Acceleration of a Domain Decomposition Method for Advection-Diffusion Problems

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Summary. For advection-diffusion problems we show that a non-overlapping domain decomposition method with interface conditions of Robin type can be accelerated by using a critical parameter of the transmission condition in a cyclic way.

1 Introduction

We consider a non-overlapping domain decomposition method (DDM) with Robin transmission conditions for the advection-diffusion-reaction model

$$Lu := -\epsilon \Delta u + (\mathbf{b} \cdot \boldsymbol{\nabla}) u + cu = f \qquad \text{in } \Omega \tag{1}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{2}$$

in a bounded polyhedral domain $\Omega \subset \mathbf{R}^d$ with a Lipschitz boundary $\partial\Omega$ and $0 < \epsilon \leq 1, \mathbf{b} \in [H^1(\Omega) \cap L^\infty(\Omega)]^d,\ c \in L^\infty(\Omega), f \in L^2(\Omega),\ c - \frac{1}{2}\nabla \cdot \mathbf{b} \geq 0$.

Let $\{\Omega_k\}$ be a non-overlapping macro partition with $\overline{\Omega} = \cup_{k=1}^N \overline{\Omega}_k$. The goal of the well-known DDM of Robin type (Lions [1990]) is to enforce (in appropriate trace spaces) continuity of the solution u and of the diffusive and advective fluxes $\epsilon \nabla u \cdot \mathbf{n}_{kj}$ resp. $-\frac{1}{2}(\mathbf{b} \cdot \mathbf{n}_{kj}) u$ on the interfaces $\Gamma_{kj} := \partial\Omega_k \cap \partial\Omega_j$. The algorithm reads in strong form:

For given $u_k^n,\ n \in \mathbf{N}_0$, in each Ω_k, find (in parallel) u_k^{n+1}, such that

$$L u_k^{n+1} = f \qquad \text{in } \Omega_k \tag{3}$$

$$u_k^{n+1} = 0 \qquad \text{on } \partial\Omega_k \cap \partial\Omega \tag{4}$$

$$\Phi_k(u_k^{n+1}) = \Phi_k(u_j^n) \qquad \text{on } \Gamma_{kj} \tag{5}$$

with $\Phi_k(u) = \epsilon \boldsymbol{\nabla} u \cdot \mathbf{n}_{kj} + (z_k - \frac{1}{2}\mathbf{b} \cdot \mathbf{n}_{kj}) u$ on $\Gamma_{kj},\ k \neq j$ and the outer normal vector $\mathbf{n}_{kj}$ on $\partial\Omega_k \cap \partial\Omega_j$. By determining the interface parameter $z_k > 0$ in a proper way the convergence of the method (3)-(5) can be accelerated.

Let $\mathcal{T}_h$ be an admissible, quasi-uniform triangulation of Ω being aligned with the macro partition. $V_h := \{v \in H_0^1(\Omega) \mid v|_K \in \mathcal{P}_l(K)\ \forall K \in \mathcal{T}_h\}$ denotes a conforming finite element (FE) space of order l. The stabilized FE method

$$\textit{Find } u_h \in V_h, \textit{ such that}: \quad a^s(u_h, v_h) = l^s(v_h) \qquad \forall v_h \in V_h \ , \tag{6}$$

$$a^s(u,v) = (\epsilon \boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_\Omega + (\mathbf{b} \cdot \boldsymbol{\nabla} u \ + c\, u,\ v)_\Omega + \sum_{T \in \mathcal{T}_h} (\delta_T L u,\ \mathbf{b} \cdot \boldsymbol{\nabla} v)_T,$$

$$l^s(v) = (f,\ v)_\Omega + \sum_{T \in \mathcal{T}_h} (\delta_T f,\ \mathbf{b} \cdot \boldsymbol{\nabla} v)_T, \qquad \delta_T \sim h_T^2 (\epsilon + h_T ||\mathbf{b}||_{\infty,T})^{-1}$$

with residual stabilization provides improved stability and accuracy (w.r.t. the streamline derivative $\mathbf{b} \cdot \nabla u$) in the hyperbolic limit $\epsilon \to 0$ of (1)-(2).

Assume, for simplicity, that the macro partition has no interior cross-points. Then we restrict the bilinear and linear forms a^s and l^s to each subdomain by $a_k^s = a^s|_{\Omega_k}$ and $l_k^s = l^s|_{\Omega_k}$. Moreover, we define $V_{k,h} = V_h|_{\Omega_k}$ and the trace space $W_{kj,h} = V_h|_{\Gamma_{kj}}$. Finally, we denote by $\langle \cdot, \cdot \rangle_{\Gamma_{kj}}$ the dual product on $(W_{kj,h})^* \times W_{kj,h}$. Then the weak formulation of the fully discretized DDM is given by

(I) Parallel computation step :
For $k = 1, \ldots, N$ *find* $u_{h,k}^{n+1} \in V_{k,h}$ *such that for all* $v_k \in V_{k,h}$:

$$a_k^s(u_{h,k}^{n+1}, v_k) + \langle z_k u_{h,k}^{n+1}, v_k \rangle_{\Gamma_k} = l_k^s(v_k) + \sum_{j(\neq k)} \langle \Lambda_{jk}^n, v_k \rangle_{\Gamma_{kj}}. \tag{7}$$

(II) Communication step :
For all $k \neq j$, *update Lagrange multipliers* $\Lambda_{kj}^{n+1} \in W_{kj,h}^*$ *such that:*

$$\langle \Lambda_{kj}^{n+1}, \phi \rangle_{\Gamma_{kj}} = \langle (z_k + z_j) u_{h,k}^{n+1} - \Lambda_{jk}^n, \phi \rangle_{\Gamma_{kj}}, \qquad \forall \phi \in W_{kj,h}. \tag{8}$$

This method is very easy to implement. It is used as a building block in a parallelized flow solver for the thermally driven incompressible Navier-Stokes problem, cf. Knopp et al. [2002]. A fast convergence of the DDM is desirable, in particular for time-dependent problems.

It is well-known that the algorithm (7)-(8) is well-posed if $z_k = z_j > 0$. Moreover, the sequence $\{u_k^n\}_{n \in \mathbf{N}}$ is strongly convergent according to

$$\lim_{n \to \infty} ||| \ u_{h,k}^n - u_h|_{\Omega_k} |||_{\Omega_k} = 0 \tag{9}$$

where $|||v|||_{\Omega_k} := \sqrt{a_k^s(v,v)}$, Lube et al. [2000]. The convergence speed depends in a sensitive way on the parameters z_k which have to be optimized. In Sec. 2 we review an *a priori* optimization introduced by Nataf and Gander. Sec. 3 is devoted to an *a posteriori based* approach.

2 A-priori optimization of the interface condition

The convergence of the Robin DDM (3)-(5) can be described in simple cases using a Fourier analysis. Nataf and Gander proposed a semi-continuous *a priori optimization* of the interface parameter z over a relevant range $S = (s_{min}, s_{max})$ of Fourier modes. An optimization is important for highly oscillatory solutions, e.g. for the Helmholtz equation (1)-(2) with $\mathbf{b} \equiv \mathbf{0}$ and $c/\epsilon \ll -1$. An improved idea is a *cyclic* change of z for appropriate frequency ranges. For the *symmetric* case with $\mathbf{b} = \mathbf{0}$ and $\epsilon = 1$, Gander and Golub [2002] proposed the following variant of the DDM (3)-(5) in $\Omega = \mathbf{R}^2$, $\Omega_1 = \mathbf{R}^+ \times \mathbf{R}$, $\Omega_2 = \mathbf{R}^- \times \mathbf{R}$ with the *cyclic* Robin condition

$$\nabla u_1^{n+1} \cdot \mathbf{n}_1 + z^{n\, mod(m)} u_1^{n+1} = \nabla u_2^n \cdot \mathbf{n}_1 + z^{n\, mod(m)} u_2^n \tag{10}$$

and similarly for u_2^{n+1} on $\Gamma = \{0\} \times \mathbf{R}$ for $m = 2^l$ and appropriate chosen $z^0, \ldots, z^{m-1}$. For $l = 0$, Gander and Golub [2002] obtain the following contraction rate

$$\rho(s,z) = \left(\frac{\sqrt{c+s^2} - z^0}{\sqrt{c+s^2} + z^0} \right)^2$$

for the $s-$th Fourier mode. For the mesh parameter h, an optimization over $S = (s_{min}, \pi/h)$ gives $\min_{z^0 \geq 0} \left(\max_{s \in S} \rho(s, z^0) \right) = 1 - \mathcal{O}(\sqrt{h})$. In the cyclic case $m = 2^l$ one gets the rate

$$\rho(m,s,z) = \prod_{j=1}^{m} \left(\frac{\sqrt{c+s^2} - z^{n\, mod(m)}}{\sqrt{c+s^2} - z^{n\, mod(m)}} \right)^{2/m}$$

and the optimized result

$$\min_{z \geq 0} \left(\max_{s \in S} \rho(m,s,z) \right) \approx 1 - \frac{4}{m} \left[\frac{(c + s_{min}^2) h^2}{16\pi^2} \right]^{\frac{1}{4m}}, \quad h \to +0.$$

This result is useful for meshes with $\frac{(c+s_{min}^2)h^2}{16\pi^2} \leq 1$, but this estimate deteriorates in the singularly perturbed case, i.e. for fixed h and $c \to +\infty$.

For the *non-symmetric case*, the optimization of the Schwarz method corresponding to $l = 0$ can be found in Nataf [2001]. The extension to a cyclic Schwarz method with $l \geq 1$ is open.

3 A posteriori based design of the interface condition

As an alternative to the a priori based design of the interface parameter we propose an approach based on an *a posteriori* estimate. Consider a simplified situation with $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \subset \mathbf{R}^2$ with $\text{meas}_1(\partial\Omega \cap \partial\Omega_i) > 0$ and straight interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ of size $H = \text{meas}\ (\Gamma) \sim \text{diam}(\Omega_i)$, $i = 1, 2$. Set $W = H_{00}^{\frac{1}{2}}(\Gamma)$. We assume constant data ϵ, $\mathbf{b}$, c. In Lube et al. [2000] we proved

Theorem 1. *Let u_h be the solution of (6). The DDM-subdomain error $e_{h,k}^n = u_{h,k}^n - u_h|_{\Omega_k}$, $k \in \{1, 2\}$, can be controlled via (computable) interface data:*

$$|||e_{h,k}^{n+1}|||_{\Omega_k} \leq A_j \ \|u_{h,k}^n - u_{h,j}^{n+1}\|_W + B_j \left| z_k - \frac{\mathbf{b} \cdot \mathbf{n}_k}{2} \right| \|u_{h,k}^n - u_{h,j}^{n+1}\|_{L^2(\Gamma)} \tag{11}$$

for $j = 3 - k$ and with data-dependent constants

$$A_j = \sqrt{\epsilon}\left(1 + \sqrt{\frac{c}{\epsilon}}H + \min\left[\frac{\|\mathbf{b}\|_\infty H}{\epsilon}; \frac{\|\mathbf{b}\|_\infty}{\sqrt{c\epsilon}}\right]\right), \qquad B_j = \sqrt{\frac{H}{\epsilon}}. \tag{12}$$

This result motivates to equilibrate the two right-hand side terms in (11) in order to obtain information about the design of the interface parameter z_k. In Lube et al. [2000] we considered the estimate

$$|||e_{h,k}^{n+1}|||_{s,\Omega_k} \leq \max\left(A_j;\ B_j C\sqrt{H}\left|z_k - \frac{1}{2}\mathbf{b}\cdot\mathbf{n}_k\right|\right)\|u_{h,k}^n - u_{h,j}^{n+1}\|_W \tag{13}$$

using the continuous embedding result $\|\phi\|_{L^2(\Gamma)} \leq C\sqrt{H}\|\phi\|_W$ for all $\phi \in W$. On the other hand, an inverse estimate in (11) leads to

$$|||e_{h,k}^{n+1}|||_{s,\Omega_k} \leq \max\left(CA_j h^{-\frac{1}{2}}; B_j\left|z_k - \frac{1}{2}\mathbf{b}\cdot\mathbf{n}_k\right|\right)\|u_{h,k}^n - u_{h,j}^{n+1}\|_{L^2(\Gamma)}. \tag{14}$$

In the *symmetric* case $\mathbf{b} = \mathbf{0}$ we get from (13) and (14)

$$z_k \sim \frac{\epsilon}{H}\sqrt{\frac{H}{L}}\left(1 + H\sqrt{\frac{c}{\epsilon}}\right), \qquad L \in \{h, H\}. \tag{15}$$

In the *non-symmetric case* $\mathbf{b} \neq \mathbf{0}$, the design of z_k has to match the hyperbolic limit of the Robin condition, i.e.

$$0 = \lim_{\epsilon\to 0}\Phi_k(u) = (-\frac{1}{2}\mathbf{b}\cdot\mathbf{n}_k + \lim_{\epsilon\to 0} z_k)u \quad \text{if} \quad \mathbf{b}\cdot\mathbf{n}_k \geq 0.$$

By extending this condition to the inflow part of $\partial\Omega_k$ with $\mathbf{b}\cdot\mathbf{n}_k < 0$, we obtain from (13)-(14) as a reasonable choice

$$z_k = \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}_k| + R_k(L), \qquad L \in \{h, H\}, \tag{16}$$

$$R_k(L) \sim \frac{\epsilon}{H}\sqrt{\frac{H}{L}}\left(1 + H\sqrt{\frac{c}{\epsilon}} + \min\left[\frac{H\|\mathbf{b}\|}{\epsilon}; \frac{\|\mathbf{b}\|}{\sqrt{c\epsilon}}\right]\right). \tag{17}$$

Inserting (16), (17) with $L = H$ in (13) and applying an inverse inequality, we obtain the *optimized a posteriori* estimates

$$|||e_{h,k}^{n+1}|||_{\Omega_k} \leq A_j\|u_{h,k}^n - u_{h,j}^{n+1}\|_W \leq CA_j h^{-\frac{1}{2}}\|u_{h,k}^n - u_{h,j}^{n+1}\|_{L^2(\Gamma)}. \tag{18}$$

The last estimate also follows directly by inserting (16), (17) with $L = h$ in (14). Therefore we propose to extend the condition (16) to $L \in [h, H]$.

In Lube et al. [2000] we considered the case $L = H$. This choice usually allows a fast error reduction down to the discretization error level if the solution has no highly oscillatory behaviour. Fortunately, the latter case is rare for problem (1)-(2) with $c - \frac{1}{2}\nabla\cdot\mathbf{b} \geq 0$.

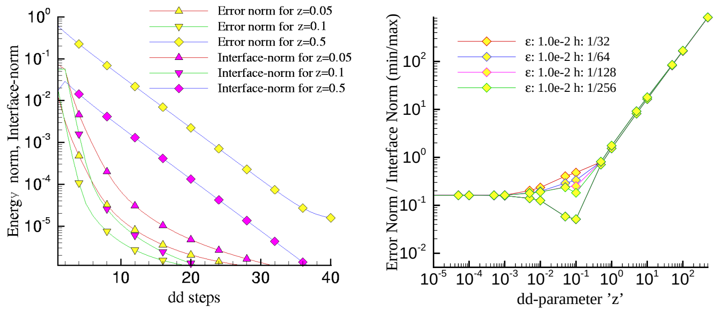

Fig. 1. Reliability of the a posteriori estimate for $h = \frac{1}{128}$ (left), Control of $\max_n / \ \min_n \sum_{i=1}^{4} |||e_{h,k}^{n+1}|||_{\Omega_i} / \sum_{i \neq j} \|u_{h,i}^{n+1} - u_{h,j}^{n}\|_{L^2(\Gamma_{ij})}$ vs. z (right).

Example 1. Consider the problem (1)-(2) with $\mathbf{b} \equiv 0$, $\epsilon = 10^{-2}$, $c = 1$ in $\Omega = (0,1)^2$. The exact (smooth) solution is $u = x_1(1-x_1)x_2(1-x_2)e^{x_1 x_2}$. We denote the solution of (6) with $\mathcal{P}_1$-elements and $h = \frac{1}{128}$ by $u_k = u_h|_{\Omega_k}$. The DDM on an equidistant 2×2 macro partition with an initial guess $\Lambda_{jk}^0 = 0$ leads to the sequence $u_{h,k}^n$. The stopping criterion $\sum_k |||u_{h,k}^n - u_h|_{\Omega_k}|||_{\Omega_k} \leq 10^{-6}$ has a tolerance beyond the discretization error level.

Fig. 1 (left) shows that the subdomain error $|||\cdot|||_{\Omega_k}$ is clearly controlled by the $L^2(\Gamma)$ interface error according to Theorem 1. Moreover, the convergence of the DD-iteration depends strongly on z. The fast error reduction in the first phase corresponds to a fast reduction of "low" frequencies; but then a (very) slow reduction of "higher" modes can be seen. In Fig. 1 (right) we control the maximal/minimal (w.r.t. to the number n of DD steps) ratio between the subdomain and interface errors for varying h. The value $z_k \sim \frac{1}{10}$ corresponding to the minimum of this ratio for $h = \frac{1}{256}$ is in agreement with the value predicted by (15) with $L = H$. As predicted by Theorem 1, we observe a linear dependence of the error on z for increasing z. □

Obviously, the results of Example 1 with the optimized value of z according to (16), (17) with $L = H$ depend only on the data of the problem (1)-(2) and not on h. We want to check this result for other typical cases.

Example 2. Let be Ω and the solution u as in Example 1. The FEM solution u_h of (6) is computed with $\mathcal{P}_1$-elements on a fine mesh with $h = \frac{1}{256}$ and with SUPG stabilization in advection-dominated cases for

A: Symmetric case: $\mathbf{b} = (0,0)$, $c = 1$, DDM with 4 subdomains,
B: Case $|\mathbf{b} \cdot \mathbf{n}_i| > 0$: $\mathbf{b} = (2,1)$, $c = 1$, DDM with 2 subdomains,
C: Case $|\mathbf{b} \cdot \mathbf{n}_i| \equiv 0$: $\mathbf{b} = (0,1)$, $c = 1$, DDM with 2 subdomains .

The initial guess for the Lagrange multipliers is $\Lambda_{ij}^0 = 0$. The stopping criterion for the error between the discrete solutions with and without DDM is $\sum_k \|u_{h,k}^n - u_h|_{\Omega_k}\|_{L^2(\Omega_k)} \leq 10^{-6}$. The convergence in this range is predicted

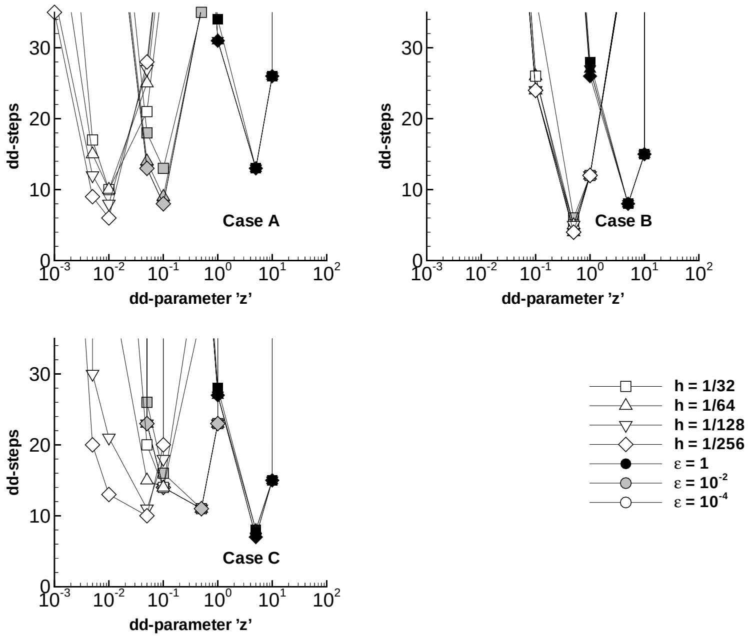

Fig. 2. Optimization of the interface parameter z with one-level approach.

by the data of (1)-(2) and is h-independent. The optimal values of z_k are predicted by the optimized z_k from (16), (17) with $L = H$, see Fig. 2. □

The nice convergence behaviour can be explained by the smoothness of the solutions and of the initial guess Λ^0_{ij}. Moreover, in our experiments we never found problems for singularly perturbed problems with sharp layers.

Nevertheless, the convergence behaviour of the Robin-DDM is not satisfactory beyond the discretization error level. Moreover, regarding our application to flow problems (Knopp et al. [2002]), in the turbulent case the solution usually has high-frequent components which may not be efficiently damped in our previous approach. As a remedy we propose a combination of the a posteriori control of the interface error with a *cyclic multi-level* version of the DDM:

Step 1: (optionally) Apply (7)-(8) with the optimized z_k from (16)-(17) with $L = H$ until reduction of the interface error down to discretization error level, e.g. $\|u^n_{h,i} - u^{n+1}_{h,j}\|_{L^2(\Gamma)} \leq \kappa h^{l+1/2}$ for $\mathcal{P}_l$ elements.

Step 2: Apply (7)-(8) in a cyclic way with p levels (see below) using (16)-(17) with $z^1_k, ..., z^p_k$ related to $L = H$ (for z^1_k) and $l = h$ (for z^p_k), resp., and an even number (to our experience, 4 or 6 are sufficient) of DD steps per level until $\|u^n_{h,i} - u^{n+1}_{h,j}\|_{L^2(\Gamma)} \leq TOL$.

Let us discuss this approach for some cases of Example 2. First of all, we have to fix the number p of levels. Assume a dyadic representation of the coarse and of the fine mesh of the domain $\Omega = (0,1)^2$ with $H = 2^{-s},\ h = 2^{-t},\ s,t \in \mathbf{N}$. From (16)-(17) we obtain $R_k(L) \le R_k(h) \sim \sqrt{H/h}$, i.e. a mild dependence on $\sqrt{H/h}$. We propose the following rule: For $2^p < \sqrt{H/h} \le 2^{p+1}$, take p levels. Thus we obtain for a very fine mesh width $h = 2^{-10}$ a number of two levels for a coarse grid width $H = 2^{-5}$ and of four levels for $H = 2^{-1}$.

Example 3. Consider the situation of Example 2 with a 2×2 macro partition with $H = \frac{1}{2}$ and a fine mesh with $h = 2^{-6}$. This leads to $p = 2$ levels. We start with the symmetric case of (1)-(2) with $\epsilon = 1, \mathbf{b} = \mathbf{0},\ c = 1$. The fast error reduction within the first steps is followed by a very slow reduction in the one-level case, cf. Fig. 3 (left). Here u^h and u^h_{seq} denote the solutions with and without DDM. The two-level method with 6 DD-steps per level leads to a dramatic acceleration, cf. Fig. 3 (right).

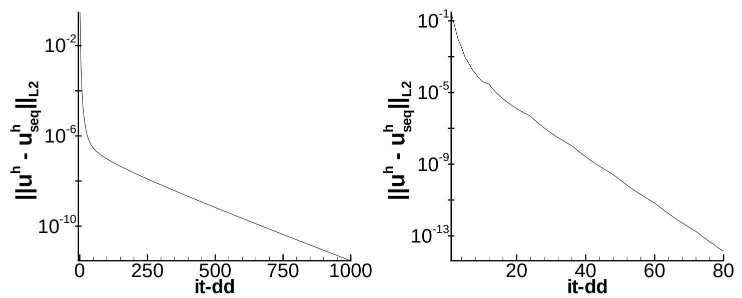

Fig. 3. Error reduction for the *symmetric* case: $p = 1$ (left), $p = 2$ (right)

Consider now the non-symmetric and advection-dominated case with $\epsilon = 10^{-5}, \mathbf{b} = \frac{(1,2)^T}{\sqrt{5}},\ c = 0$. In Fig. 4 we observe a similar behaviour of the proposed approach with $p = 1$ (left) and $p = 2$ (right) levels, although the acceleration is not so dramatic as in the symmetric case. □

Finally, let us note an observation of Gander and Golub [2002] for the symmetric case: The quality of the cyclic DDM (3)-(5) with an optimized condition (10) as a solver increases with the number of levels such that no improvement can be found with Krylov acceleration. A similar behaviour is very likely in the non-symmetric case.

4 Summary

Considerable progress has been reached for Schwarz methods with (a priori) optimized transmission conditions. We propose an approach based on a refined a posteriori error estimate for a DDM with transmission conditions of Robin type. For the one-level variant, this condition can be optimized in such a

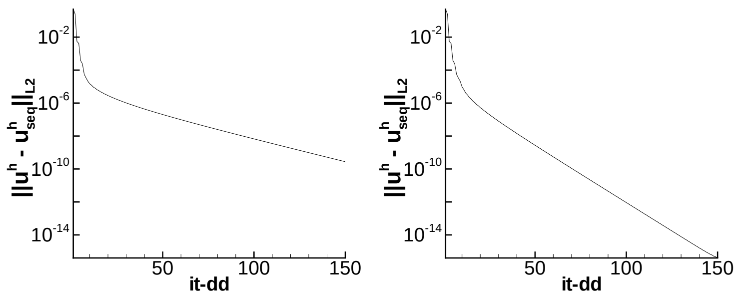

Fig. 4. Error reduction for the *non-symmetric* case: $p = 1$ (left), $p = 2$ (right)

way that the convergence is very reasonable down to the discretization error level; but then one observes a rapid slow-down of error reduction for higher error modes. This is valid for "smooth" solutions and is in contrast to highly oscillatory solutions typically appearing, e.g., for turbulent flows.

A multilevel-type method with optimized interface parameters allows a strong acceleration of the convergence. The approach is motivated by theoretical results, but more efforts are necessary to improve its present state. An advantage of the method over a priori optimized methods is the control of the convergence within the iteration. Moreover, a combination with adaptive mesh refinement is possible. It remains open whether the method is linearly convergent. Moreover, a genuinely multilevel-type implementation might be possible. Finally, the extension to incompressible flows has to be done.

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