathbf{f})^{n+1}}{\partial x} = 0$$

- Momentum (\*\*)

$$\frac{(u \mathbf{f})^{n+1} - (u \mathbf{f})^n}{t} + \frac{\partial (u^2 \mathbf{f})^n}{\partial x} + g \mathbf{f}^n \frac{\partial \mathbf{f}^{n+1}}{\partial x} = 0$$

- Solving (\*\*) for  $(u \mathbf{f})^{n+1}$  and substituting into (\*),

$$\mathbf{f}^{n+1} - g t^2 \frac{\partial}{\partial x} \left( \mathbf{f}^n \frac{\partial \mathbf{f}^{n+1}}{\partial x} \right) = \mathbf{f}^n + \frac{\partial S^n}{\partial x}$$

where

$$S^n = (u \mathbf{f})^n - t \frac{\partial (u^2 \mathbf{f})^n}{\partial x}$$



# 1D shallow water equations, cont.

- After the parabolic equation is spatially discretized and solved for  $\mathbf{f}^{n+1}$ , then  $(u \mathbf{f})^{n+1}$  can be found from

$$(u \mathbf{f})^{n+1} = -t g \mathbf{f}^n \frac{\partial \mathbf{f}^{n+1}}{\partial x} + S^n$$

- *One scalar parabolic solve and one scalar explicit update* replace an *implicit hyperbolic system*
  - This *semi-implicit* operator splitting is foundational to multiple scales problems in geophysical modeling
  - Similar tricks are employed in aerodynamics (*sound waves*), MHD (*multiple Alfvén waves*), reacting flows (*fast kinetics*), etc.
  - Temporal truncation error remains due to the lagging of the advection in (\*\*)
- ← To be dealt with shortly



# 1D Shallow water preconditioning

- Define continuity residual for each timestep:

$$R_{\mathbf{f}} \equiv \frac{\boxed{\mathbf{f}^{n+1}} - \mathbf{f}^n}{t} + \frac{\partial \boxed{(u \mathbf{f})^{n+1}}}{\partial x}$$

- Define momentum residual for each timestep:

$$R_{u \mathbf{f}} \equiv \frac{\boxed{(u \mathbf{f})^{n+1}} - (u \mathbf{f})^n}{t} + \frac{\partial (u^2 \mathbf{f})^n}{\partial x} + g \mathbf{f}^n \frac{\partial \boxed{\mathbf{f}^{n+1}}}{\partial x}$$

- Continuity delta-form (\*):

$$\frac{\boxed{d\mathbf{f}}}{t} + \frac{\partial \boxed{d(u \mathbf{f})}}{\partial x} = -R_{\mathbf{f}}$$

- Momentum delta form (\*\*):

$$\frac{\boxed{d(u \mathbf{f})}}{t} + g \mathbf{f}^n \frac{\partial \boxed{d\mathbf{f}}}{\partial x} = -R_{u \mathbf{f}}$$



# 1D Shallow water preconditioning, cont.

- Solving (\*\*) for  $\mathbf{d}(u\mathbf{f})$  and substituting into (\*),

$$d\mathbf{f} - g t^2 \frac{\partial}{\partial x} \left( \mathbf{f}^n \frac{\partial [d\mathbf{f}]}{\partial x} \right) = -R_{-}\mathbf{f} + t^2 \frac{\partial}{\partial x} (R_{-}u\mathbf{f})$$

- After this parabolic equation is solved for  $d\mathbf{f}$ , we have

$$\mathbf{d}(u\mathbf{f}) = -t g \mathbf{f}^n \frac{\partial [d\mathbf{f}]}{\partial x} - R_{-}u\mathbf{f}$$

- This completes the application of the preconditioner to one Newton-Krylov iteration at one timestep
- Of course, the parabolic solve need not be done exactly; one sweep of multigrid can be used
- See paper by Mousseau et al. (2002) for impressive results for longtime weather integration



# Physics-based preconditioning update

- So far, physics-based preconditioning has been applied to several codes at Los Alamos, in an effort led by D. Knoll
- Summarized in new J. Comp. Phys. paper by Knoll & Keyes (2002, under review)
- PETSc's "shell preconditioner" is ideal for inserting physics-based preconditioners, and PETSc's solvers underneath are ideal building blocks, but there are not yet examples in the public release



# SciDAC philosophy on PDEs

- **Solution of a system of PDEs is rarely a goal in itself**
  - **PDEs are solved to derive various outputs from specified inputs**
  - **actual goal is characterization of a response surface or a design or control strategy**
  - **together with analysis, sensitivities and stability are often desired**

**⇒ Tools for PDE solution should also support these related desires**



# PDE-constrained optimization

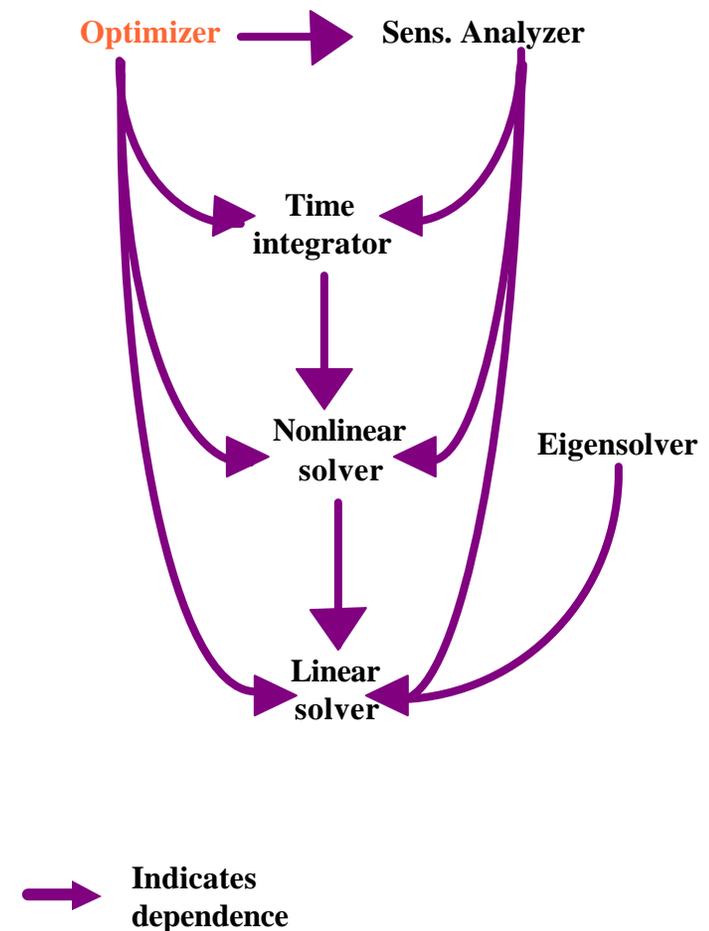
- **PDE-constrained optimization: a relatively new horizon**
  - **... for large-scale PDE solution**
    - ◆ next step after reducing to practice parallel implicit solvers for coupled systems of (steady-state) PDEs
    - ◆ now “routine” to solve systems of PDEs with millions of DOFs on thousands of processors
  - **... for constrained optimization**
    - ◆ complexity of a single projection to the constraint manifold for million-DOF PDE is too expensive for an inner loop of traditional RSQP method
    - ◆ must devise new “all-at-once” algorithms that seek “exact” feasibility only at optimality
- **Our approach starts from the iterative PDE solver side**



# Optimizers

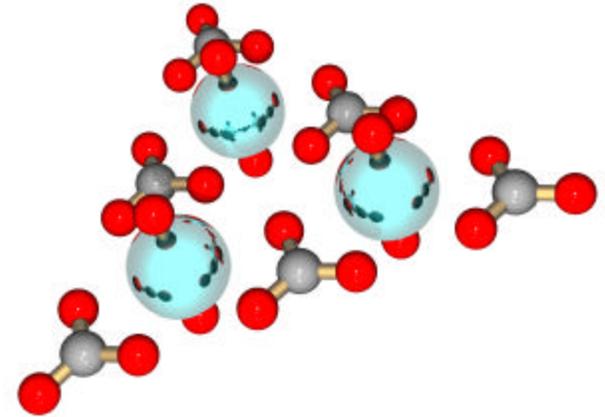
$$\min_u \mathbf{f}(x, u) \text{ s.t. } F(x, u) = 0, u \geq C$$

- Many simulations are properly posed as optimization problems, but this may not always be recognized
- Unconstrained or bound-constrained applications use **TAO**
- PDE-constrained problems use **Veltisto**
- Both are built on **PETSc** solvers (and **Hypre** preconditioners)
- **TAO** makes heavy use of **AD**, freeing user from much coding
- **Veltisto**, based on **RSQP**, switches as soon as possible to an “all-at-once” method and minimizes the number of PDE solution “work units”



# Recent optimization progress

- **Unconstrained or bound-constrained optimization**
  - **TAO-PETSc** used in quantum chemistry energy minimization
- **PDE-constrained optimization**
  - **Veltisto-PETSc** used in flow control application, to straighten out wingtip vortex by wing surface blowing and suction; performs full optimization in the time of just five N-S solves
- **“Best technical paper” at SC2002** went to our SciDAC colleagues at CMU (Ghattas, PI, speaking here Thursday morning):
  - Inverse wave propagation employed to infer hidden geometry

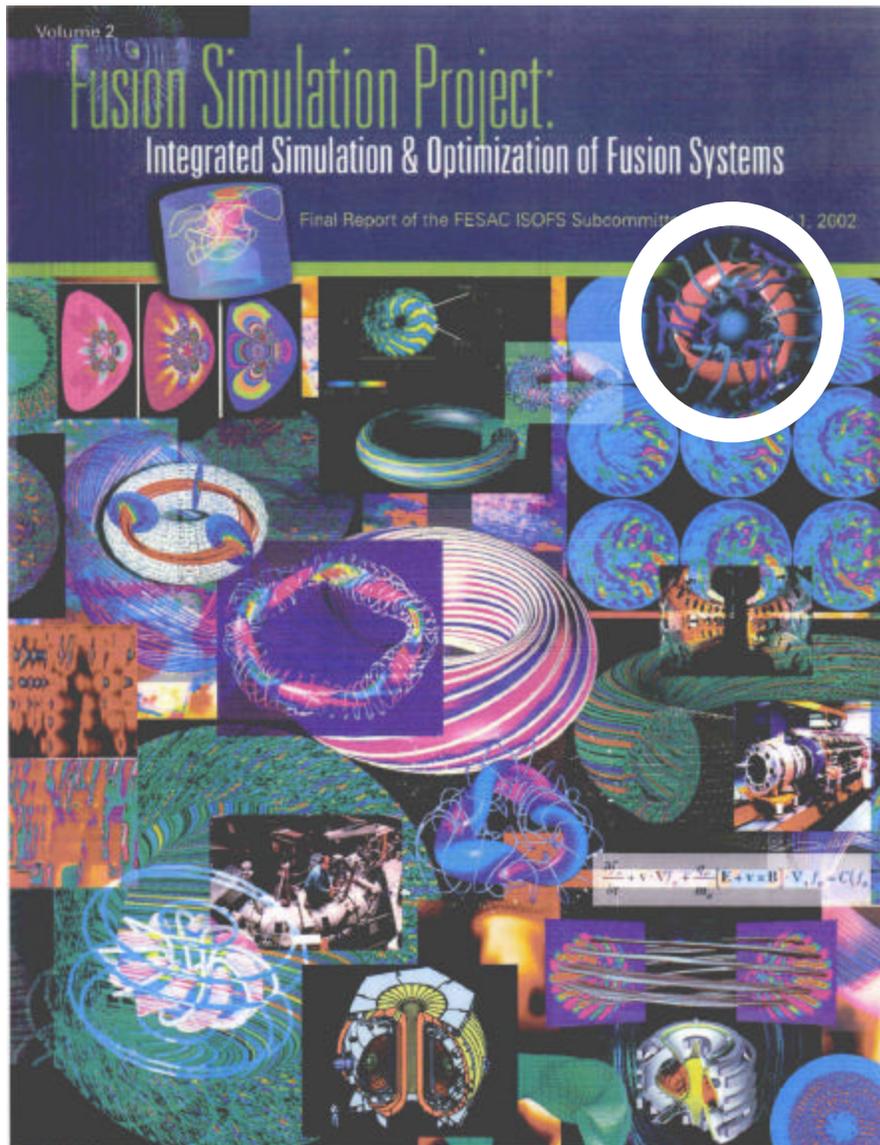


# Motivating examples for simulation-based optimization

- **Stellarator design**
- **Materials design and molecular structure determination**
- **Source inversion**



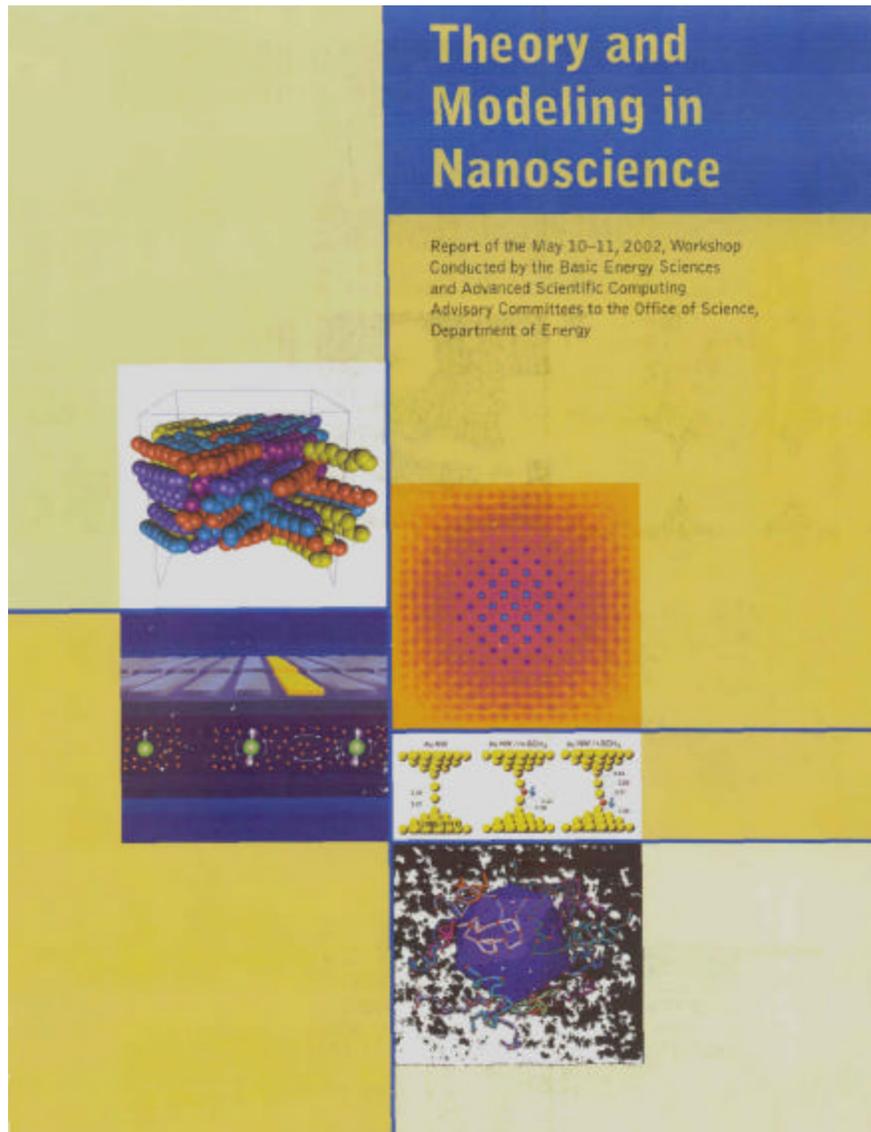
# Stellarator design



- Dec 2002 report to DOE
- Multiphysics simulation for shape optimization of magnetic confinement fusion devices featured as a *key technology* for fusion energy
- Currently genetic algorithms used to optimize single-physics subsystems, e.g., magnetic flux surfaces of the plasma, magnetic coil shape of the controls – many analysis runs



# Molecular-level materials design



- Jul 2002 report to DOE
- Optimization methods frequently invoked by nanoscientists as a *pressing need* – though there is much work ahead to make this concrete mathematically, except for ...
- ... Multilevel optimization for energy minimization in molecular structure determination – well along in protein folding, etc.



# Source inversion

## SAND REPORT

SAND2002-3198  
Unlimited Release  
October 2002

### Large Scale Non-Linear Programming for PDE Constrained Optimization.

Bart van Bloemen Waanders, Roscoe Bartlett, Kevin Long, Paul Boggs,  
Andrew Salinger  
Sandia National Laboratories

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- **Oct 2002 Sandia LDRD final report**
- **Model source inversion problem solved by multilevel optimization, using Sundance/rSQP++**
- **Simultaneous Analysis and Design (SAND) framework exploited to save an order of magnitude in execution time, relative to blackbox methods**



# Constrained optimization w/Lagrangian

- Consider Newton's method for solving the nonlinear rootfinding problem derived from the necessary conditions for constrained optimization
- **Constraint**  $c(x, u) = 0 ; x \in \mathfrak{R}^N ; u \in \mathfrak{R}^M ; c \in \mathfrak{R}^N$
- **Objective**  $\min_u f(x, u) ; f \in \mathfrak{R}$
- **Lagrangian**  $f(x, u) + \mathbf{I}^T c(x, u) ; \mathbf{I} \in \mathfrak{R}^N$
- Form the gradient of the Lagrangian with respect to each of  $x, u$ , and  $\lambda$ :

$$f_x(x, u) + \mathbf{I}^T c_x(x, u) = 0$$

$$f_u(x, u) + \mathbf{I}^T c_u(x, u) = 0$$

$$c(x, u) = 0$$



# Newton on first-order conditions

- Equality constrained optimization leads to the KKT system for states  $x$ , designs  $u$ , and multipliers  $l$

$$\begin{bmatrix} W_{xx} & W_{ux}^T & J_x^T \\ W_{ux} & W_{uu} & J_u^T \\ J_x & J_u & 0 \end{bmatrix} \begin{bmatrix} dx \\ du \\ dl \end{bmatrix} = - \begin{bmatrix} g_x \\ g_u \\ c \end{bmatrix}$$

- Newton Reduced SQP solves the Schur complement system  $H du = g$ , where  $H$  is the reduced Hessian

$$H = W_{uu} - J_u^T J_x^{-T} W_{ux}^T - (J_u^T J_x^{-T} W_{xx} - W_{ux}^T) J_x^{-1} J_u$$

$$g = -g_u + J_u^T J_x^{-T} g_x - (J_u^T J_x^{-T} W_{xx} - W_{ux}^T) J_x^{-1} c$$

- Then
 
$$J_x dx = -c - J_u du$$

$$J_x^T dl = -g_x - W_{xx} dx - W_{ux}^T du$$



# RSQP when constraints are PDEs

- **Problems**

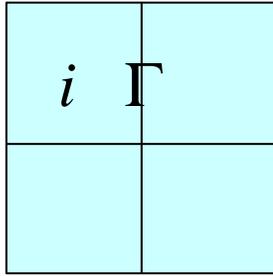
- ➔ ■  $J_x$  is the Jacobian of a PDE  $\mathcal{P}$  huge!
- ➔ ■  $W_{ab}$  involve Hessians of objective and constraints  $\mathcal{P}$  second derivatives and huge
- ➔ ■  $H$  is unreasonable to form, store, or invert

- **Proposed solution: Schwarz inside Schur!**

- ➔ ■ form approximate inverse action of state Jacobian and its transpose in parallel by Schwarz/multilevel methods
- ➔ ■ form forward action of Hessians by automatic differentiation; exact action needed only on vectors (JFNK)
- ➔ ■ do not eliminate exactly; use Schur preconditioning on *full* system



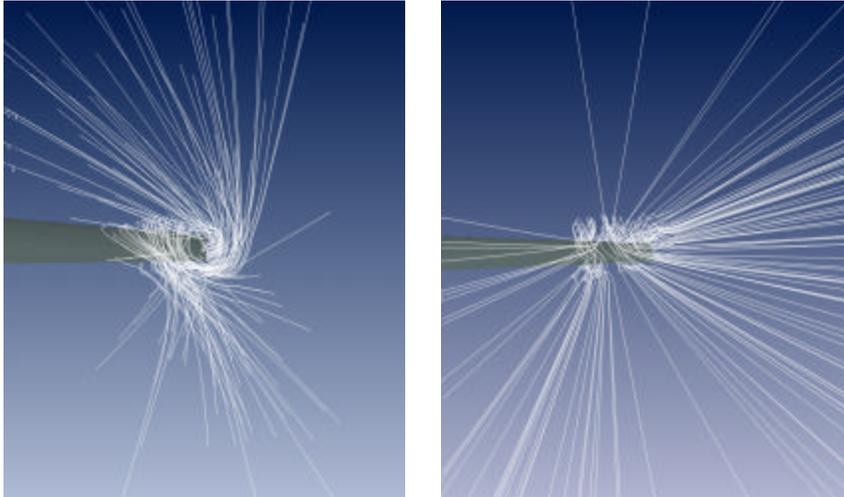
# Schur preconditioning from DD theory

- Given a partition  $\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_i \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} f_i \\ f_\Gamma \end{bmatrix}$  
- Condense:  $Su_\Gamma = g \quad S \equiv A_{\Gamma\Gamma} - A_{\Gamma i}A_{ii}^{-1}A_{i\Gamma} \quad g \equiv f_\Gamma - A_{\Gamma i}A_{ii}^{-1}f_i$
- Let  $M$  be a good preconditioner for  $S$
- Then  $\begin{bmatrix} A_{ii} & 0 \\ A_{\Gamma i} & I \end{bmatrix} \begin{bmatrix} I & A_{ii}^{-1}A_{i\Gamma} \\ 0 & M \end{bmatrix}$  is a preconditioner for  $A$
- Moreover, solves with  $A_{ii}$  may be approximate if all degrees of freedom are retained (e.g., a single V-cycle)
- Algebraic analogy from constrained optimization: “ $i$ ” is state-like, “ $\Gamma$ ” is decision-like



# PDE-constrained Optimization

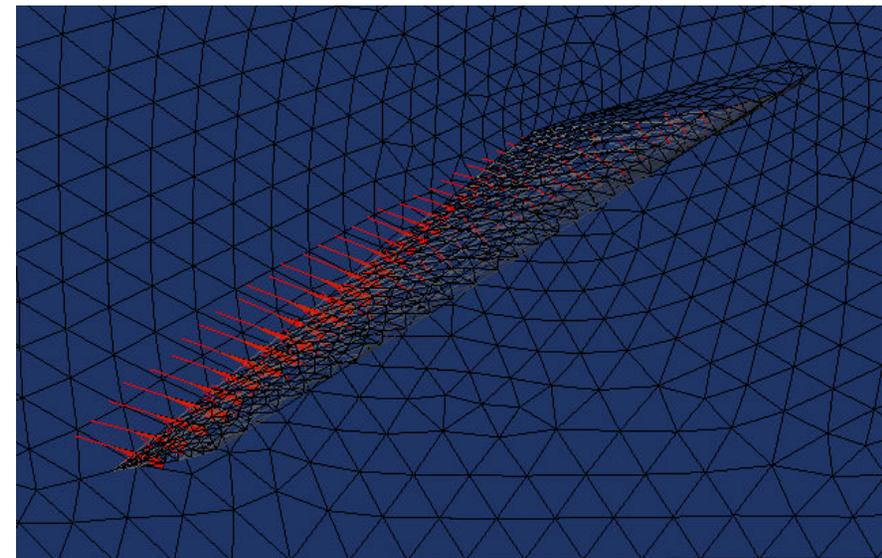
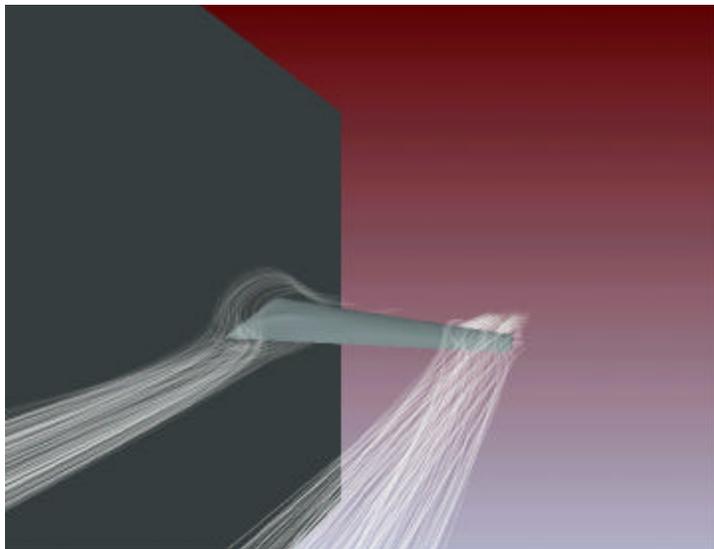
Lagrange-Newton-Krylov-Schur implemented in Veltisto/PETSc



wing tip vortices, no control (l); optimal control (r)

- **Optimal control of laminar viscous flow**

- optimization variables are surface suction/injection
- objective is minimum drag
- 700,000 states; 4,000 controls
- 128 Cray T3E processors
- ~5 hrs for optimal solution (~1 hr for analysis)



optimal boundary controls shown as velocity vectors

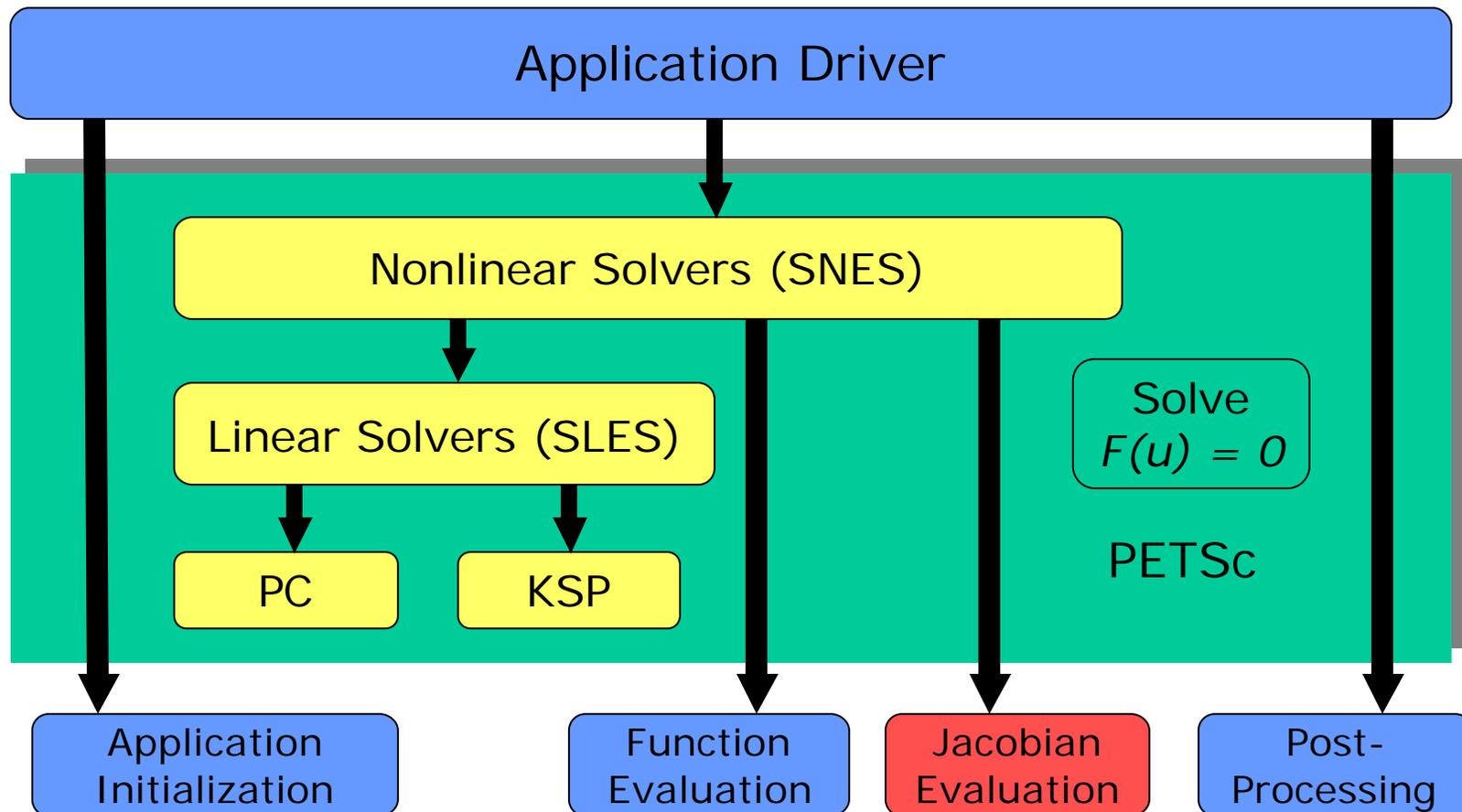
*c/o G. Biros and O. Ghattas*

[www.cs.nyu.edu/~biros/veltisto/](http://www.cs.nyu.edu/~biros/veltisto/)

**Carnegie Mellon**



# Nonlinear PDE solution w/PETSc



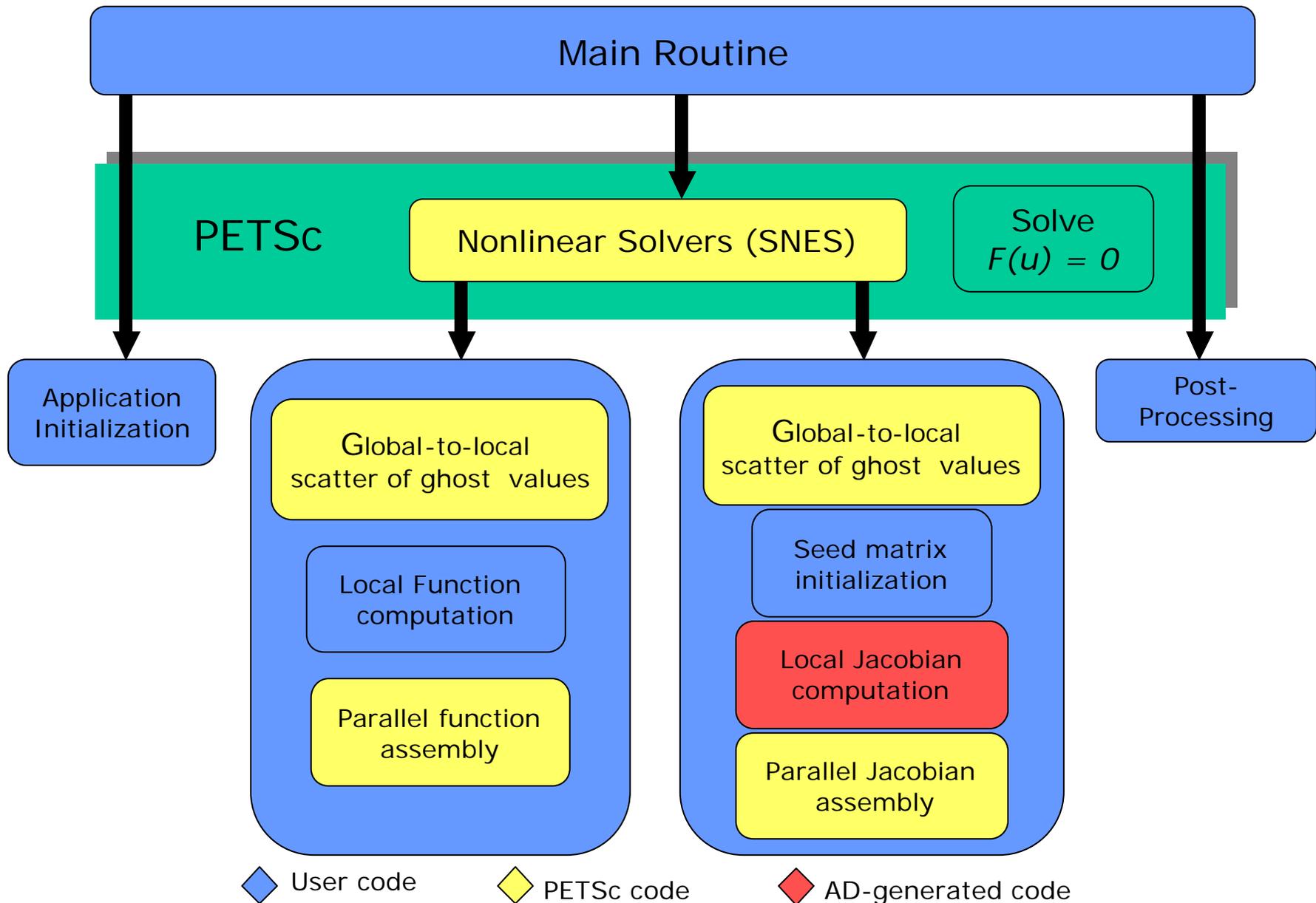
◆ User code    ◆ PETSc code    ◆ AD-generated code

• **Automatic Differentiation (AD):** a technology for automatically augmenting computer programs, including arbitrarily complex simulations, with statements for the computation of derivatives, also known as sensitivities.

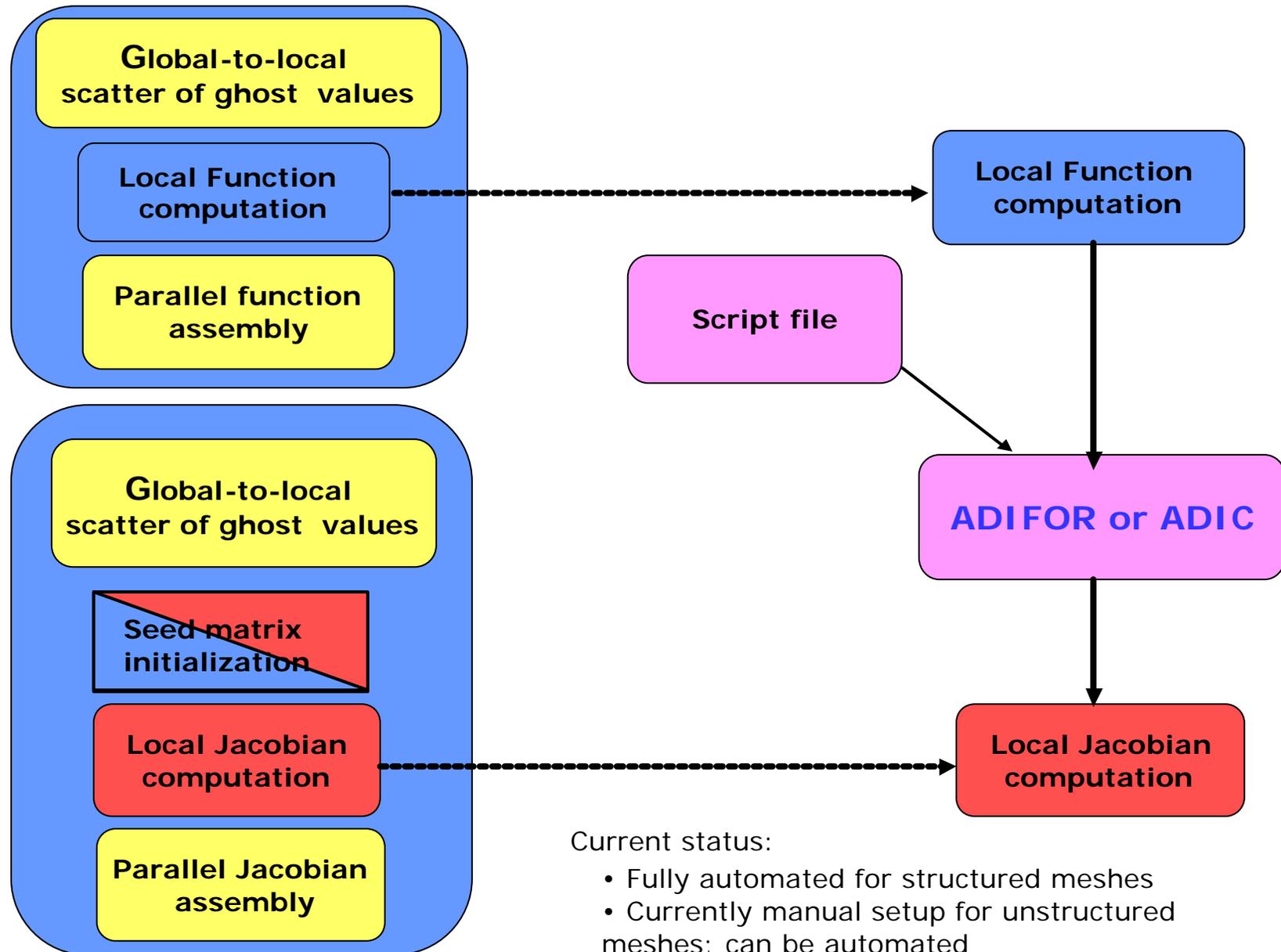
• **AD Collaborators:** P. Hovland and B. Norris (<http://www.mcs.anl.gov/autodiff>)



# Zoom on user routine structure



# Using AD with PETSc: Creation of AD Jacobian from function



Current status:

- Fully automated for structured meshes
- Currently manual setup for unstructured meshes; can be automated



# Parameter identification model

- **Nonlinear diffusion PDE BVP:**  $\nabla \cdot (\mathbf{a}(x)T^b \nabla T) = 0$
- **Parameters to be identified:**  $\mathbf{a}(x)$ ,  $b$
- **Dirichlet conditions in  $x$ , homogeneous Neumann in all other dimensions (so solution has 1D character but arbitrarily large parallel test cases can be set up)**
- **Objective:**  $\Phi = ||T(x) - \bar{T}(x)||^2$  where  $\bar{T}(x)$  is synthetic data specified from *a priori* solution with given  $\mathbf{a}(x)$  piecewise constant,  $b=2.5$  (**Brisk-Spitzer approximation for radiation diffusion**)



## Progress to date (Samyono thesis)

- **Parallel implementation using PETSc's “shell preconditioner” functionality to build the block factored KKT preconditioner recursively**
- **Solution method: LNKS with Schwarz preconditioning of the PDE Jacobian blocks, ILU on Schwarz subdomains**
- **ADIC generates Jacobian blocks from user functions**
- **Newton-like convergence for PDE analysis**
- **Rougher, but monotone, convergence for parameter identification problems**

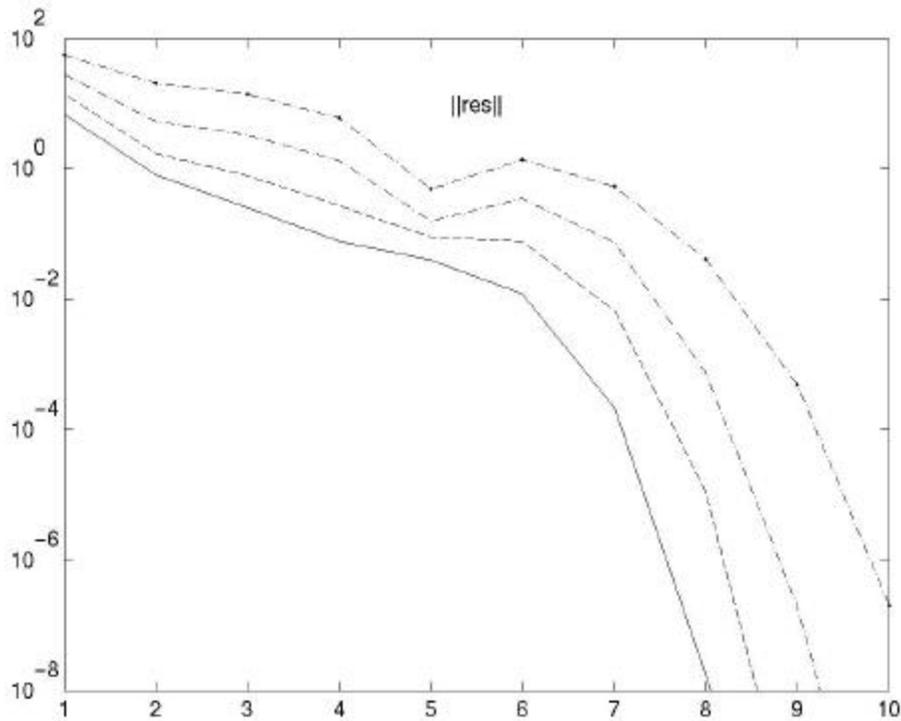


# Implementation

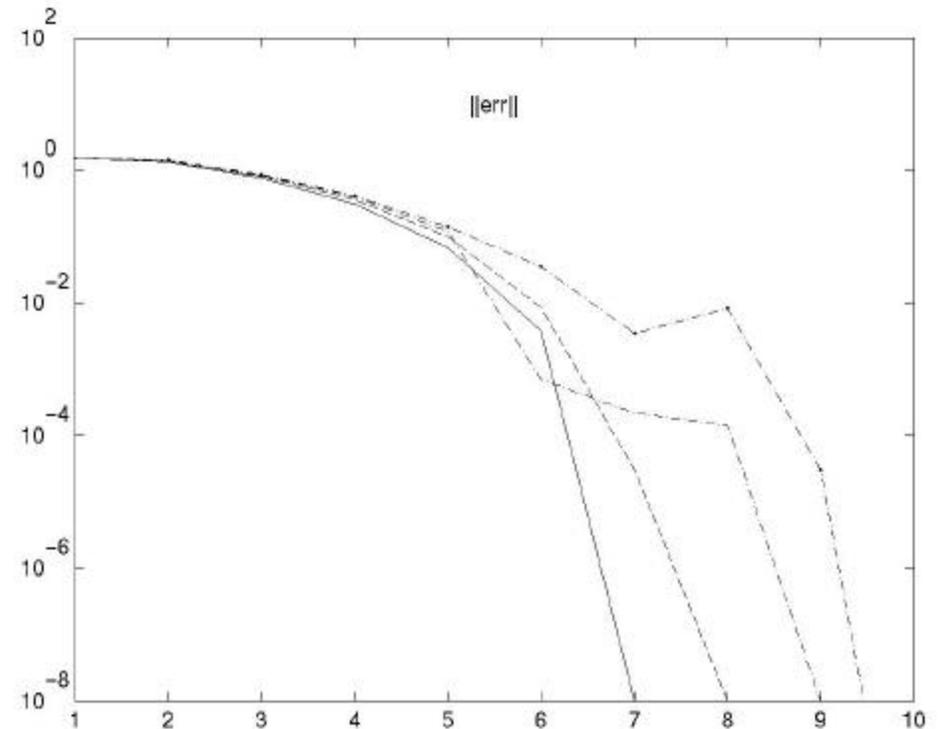
- PETSc's “shell preconditioner” functionality used to build the block factored KKT preconditioner recursively
- Solution method: LNKS with Schwarz preconditioning of the PDE Jacobian blocks, ILU on Schwarz subdomains
- MPI-based parallelization
- ADIC generates Jacobian blocks from user functions
- Illustrative results (next slide) fix  $a(x)$  and identify exponent  $b$  *only*, while uniform mesh density is refined in 2D; have also identified  $a(x)$  throughout full domain
- Newton-like, mesh-independent convergence for overall residual



# Illustrative results



**2-norm of residual of full system  
 $F(T(x), b, l(x))$  vs. iteration**



**$\|b - b^*\|$  vs. iteration**

[**solid**: 25x25 2Dmesh, **dash**: 50x50, **dot-dash**: 100x100, **dot-dot-dash**:200x200]



# Related URLs

- **Personal homepage: papers, talks, etc.**

<http://www.math.odu.edu/~keyes>

- **SciDAC initiative**

<http://www.science.doe.gov/scidac>

- **TOPS project**

<http://www.math.odu.edu/~keyes/scidac>

- **PETSc project**

<http://www.mcs.anl.gov/petsc>

- **Hypre project**

<http://www.llnl.gov/CASC/hypre>

- **ASCI platforms**

<http://www.llnl.gov/asci/platforms>



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