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MS03 Recent Developments for Schwarz Methods

Organized by: Martin J. Gander

The Schwarz methods are the oldest domain decomposition methods, but over the last decade several interesting developments happened. This mini-symposium gives an overview of recent results and new directions of research for Schwarz methods. It includes Schwarz methods with new transmission conditions, optimized Schwarz methods, restricted Schwarz methods, Schwarz waveform relaxation methods for evolution and delay differential equations and Schwarz methods for systems of equations.

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Martin J. Gander, Evridiki Efstathiou

RAS: Understanding Restricted Additive Schwarz

Location: Room 049, **Time:** Tuesday, 22 July, 16:00

Recently a variant of the additive Schwarz (AS) preconditioner, the restricted additive Schwarz (RAS) preconditioner has been introduced, and numerical experiments showed that RAS converges faster and requires less communication than AS. We show in this talk how RAS, which is defined at the matrix level, can be interpreted as an iteration at the continuous level of the partial differential equation. This interpretation reveals why RAS converges faster than classical AS and shows that RAS coincides with a parallel version of the alternating Schwarz method introduced by Lions.

Gert Lube, Tobias Knopp, Gerd Rapin

Acceleration of an Iterative Substructuring Method for Singularly Perturbed Elliptic Problems

Location: Room 049, **Time:** Tuesday, 22 July, 16:25

Iterative substructuring methods with transmission conditions of Robin type allow the parallel solution of elliptic equations, e.g. of advection-diffusion problems [1]. The implementation of such methods is relatively easy but the convergence speed is basically influenced by a properly optimized transmission condition, for a nice overview see [2]. For symmetric elliptic problems, it is shown in [2] that the simplest variant of the non-overlapping method with Robin transmission condition can be dramatically accelerated if a critical parameter of the transmission condition is adapted in a cyclic way (or even in a multilevel version).

First we reconsider the method of [2] for the singularly perturbed symmetric case. The optimization of the transmission condition is more complicated for advection-diffusion problems. In [3] we extracted from an a-posteriori estimate some information on a proper design of the critical parameter of the Robin transmission condition. The method provides a fast reduction of the interface error in the first steps; but, at a certain level (related to the discretization error), the convergence slows down. Inspirited by the approach in [2], we investigate whether a proper adaption of the critical parameter allows to accelerate the convergence speed.

[1] Achdou, Y. Le Tallec, P., Nataf, F., Vidrascu, M.: *A domain decomposition preconditioner for an advection-diffusion problem*, Comput. Meths. Appl. Mech. Engrg. 184 (2000), 145-170

[2] Gander, M.J., Golub, G.H.: *A non-overlapping optimized Schwarz method which converges with arbitrarily weak dependence on h* , in: Proc. Fourteenth Intern. Conf. on Domain Decomposition Methods, I. Herrera et.al. (eds.), 2002 DDM.org

[3] Lube, G., Müller, L., Otto, F.C.: *A non-overlapping domain decomposition method for the advection-diffusion problem*, Computing 64 (2000), 49-68

Luca Gerardo-Giorda

Modified Schwarz Algorithms without Overlap for the Harmonic Maxwell's System

Location: Room 049, **Time:** Tuesday, 22 July, 16:50

The earliest non-overlapping Schwarz algorithm for the time-harmonic Maxwell's system was introduced by B. Després *et al.* in [1]. We show that the original method, based on an interface transmission condition of radiation type, is convergent only for propagative modes. We then consider a modified interface condition, where we add a tangential second order operator of **curl-curl** type to the original Després' condition, and we prove that such algorithm is convergent for all modes. The resulting discrete system is then solved by means of a Krylov subspace method, using the Schwarz algorithm as a preconditioner, and some numerical results are also addressed.

[1] Bruno Després and Patrick Joly and Jean E. Roberts, *A domain decomposition method for the harmonic Maxwell equations*, "Iterative methods in linear algebra (Brussels, 1991)", pp. 475–484, North-Holland, Amsterdam, 1992.

[2] Martin J. Gander and Frédéric Magoulès and Frédéric Nataf, *Optimized Schwarz Methods without Overlap for the Helmholtz Equation*, SIAM J. Sci. Comput, 2001.

Frédéric Nataf, I. Faille, L. Saas, F. Willien

Finite Volume Methods on Non-Matching Grids with Arbitrary Interface Conditions

Location: Room 049, **Time:** Tuesday, 22 July, 17:15

Many works have been carried out on conforming grids either for finite element methods (see e.g. [5]), or mixed finite element methods (see e.g. [2]) and to a smaller extent for finite volume methods (see e.g. [6]).

Another important related issue is the associated linear solver which is very often based on domain decomposition methods. It is then very convenient to be able to use not only Dirichlet or Neumann boundary at the subdomain interfaces but arbitrary conditions as well (e.g. Robin boundary conditions), see [8], [7] or [2].

So, there is a need for a discretization method which allows for non matching grids and arbitrary boundary conditions at the subdomain interfaces without loosing the accuracy of the original discretization scheme. This question has been addressed in [4] and [1]. The main drawback of these papers is that the discrete solution depends on the parameters of the boundary condition used in the domain decomposition solver at the interface between the subdomains.

The main novelty of the submitted work is to introduce and analyse a method which enables to decouple in an efficient way the problem of the discretization scheme on non-matching grids and the use of arbitrary conditions at the subdomain interfaces. The analysis is valid for a convection-diffusion equation. Numerical comparisons with other discretization schemes (e.g. the standard TPFA (two point flux approximation) method standard in the petroleum community) will be shown.

[1] Y. Achdou, C. Japhet, Y. Maday, and F. Nataf. *A new cement to glue non-conforming grids with Robin interface conditions: the finite volume case.* Numer. Math., 92(4):593–620, 2002.

[2] Y. Achdou, P. Le Tallec, F. Nataf, and M. Vidrascu. *A domain decoposition preconditioner for an advection-diffusion problem.* Comp. Meth. Appl. Mech. Engrg, 184:145–170, 2000.

[3] T. Arbogast, L.C. Cowsar, M.F. Wheeler, and I. Yotov. *Mixed finite element methods on non-matching multiblock grids.* SIAM J. Numer. Anal., 37(4):1295–1315, 2000.

[4] T. Arbogast and I. Yotov. A non-mortar mixed finite element method for elliptic problems on non-matching multiblock grids. Comput. Methods Appl. Mech. Engrg., 149(1-4):255–265, 1997.

[5] Christine Bernardi, Yvon Maday, and Anthony T. Patera. *A new non conforming approach to domain decomposition: The mortar element method.* In Haim Brezis and Jacques-Louis Lions, editors, Collège de France Seminar. Pitman, 1994. This paper appeared as a technical report about five years earlier.

[6] R.E. Ewing, R.D. Lazarov, and P.S. Vassilevski. Local refinement techniques for elliptic problems on cell-centered grids. I: Error analysis. Math. Comput., 56(194):437–461, 1991.

[7] Martin J. Gander, Frédéric Magoulès, and Frédéric Nataf. *Optimized Schwarz methods without overlap for the Helmholtz equation.* SIAM J. Sci. Comput., 2002.

[8] Alfio Quarteroni and Alberto Valli. *Domain Decomposition Methods for Partial Differential Equations.* Oxford Science Publications, 1999.

Victorita Dolean, Frédéric Nataf

A Non-Overlapping Schwarz Type Algorithm for the Resolution of the Euler Equations

Location: Room 049, **Time:** Wednesday, 23 July, 11:00

In a previous paper [1] we reported on numerical experiments with a non-overlapping domain decomposition method that has been specifically designed for the calculation of steady compressible inviscid flows governed by the two-dimensional Euler equations. We have also studied this method from the theoretical standpoint [3] and by using different type of interface conditions [2]. The proposed method relies on the formulation of an additive Schwarz algorithm which is based on interface conditions that are Dirichlet conditions for the characteristic variables corresponding to incoming waves (often referred as *natural* or *classical* interface conditions), thus taking into account the hyperbolic nature of the Euler equations. The convergence of the additive Schwarz algorithm has been analyzed in two- and three-dimensional cases by considering the linearized equations and applying a Fourier analysis. We found out that besides the fact that the algorithm is always convergent, surprisingly, there exists flow conditions for which the asymptotic convergence rate is equal to zero. Moreover, this behaviour is independent of the space dimension.

For the discretized mesh we use an non-overlapping element-based partitioning where the elements are the finite volume cells. This corresponds to a "minimum" overlap partition of the set of all nodes of the mesh as seen in [4]. Therefore even if we considered a non-overlapping decomposition from the continuous point of view, by studying the discrete counterpart of the "non-overlapping" additive Schwarz type algorithm considered before on a regular geometry we find out that the result corresponds rather to an "overlapping" expression of the continuous problem. This proves that the decomposition as well as the discrete scheme used impose a naturally "overlapping" discrete formulation of the continuous non-overlapping algorithm. In this work we will build a real non-overlapping discrete formulation of the algorithm which will lead to local well posed problems and will improve the convergence for certain flow regimes.

[1] Dolean, V. and Lanteri, S., *A domain decomposition approach to finite volume solutions of the Euler equations on unstructured triangular meshes*, Int. J. Numer. Meth. Fluids,vol. 37,pp. 625-656, 2001

[2] Dolean, V. and Nataf, F. and Lanteri, S., *Construction of interface conditions for solving the compressible Euler equations by non-overlapping domain decomposition methods*,Proceedings of the LMS Workshop on Domain Decomposition Methods for Fluid Mechanics, Greenwich,Editor Lai, C.-H.,2001

[3] Dolean, V. and Lanteri, S. and Nataf, F.,*Convergence analysis of a Schwarz type domain decomposition method for the solution of the Euler equations*,INRIA Tech. Report,RR 3916, 2000.

[4] Cai, X.-C. and Farhat, C. and Sarkis, M.,*A minimum overlap restricted additive Schwarz preconditioner and appication in 3D flow simulations*,Proceedings of the 10th Domain Decomposition Methods in Sciences and Engineering,Editors J. Mandel, C. Farhat and X.-C. Cai,Contemporary Mathematics,vol. 218,AMS,pp. 479-485,1998

Christian Rohde, Martin J. Gander

Overlapping Schwarz Waveform Relaxation for Convection Dominated Nonlinear Conservation Laws

Location: Room 049, **Time:** Wednesday, 23 July, 11:25

Schwarz waveform relaxation methods have been successfully applied a solve a wide class of linear and nonlinear evolution problems. On the theoretical side the algorithm has been analyzed for, e.g., the heat equation or the wave equation including even nonlinear source terms. Less is known for problems with strong nonlinearities. In the talk we consider an instance of this class: convection dominated problems with nonlinear fluxes. New phenomena for these problems are genuinely nonlinear waves like viscous shocks and rarefactions. First we present a superlinear convergence result on bounded time intervals. In particular we observe that the convergence rate gets better with decreasing viscosity parameter. This is in agreement with the fact that the algorithm converges in a finite number of steps for the hyperbolic limit problem. Surprisingly the situation changes for a corresponding steady state equation: The (linear) convergence rate becomes arbitrarily bad if viscosity tends to 0. We will show that this striking discrepancy is caused by nonlinear fluxes. The results are illustrated with numerical experiments.

Véronique Martin

Domain Decomposition Methods for Unsteady Convection Diffusion Equation

Location: Room 049, **Time:** Wednesday, 23 July, 11:50

Classical Schwarz methods have been first applied to stationary problems and for overlapping subdomains. The first idea was developed by Schwarz who introduced an alternating algorithm with Dirichlet transmission conditions at the interface. More recently, new types of transmission conditions have been introduced which apply also to non-overlapping problems : the simplest conditions are of Robin type *i.e.* they involve normal derivatives. And it is possible to choose these transmission conditions such as to minimize the convergence rate. This strategy proved to be very useful for many steady problems as for instance convection diffusion, Euler or Helmholtz equations.

In this talk we are interested in solving time dependent equations, formulating algorithms directly for the original problem without first discretizing in time, contrary to the classical approach.

We apply this method to the convection diffusion equation in two dimensions and we write an algorithm with optimized transmission conditions. We prove the well-posedness and the convergence of this algorithm and show numerical results which illustrate the efficiency of the method.

Gerd Rapin, Gert Lube

A Stabilized Three Field Domain Decomposition Formulation for Advection-Diffusion Problems and its Iterative Decoupling

Location: Room 049, **Time:** Wednesday, 23 July, 12:15

Based on the ideas of [1] a stabilized three field formulation for advection-diffusion problems is presented. Due to the stabilization it is possible to choose the discrete function spaces almost arbitrary. Stability and an optimal a priori estimate are proven for the scheme. By minimizing the right hand side of the a priori estimate a suitable choice of the stabilization parameter is given. Numerical tests confirm our results.

Following the line of [3] the scheme is decoupled by an alternating Schwarz method for nonmatching grids. The resulting iterative scheme is similar to the algorithms in [2,4,5]. In these papers the key problem is the optimal choice of some acceleration parameters. Here the parameters are determined by the a priori estimate in a very intrinsic way. Finally some numerical experiments corresponding to the choice of the acceleration parameters are presented.

[1] C. Baiocchi, F. Brezzi, and L.D. Marini. *Stabilization of Galerkin methods and applications to domain decomposition.* In Future Tendencies in Computer Science, Control and Applied Mathematics, pages 345–355. Springer-Verlag, 1992.

[2] G. Lube, L. Müller, and F.C. Otto. *A non-overlapping domain decomposition method for the advection-diffusion problem.* Computing, 64:49–68, 2000.

[3] P. Le Tallec and T. Sassi. *Domain Decomposition with nonmatching grids: Augmented Lagrangian Approach.* Math. Comp., 64:1367–1396, 1995.

[4] F. Nataf and F. Rogier. *Factorization of the convection-diffusion operator and the Schwarz algorithm.* Math. Models Methods Appl. Sci., 5:67–93, 1995.

[5] C. Japhet, F. Nataf, and F. Rogier. *The Optimized Order 2 Method. Application to Convection-Diffusion Problems.* Future Generation Computer Systems., Vol. 18, Elsevier Science, 2000. no. 1.

Franois-Xavier Roux

Approximation of Optimal Interface Boundary Conditions for Two-Lagrange Multiplier FETI Method

Location: Room 049, **Time:** Friday, 25 July, 9:00

Interface boundary conditions are the key ingredient to design efficient domain decomposition methods. Without a global preconditioner, convergence cannot be obtained for any method in a number of iterations less than the number of subdomains minus one in the case of a one-way splitting. For the two-Lagrange multiplier FETI method, this optimal convergence can be obtained with generalized Robin type boundary conditions associated with an operator equal to the Schur complement of the outer domain.

In practice however this optimal condition cannot be implemented since the Schur complement is too expensive to compute exactly. Furthermore, the Schur complement is a dense matrix on each interface and even if it were computed, using it would create a very large increase of the bandwidth of the local stiffness matrix.

Hence the issue is how to build a sparse approximation of the Schur complement that is not expensive to compute and that gives good convergence for the two-Lagrange multiplier FETI method. Different approaches based on approximate factorization or inverse computation of the inner stiffness matrix have been tested. None of them is actually general and robust.

A new approach based on the computation of the exact Schur complement for a small patch around each interface node appears to be a very efficient method for designing approximations of the complete Schur complement that give very good convergence for many different kinds of problems. Furthermore this approach can be easily implemented without any other information than the local stiffness matrix in each subdomain.

In this paper, the results obtained with this new method will be presented for various applications in structural analysis for both static and time harmonic kinds of problems and in acoustics.

Stefan Vandewalle, Martin J. Gander

Optimized Overlapping Schwarz Methods for Parabolic PDEs with Time-Delay.

Location: Room 049, **Time:** Friday, 25 July, 9:25

Parabolic delay partial differential equations model physical systems for which the evolution does not only depend on the present state of the system but also on the past history. Such models are found, for example, in population dynamics and epidemiology, where the delay is due to a gestation or maturation period, or in numerical control, where the delay arises from the processing in the controller feedback loop.

In the first part of the talk we will study the analytical properties of the solutions of parabolic delay PDEs. Two model problems will be considered in particular: the heat equation with a fixed delay term, and the heat equation with a distributed delay in the form of an integral over the past. It will be shown that the dynamics of delay PDEs is fundamentally different from that of regular time-dependent PDEs without time delay.

Next, we will study the numerical solution of the above model problems with overlapping Schwarz methods. The considered methods are of waveform relaxation type: they compute the local solution in each subdomain over many time-levels before exchanging boundary information to neigbouring subdomains. We analyse the effect of the overlap width and we derive optimized transmission boundary conditions of Robin type. The theoretical results and convergence estimates are verified through some numerical experiments.

Lahcen Laayouni, Martin J. Gander, S. Loisel

Non-Overlapping Optimized Domain Decomposition Methods in Spherical Coordinates

Location: Room 049, **Time:** Friday, 25 July, 9:50

We investigate the performance of non-overlapping domain decomposition methods for solving Possoin equations on the sphere. This equation arises in a global weather model to improve the precipitation forecast over Canada. We consider two different types of non-overlapping algorithms: the Dirichlet-Neumann algorithm and an optimized Schwarz method. We show that both algorithms applied to a simple two subdomain decomposition converge in two iterations. This is however not true for a more general subdomain decomposition. Using numerical experiments, we compare the performance of the Dirichlet-Neumann algorithm with the optimal relaxation parameter found for the two subdomain problem, and the optimized Schwarz method with Robin transmission conditions optimized for the two subdomain problem.

Caroline Japhet, Yvon Maday, Frédéric Nataf

A New Cement to Glue Nonconforming Grids with Robin Interface Conditions: The Finite Element Case

Location: Room 049, **Time:** Friday, 25 July, 10:15

We design and analyse a new nonconforming domain decomposition method based on Schwarz type algorithms that allows for an extension to optimized interface conditions on nonconforming grids. Such interface conditions are developed in [2]. When the grids are conforming, the implementation of such interface conditions on the discretized problem has already been considered in [2].

On the other hand, using nonconforming grids allows for parallel generation of meshes, for local adaptative meshes and fast and independent solvers. The mortar method, first introduced in [1], enables the use of nonconforming grids.

It is also well suited to the use of “Dirichlet-Neumann” or “Neumann-Neumann” preconditioned conjugate gradient method applied to the Schur complement matrix. But the mortar method can’t be used easily with optimized interface conditions in the framework of Schwarz type methods.

We design and study a nonconforming domain decomposition method which allows for the use of Robin interface conditions for Schwarz type methods. We consider a diffusion equation discretized by a finite element method. The nonconforming domain decomposition method is proved to be well posed, and the iterative solver to converge. The error analysis is performed.

Then we present numerical results that illustrate the method.

[1] C. Bernardi, Y. Maday and A. Patera, *A new nonconforming approach to domain decomposition: the mortar element method*, Nonlinear Partial Differential Equations and their Applications, eds H. Brezis and J.L. Lions (Pitman, 1989).

[2] C. Japhet, F. Nataf and F. Rogier, *The Optimized Order 2 Method. Application to convection-diffusion problems*, Future Generation Computer Systems, 18(1) (2001), pp. 17-30, Elsevier Science.