Approximation of Optimal Interface Boundary Conditions for Two-Lagrange Multiplier FETI Method

F.-X. Roux, F. Magoulès, L. Series, Y. Boubendir

ONERA, 29 av. de la Division Leclerc, BP72, 92322 Châtillon, France,
<roux@onera.fr>, <series@onera.fr>, <boubendir@onera.fr>
Univ. Henri Poincaré, BP239, 54506 Vandoeuvre-les-Nancy, France,
<magoules@iecn.u-nancy.fr>

Summary. Interface boundary conditions are the key ingredient to design efficient domain decomposition methods. However, convergence cannot be obtained for any method in a number of iterations less than the number of subdomains minus one in the case of a one-way splitting. This optimal convergence can be obtained with generalized Robin type boundary conditions associated with an operator equal to the Schur complement of the outer domain. Since the Schur complement is too expensive to compute exactly, a new approach based on the computation of the exact Schur complement for a small patch around each interface node is presented for the two-Lagrange multiplier FETI method.

1 Introduction

Interface boundary conditions are the key ingredient to design efficient domain decomposition methods, see Chevalier and Nataf [1998], Benamou and Després [1997], Gander et al. [2002]. However, convergence cannot be obtained for any method in a number of iterations less than the number of subdomains minus one in the case of a one-way splitting. For the two-Lagrange multiplier FETI method, this optimal convergence can be obtained with generalized Robin type boundary conditions associated with an operator equal to the Schur complement of the outer domain, see Roux et al. [2002]. In practice this optimal condition cannot be implemented since the Schur complement is too expensive to compute exactly. Furthermore, the Schur complement is a dense matrix on each interface and even if it were computed, using it would create a very large increase of the bandwidth of the local subproblem matrix. Hence the issue is how to build a sparse approximation of the Schur complement that is not expensive to compute and that leads to good convergence properties of the two-Lagrange multiplier FETI iterative method.

Different approaches based on approximate factorization or inverse computation of the subproblem matrix have been tested, see Roux et al. [2002]. Here,

a new approach based on the computation of the exact Schur complement for a small patch around each interface node appears to be a very efficient method for designing approximations of the complete Schur complement. Furthermore this approach can be easily implemented without any other information than the local matrix in each subdomain.

2 Review of the Two-Lagrange Multiplier FETI Method

2.1 Introduction of Two-Lagrange Multiplier on the Interface

Consider a splitting of the domain Ω as in Figure 1 and note by subscripts i and p the degrees of freedom located inside subdomain $\Omega^{(s)}$, $s = 1, 2$, and on the interface Γ. Then, the contribution of subdomain $\Omega^{(s)}$, $s = 1, 2$ to the matrix and the right-hand side of a finite element discretization of a linear partial differential equation on Ω can be written as follows:

$$K^{(s)} = \begin{bmatrix} K_{ii}^{(s)} & K_{ip}^{(s)} \\ K_{pi}^{(s)} & K_{pp}^{(s)} \end{bmatrix}, \quad b^{(s)} = \begin{bmatrix} b_i^{(s)} \\ b_p^{(s)} \end{bmatrix}$$

where $K_{pp}^{(1)}$ and $K_{pp}^{(2)}$ represent the interaction matrices between the nodes on the interface obtained by integration on $\Omega^{(1)}$ and on $\Omega^{(2)}$. The global problem is a block system obtained by assembling local contribution of each subdomain:

$$\begin{bmatrix} K_{ii}^{(1)} & 0 & K_{ip}^{(1)} \\ 0 & K_{ii}^{(2)} & K_{ip}^{(2)} \\ K_{pi}^{(1)} & K_{pi}^{(2)} & K_{pp} \end{bmatrix} \begin{bmatrix} x_i^{(1)} \\ x_i^{(2)} \\ x_p \end{bmatrix} = \begin{bmatrix} b_i^{(1)} \\ b_i^{(2)} \\ b_p \end{bmatrix}. \tag{1}$$

The block K_{pp} is the sum of the two blocks $K_{pp}^{(1)}$ and $K_{pp}^{(2)}$. In the same way, $b_p = b_p^{(1)} + b_p^{(2)}$ is obtained by local integration in each subdomain and sum on the interface.

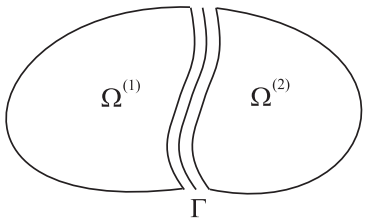

Fig. 1. Non-overlapping domain splitting.

The two-Lagrange multiplier FETI method, see Farhat et al. [2000], is an iterative based domain decomposition method which consists to determine the solution of the following coupled problem:

$$\begin{bmatrix} K_{ii}^{(1)} & K_{ip}^{(1)} \\ K_{pi}^{(1)} & K_{pp}^{(1)} + A^{(1)} \end{bmatrix} \begin{bmatrix} x_i^{(1)} \\ x_p^{(1)} \end{bmatrix} = \begin{bmatrix} b_i^{(1)} \\ b_p^{(1)} + \lambda^{(1)} \end{bmatrix}$$
$$\begin{bmatrix} K_{ii}^{(2)} & K_{ip}^{(2)} \\ K_{pi}^{(2)} & K_{pp}^{(2)} + A^{(2)} \end{bmatrix} \begin{bmatrix} x_i^{(2)} \\ x_p^{(2)} \end{bmatrix} = \begin{bmatrix} b_i^{(2)} \\ b_p^{(2)} + \lambda^{(2)} \end{bmatrix}$$
$$\lambda^{(1)} + \lambda^{(2)} - (A^{(1)} + A^{(2)}) x_p^{(1)} = 0$$
$$\lambda^{(1)} + \lambda^{(2)} - (A^{(1)} + A^{(2)}) x_p^{(2)} = 0$$

where the free matrices $A^{(1)}$ and $A^{(2)}$ are to be determined for the best performance of the algorithm. It is clear that this coupled problem is equivalent to the global problem (1), see Roux et al. [2002]. The elimination of $x_i^{(s)}$ in favor of $x_p^{(s)}$ in the two first equations and substitution in the two last equations leads to the following linear system upon the variable $\lambda := (\lambda^{(1)}, \lambda^{(2)})^T$:

$$F\lambda = d \tag{2}$$

with F and d the matrix and right hand side defined as:

$$F := \begin{bmatrix} I & I - (A^{(1)} + A^{(2)})[S^{(2)} + A^{(2)}]^{-1} \\ I - (A^{(1)} + A^{(2)})[S^{(1)} + A^{(1)}]^{-1} & I \end{bmatrix}$$
$$d := \begin{bmatrix} (A^{(1)} + A^{(2)})[S^{(2)} + A^{(2)}]^{-1} c_p^{(2)} \\ (A^{(1)} + A^{(2)})[S^{(1)} + A^{(1)}]^{-1} c_p^{(1)} \end{bmatrix}$$

The iterative solution of this system is usually performs with a Krylov method.

2.2 Optimal Interface Boundary Conditions

It is shown in Roux et al. [2002] that the best choice for the free matrices $A^{(s)}$, $s = 1, 2$ corresponds to the complete outer Schur complement, i.e. the discretization of the optimal continuous boundary conditions associated to the Steklov-Poincaré operator, see Ghanemi [1997], Collino et al. [2000] and Boubendir [2002]. An extension of this result in the case of a one way splitting can be obtained in the discrete case, see Roux et al. [2002], and in the continuous case, see Nataf et al. [1994].

Theorem 1. *In a case of a two-domain splitting, the Jacobi iterative algorithm for the two-Lagrange multiplier FETI method with augmented term equal to the complete outer Schur complement converges in one iteration at most.*

Theorem 2. *In a case of a one way splitting, the Jacobi iterative algorithm for the two-Lagrange multiplier FETI method with augmented term equal to the complete outer Schur complement converges in a number of iteration equal to the number of subdomain minus one.*

3 Approximation of Optimal Interface Boundary Conditions

In the previous section, we have recalled that the best choice for the augmented matrix in the case of a one way splitting domain decomposition is the complete outer Schur complement matrix. This choice can not be done in practice since the computational cost of the complete outer Schur complement matrix is too expensive.

3.1 Neighbor Schur Complement

From a physical point of view, the complete outer Schur complement matrix represent the interactions of all the degree of freedom of the subdomains condensed on the interfaces. The restriction of the interactions only with the neighboring subdomains, leads to approximate the complete outer Schur complement with the neighbor Schur complement. The computational cost and the exchange of data are thus reduced to the neighboring subdomains only. Unfortunately, this approach still leads to an expensive computational cost. Hence the issue is how to build a sparse approximation of the Schur complement that is not expensive to compute and that gives good convergence for the two-Lagrange multiplier FETI method.

3.2 Lumped Approximation

We have shown in Roux et al. [2002] that an approximation of the neighbor Schur complement matrix $K_{bb}^{(s)} - K_{bi}^{(s)}[K_{ii}^{(s)}]^{-1}K_{ib}^{(s)}$ with its first term, i.e. with the matrix $K_{bb}^{(s)}$ gives good results. Such an approximation, presents the advantage to be very easy to implement since this matrix is computed by the neighboring subdomain during the assembly procedure and the integration of the contribution of the interface nodes. Only an exchange with the neighboring subdomain is required for this regularization procedure.

3.3 Sparse Approximation based on Overlapping Layers

In this section we present a new approach for the approximation of the neighboring Schur complement with a sparse matrix, which leads to a better approximation than the lumped approximation, as shown in the numerical results. The goal is to obtained a spectral density of the approximated matrix close to the spectral density of the neighbor Schur complement matrix. We first define the following subsets of indexes:

$$
\begin{aligned}
V_{\Omega^{(2)}} &= \{\text{indexes of nodes inside the subdomain } \Omega^{(2)}\} \\
V_{\Gamma} &= \{\text{indexes of nodes on the interface } \Gamma\} \\
V_i^l &= \{\text{indexes of the nodes } j \text{ such that the minimum} \\
&\quad \text{connectivity distance between } i \text{ and } j \text{ is lower or} \\
&\quad \text{equal than } l, l \in \mathbb{N}\} \\
V_{\Gamma,i}^l &= V_{\Gamma} \cap V_i^l
\end{aligned}
$$

The sparse approximation investigated here consist to define a sparse augmented matrix obtained through an extraction of some coefficients and local condensation along the interface. The complete algorithm to compute the augmented matrix—in the case of a two domain splitting—in subdomain $\Omega^{(1)}$ can be define as:

Algorithm 1. [**sparse approximation**]

1. construction of the structure of the interface matrix $A_1 \in \mathbb{R}^{dimV_\Gamma \times dimV_\Gamma}$.
2. construction of the sparse structure of the subdomain matrix $K^{(2)} \in \mathbb{R}^{dimV_{\Omega^{(2)}} \times dimV_{\Omega^{(2)}}}$.
3. assembly of the matrix $K^{(2)}$.
4. for all i in V_Γ do
   4.1. extraction of the coefficients K_{mn}, $(m,n) \in V_i^l \times V_i^l$, and construction of the sparse matrix $A_2 \in \mathbb{R}^{dimV_i^l \times dimV_i^l}$ with these coefficients.
   4.2. computation of the dense matrix $A_3 \in \mathbb{R}^{dimV_{\Gamma,i}^1 \times dimV_{\Gamma,i}^1}$ by condensation of the matrix A_2 on the degree of freedom $V_{\Gamma,i}^1$.
   4.2. extraction of the coefficients of the line associated with the node i from the matrix A_3 and insertion inside the matrix A_1 at the line associated with the node i.
5. construction of the symmetric matrix $A_4 = \frac{(A_1^T + A_1)}{2}$.
6. regularization of the matrix $K^{(1)}$ with the matrix A_4.

where l denotes the number of layers considered.

Similar calculation performed in the subdomain $\Omega^{(2)}$ gives the augmented matrix $A^{(2)}$ to add to the subdomain matrix $K^{(2)}$. As an example the regular mesh with $\mathbb{Q}_1$-finite elements presented Figure 2 leads to the subsets of

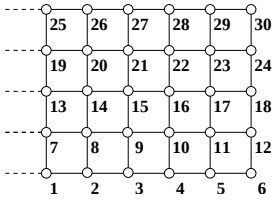

Fig. 2. Numbering of the nodes in subdomain $\Omega^{(2)}$.

indexes $V_7^1 = \{1,2,7,8,13,14\}$, $V_7^2 = \{1,2,3,7,8,9,13,14,15,19,20,21\}$ and $V_{\Gamma,7}^1 = \{1,7,13\}$. These subsets correspond to the overlapping layers represented Figure 3.

4 Numerical Results

4.1 The Model Problem

In this section, a two dimensional beam of length L_1 and high L_2 submitted to flexion is analyzed. The Poisson ratio and the Young modulus are respectively $\nu = 0.3$ and $E = 2.0\ 10^5 N/m^2$. Homogeneous Dirichlet boundary conditions

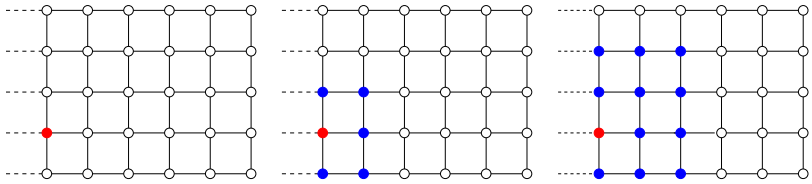

Fig. 3. On the left one interface node, on the middle one interface node with one layer, and on the right one interface node with two layers.

are imposed on the left and homogeneous Neumann boundary conditions are set on the top and on the bottom. Loading, model as non homogeneous Neumann boundary condition are imposed on the right of the structure. The beam is meshed with triangular elements and discretized with $\mathbb{P}_1$ finite elements. The domain is then split into two or ten subdomains in a one way splitting and the condensed interface problem is solved iteratively with the ORTHODIR Krylov method. The stopping criterion is set to $||r_n||_2 < 10^{-6}||r_0||_2$, where r_n and r_0 are the nth and initial global residuals.

4.2 Spectral Analysis

Figure 4 represent the spectral density of the eigenvalues of the matrix of the condensed interface problem (2) for different augmented matrices. An augmented matrix equal to the neighbor Schur complement will leads to eigenvalues equal to one which correspond to a spectral density equal to a Dirac function.

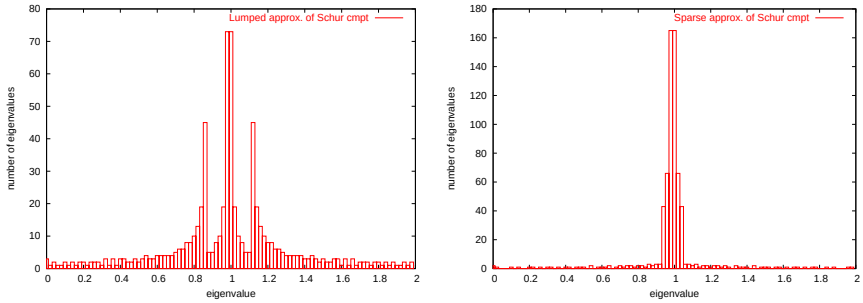

Fig. 4. Spectral density of the condensed interface problem with an augmented matrix issue from the lumped approximation (left) vs from the sparse approximation (right) ($L_1 = 10$, $L_2 = 1$, $h = 1/160$). Case of two subdomains.

We can see on Figures 4 that a sparse approximation performed with a number of layers equal to four leads to a spectral density close to a Dirac function. Opposite, a lumped approximation leads to spectrum much more different. This result can be explain by the fact that the sparse approximation is based on local condensation i.e. on local Steklov-Poincaré operators which is not the case of the lumped approximation.

4.3 Asymptotic Analysis

The asymptotic analysis of the proposed methods upon different parameters is now analyzed. The analysis upon the domain size reported Figure 5 show the respective dependence of the methods. The asymptotic behavior of the pro-

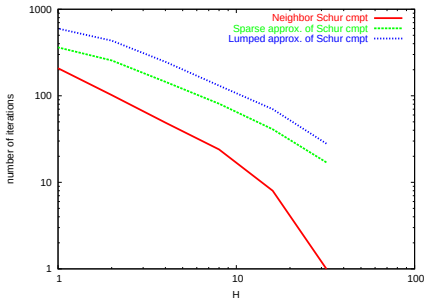

Fig. 5. Asymptotic behavior for different augmented matrices and different subdomain size. ($L_1 = 64$, $L_2 = 1$, $h = 1/20$).

posed methods upon the mesh size is presented Figure 6. On the left picture, four layers are considered for the sparse approximation. A linear dependence upon the mesh size can be noticed for all the methods. On the right picture, the number of layers of the sparse approximation increase proportionally with the mesh size. A linear dependence still occurs for the sparse approximation but the slope of the curve is lower than with a constant number of layers equal to four. The asymptotic results obtained with this last approximation

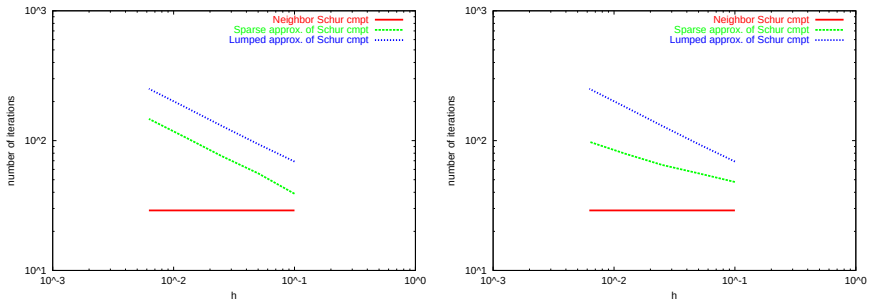

Fig. 6. Asymptotic behavior for different augmented matrices and different mesh size on the left for a constant number of layers, and on the right for a number of layers increasing proportionally with the mesh size. ($L_1 = 10$, $L_2 = 1$). Case of ten subdomains.

are still less efficient than those obtained with a continuous approach, see Gander et al. [2002], but the implementation of the previous method doesn't depends on a priori knowledge of the problem to be solved (coefficients of the

partial differential equation, mesh size, ...) and thus helps its use as a black box routine!

5 Conclusions

In this paper the principle of the two-Lagrange multiplier FETI method with optimal interface boundary conditions has been remain. A new method for the approximation of these optimal conditions has been introduced. This new method is based on the computation of the exact Schur complement for a small patch around each interface node. This method appears to be a very efficient method for designing approximations of the complete Schur complement that give robust iterative algorithms for solving many different kinds of problems.

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