A More General Version of the Hybrid-Trefftz Finite Element Model by Application of TH-Domain Decomposition

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Summary. In recent years the hybrid-Trefftz finite element (hT-FE) model, which originated in the work by Jirousek and his collaborators and makes use of an independently defined auxiliary inter-element frame, has been considerably improved. It has indeed become a highly efficient computational tool for the solution of difficult boundary value problems In parallel and to a large extent independently, a general and elegant theory of Domain Decomposition Methods (DDM) has been developed by Herrera and his coworkers, which has already produced very significant numerical results. Theirs is a general formulation of DDM, which subsumes and generalizes other standard approaches. In particular, it supplies a natural theoretical framework for Trefftz methods. To clarify further this point, it is important to spell out in greater detail than has been done so far, the relation between Herrera's theory and the procedures studied by researchers working in standard approaches to Trefftz method (Trefftz-Jirousek approach). As a contribution to this end, in this paper the hybrid-Trefftz finite element model is derived in considerable detail, from Herrera's theory of DDM. By so doing, the hT-FE model is generalized to non-symmetric systems (actually, to any linear differential equation, or system of such equations, independently of its type) and to boundary value problems with prescribed jumps. This process also yields some numerical simplifications.

1 Introduction

Trefftz [1926] method was originated by this author. However, the origins of the hybrid-Trefftz (HT) finite element (FE) model are only around twenty five years old, Jirousek and Leon [1977], Jirousek [1978]. Since then it has become a highly efficient computational tool for the solution of difficult boundary value problems, Jirousek and Wroblewski [1996], Qin [2000], with an increasing popularity among researchers and practitioners. In parallel and to a large extent independently, a general and elegant theory of domain decomposition methods (DDM) has been developed by Herrera and coworkers (Herrera et al.

[2002] and Herrera [2003]). This, throughout its different stages of development, has been known by a variety of names; mainly, localized adjoint method (LAM), Trefftz-Herrera method and unified theory of DDM. This is a general formulation, which subsumes and generalizes many other approaches. In particular, it seems to be the natural framework for Trefftz methods and several aspects of that theory have been recognized as fundamental by some of the most conspicuous researchers of these methodologies (Jirousek and Wroblewski [1996], Zielinski [1995] and Jirousek and Zielinski [1997]). However, it is important to spell out in greater detail than thus far, the relation between Herrera's theory and the procedures of Trefftz-Jirousek approach, Jirousek and Wroblewski [1996], which are extensively used by the researchers working in Trefftz method.

In particular, to this end, in the present paper a detailed analysis and comparison of the hybrid-Trefftz finite element model is carried out using Herrera's theory. In this manner, the HT-FE approach is generalized to problems with prescribed jumps and to non-symmetric operators. Also, a manner in which a significant reduction of the number of degrees of freedom involved in the HT-FE global equations is indicated. Although only problems formulated in terms Laplace operator are considered, the results can be extended to very general classes of differential operators using Herrera's general framework, as it will be explained in a paper now being prepared.

2 Notations and auxiliary results

Since the main purpose of this paper is to clarify the relation between Herrera's theory and Trefftz-Jirousek approach, as was stated in the Introduction, the notation that is used follows closely that which is standard is expositions of this latter approach (Qin [2000]). In addition, it is related with that which has been applied in Herrera's theory developments. A domain, Ω, is considered and one of its partitions $\{\Omega_1, ..., \Omega_E\}$, referred as *'the partition'*. In addition to the boundary Γ, of Ω, to be referred as the *'outer boundary'*, one considers the *'internal boundary'* Γ_I, which separates the subdomains from each other. The outer boundary is assumed to be the union of Γ_u and Γ_q. The boundary Γ_e, of every subdomain, Ω_e, of the partition, is assumed to be the union of $\Gamma_{eu} \equiv \Gamma_e \cap \Gamma_u$, $\Gamma_{eq} \equiv \Gamma_e \cap \Gamma_q$ and $\Gamma_{eI} \equiv \Gamma_e \cap \Gamma_I$. Trial and test functions are taken from the same linear space, D, whose members are functions defined in each one of the subdomains and, therefore, are generally discontinuous across Γ_I, together with their derivatives. Borrowing from Herrera's notation, one writes

$$[u] \equiv u_+ - u_- \quad \text{and} \quad \dot{\widehat{u}} \equiv \frac{1}{2}(u_+ + u_-) \tag{1}$$

$[u]$ and $\dot{\widehat{u}}$ are referred as the *'jump'* and the *'average'* of u, respectively. Here, u_+ and u_- are the limits from the positive and negative sides, respectively. The internal boundary is oriented by defining a unit normal vector $\underline{n}$

whose sense is chosen arbitrarily; then, the convention is that $\underline{n}$ points toward the positive side.

Given two functions, $u \in D$ and $w \in D$, the following relation between Jirousek's and Herrera's notations will be applied in the sequel

$$\sum_{e=1}^{E} \int_{\Gamma_{eI}} w \frac{\partial u}{\partial n} d\sigma \equiv - \int_{\Gamma_I} \left[w \frac{\partial u}{\partial n} \right] d\sigma \qquad (2)$$

In the left-hand side of this equation, using Jirousek's notation, the normal derivative is taken with respect to the unit normal vector that points outwards of Ω_e. Thus, when Jirousek's notation is used one has two unit normal vectors defined at each point of Γ_I, while in Herrera's notation there is only one.

3 Trefftz-Jirousek Approach

For simplicity, we restrict attention to the case when the differential operator is Laplace's operator and adopt a notation similar to that followed by Jirousek and his collaborators (see for example Qin [2000]). The boundary value problem considered in Trefftz-Jirousek method is

$$\mathcal{L}u \equiv \Delta u = \bar{b}, \quad \text{in} \quad \Omega_e, \quad e = 1, ..., E \qquad (3)$$

$$u = \bar{u} \quad \text{on} \quad \Gamma_u \quad \text{and} \quad \frac{\partial u}{\partial n} = \bar{q}_n \quad \text{on} \quad \Gamma_q \qquad (4)$$

Together with

$$[u] = \left[\frac{\partial u}{\partial n} \right] = 0 \quad \text{on} \quad \Gamma_I \qquad (5)$$

Observe that any function $u \in D$, which satisfies Eq.(3), can be written as

$$u = u_P + u_H \qquad (6)$$

Where

$$\Delta u_P = \bar{b}, \quad \Delta u_H = 0, \quad \text{in} \quad \Omega_e, \quad e = 1, ..., E \qquad (7)$$

The above equation, which is fulfilled by $u_H \in D$, is homogeneous. Therefore the set of functions that satisfy Eq.(3), constitutes a linear subspace $D_H \subset D$. In addition, $u_P \in D$ is not uniquely determined by Eq.(7). However, once $u_P \in D$ is chosen, $u_H = u - u_P$ is unique. Assuming that a function, $u_P \in D$, fulfilling Eq.(7), has been constructed the search to determine the solution $u \in D$ is carried out in the (affine) subspace $D_P \equiv u_P + D_H \subset D$.

4 Jirousek's Variational Principle

The variational principles to be applied are derived from the functional (see Qin [2000])

$$\Pi_m \equiv \frac{1}{2}\int_\Omega \left(q_1^2+q_2^2\right)^2 d\Omega - \int_{\Gamma_u} q_n \bar{u} ds + \int_{\Gamma_q} \left(\bar{q}_n - q_n\right) u ds - \sum_{e=1}^{E} \int_{\Gamma_{eI}} q_n \tilde{u} ds \quad (8)$$

Observe that Π_m is a functional of a pair: $(u, \tilde{u})$, where $u \in D$ and $\tilde{u}$, the so called *'displacement frame'*, is a function defined on Γ_I. In Herrera's notation, the above functional is

$$\Pi_m \equiv \frac{1}{2}\int_\Omega \nabla u \cdot \nabla u dx - \int_{\Gamma_u} \bar{u}\frac{\partial u}{\partial n} dx + \int_{\Gamma_q}\left(\bar{q}_n - \frac{\partial u}{\partial n}\right) u dx - \sum_{e=1}^{E}\int_{\Gamma_{eI}} \frac{\partial u}{\partial n}\tilde{u} dx \quad (9)$$

or, introducing the jumps (see the Notations Section)

$$\Pi_m \equiv \frac{1}{2}\int_\Omega \nabla u \cdot \nabla u dx - \int_{\Gamma_u} \bar{u}\frac{\partial u}{\partial n} dx + \int_{\Gamma_q}\left(\bar{q}_n - \frac{\partial u}{\partial n}\right) u ds + \int_{\Gamma_I} \tilde{u}\left[\frac{\partial u}{\partial n}\right] dx \quad (10)$$

The system of equations used in Trefftz-Jirousek method is obtained by requiring that the variation of this functional be zero, in the (affine) subspace of functions that fulfill Eq. (3), while no restriction is imposed on the frame, $\tilde{u}$. The weak formulation derived using this functional is $\delta \Pi_m = 0$, which can be written as

$$\begin{aligned}&\int_\Omega \nabla u \cdot \nabla w dx - \int_{\Gamma_u} \bar{u}\frac{\partial w}{\partial n} dx + \int_{\Gamma_q}\left\{\left(\bar{q}_n - \frac{\partial u}{\partial n}\right) w - u\frac{\partial w}{\partial n}\right\} dx \\ &+ \int_{\Gamma_I}\left\{\tilde{w}\left[\frac{\partial u}{\partial n}\right] + \tilde{u}\left[\frac{\partial w}{\partial n}\right]\right\} dx = 0\end{aligned} \quad (11)$$

Here, $w \in D_H$ and $\tilde{w}$ stand for the variation of u and $\tilde{u}$, respectively. This is the form in which it is most frequently applied. However, for our analysis it is more convenient to write it as

$$\begin{aligned}&\int_{\Gamma_u} (u - \bar{u})\frac{\partial w}{\partial n} dx - \int_{\Gamma_q} w\left(\frac{\partial u}{\partial n} - \bar{q}_n\right) dx \\ &+ \int_{\Gamma_I}\left\{\tilde{w}\left[\frac{\partial u}{\partial n}\right] - [u]\dot{\widehat{\frac{\partial w}{\partial n}}} + \left(\tilde{u} - \dot{\widehat{u}}\right)\left[\frac{\partial w}{\partial n}\right]\right\} dx = 0\end{aligned} \quad (12)$$

Therefore, the Euler equations for this variational formulation are the boundary conditions of Eqs. (4) and the continuity conditions for the function and its normal derivative of Eqs. (5), together with

$$\tilde{u} = \dot{\widehat{u}} \equiv u \quad \text{on} \quad \Gamma_I \quad (13)$$

Clearly $\dot{\widehat{u}} \equiv u$ because u is continuous across Γ_I.

In conclusion, a pair $(u, \tilde{u})$, with $u \in D_P$, that satisfies Eq. (12) for every variation $w \in D_H$, has the following properties:

1. u is solution of the BVP, and
2. $\tilde{u} = u$ on Γ_I.

Generally the linear subspace $D_H \subset D$ is infinite dimensional and therefore the search for $u \in D_P$, in the entirety of $D_P \subset D$, is not feasible. In order to make it feasible, a finite-dimensional (affine) subspace $\hat{D}_P \equiv u_P + \hat{D}_H \subset D$ is introduced as follows: a finite family of linearly independent functions $\mathcal{E} \equiv \left\{w^1, ..., w^N\right\} \subset D$ is chosen and $\hat{D}_H \subset D_H$ is defined to be

$$\hat{D}_H \equiv \text{span}\left\{w^1, ..., w^N\right\} \tag{14}$$

The system $\mathcal{E} \equiv \left\{w^1, ..., w^N\right\} \subset \hat{D}_H$, above, is usually referred as a truncated *T-complete system of homogeneous solutions*, Jirousek and Wroblewski [1996]. In addition a system of functions each one of them defined on Γ_I exclusively, $\left\{\tilde{w}^1, ..., \tilde{w}^{\tilde{N}}\right\}$, is introduced. This is referred as the frame basis. Then one approximates $u \in D_P$ and $\tilde{u}$, by

$$\hat{u} = u_P + \hat{u}_H \tag{15}$$

and

$$\widehat{\tilde{u}} = \sum_{\alpha=1}^{\tilde{N}} \tilde{c}_\alpha \tilde{w}^\alpha \tag{16}$$

respectively. Here $\hat{u}_H \in \hat{D}_H$. Clearly, Eq. (14) implies

$$\hat{u}_H = \sum_{i=1}^{N} c_i w^i \tag{17}$$

Above $\{c_1, .., c_N\}$ and $\{\tilde{c}_1, .., \tilde{c}_{\tilde{N}}\}$ are suitable choices of the coefficients. They are determined by application of the variational principle discussed before. Actually, the weak formulations of Eq. (11) or (12) are applied, with u and $\tilde{u}$ replaced by $\hat{u}$ and $\widehat{\tilde{u}}$, respectively. By inspection, it is seen that the Euler equations associated with Eq. (12) yield the following approximate relations

$$\begin{aligned} &\hat{u} \approx \bar{u} \quad \text{on} \quad \Gamma_u, \quad \tfrac{\partial \hat{u}}{\partial n} \approx \bar{q}_n \quad \text{on} \quad \Gamma_q \\ &[\hat{u}] \approx \left[\tfrac{\partial \hat{u}}{\partial n}\right] \approx 0 \quad \text{and} \quad \tilde{u} \approx \widehat{\dot{u}} \quad \text{on} \quad \Gamma_I \end{aligned} \tag{18}$$

It is relevant to observe that $\hat{u} \in \hat{D}_P$, as given by Eqs. (15) and (17), is a discontinuous function and, therefore, its average across Γ_I, in Eq. (18), can not be replaced by its value on Γ_I. Also, usually the internal Γ_I boundary is much larger than the external boundary, $\Gamma_u \cup \Gamma_q$, then the relation $N \approx 2\tilde{N}$ is fulfilled approximately and the total number of degrees of freedom is

$$N + \tilde{N} \approx \frac{3}{2} N \tag{19}$$

5 The BVPJ and Herrera's Variational Principles

The boundary value problem considered in Herrera's theory is a boundary value problem with prescribed jumps (BVPJ) in the internal boundary, Γ_I, which is the same as that considered in Section 3, except that Eq. (5) is replaced by

$$[u] = j^0_\Sigma, \quad \left[\frac{\partial u}{\partial n}\right] = j^1_\Sigma \quad \text{on} \quad \Gamma_\text{I} \tag{20}$$

where j^0_Σ and j^1_Σ are given functions defined on Γ_I. In Herrera's theory two weak formulations are introduced, Herrera [2001], Herrera [1985]: the *'weak formulation in terms of the data of the problem'*. This is the basis of the direct approach and it yields weak formulations that are quite similar those usually applied by other authors. For the BVPJ here considered, it is

$$\begin{aligned}&\langle \mathcal{L}_H u, w\rangle \equiv \\ &\int_\Omega w\Delta u dx + \int_{\Gamma_u} u\tfrac{\partial w}{\partial n}dx - \int_{\Gamma_q} w\tfrac{\partial u}{\partial n}dx + \int_{\Gamma_I}\left\{\dot{\widehat{w}}\left[\tfrac{\partial u}{\partial n}\right] - [u]\,\dot{\widehat{\tfrac{\partial w}{\partial n}}}\right\}dx = \\ &\int_\Omega w\bar{b}dx + \int_{\Gamma_u}\bar{u}\tfrac{\partial w}{\partial n}dx - \int_{\Gamma_q} w\bar{q}_n dx + \int_{\Gamma_I}\left\{\dot{\widehat{w}}j^1_\Sigma - j^0_\Sigma\dot{\widehat{\tfrac{\partial w}{\partial n}}}\right\}dx\end{aligned} \tag{21}$$

Which is equivalent to the *'weak formulation in terms of the complementary information'*

$$\begin{aligned}&\langle \mathcal{L}^*_H u, w\rangle \equiv \\ &\int_\Omega u\Delta w dx + \int_{\Gamma_u} w\tfrac{\partial u}{\partial n}dx - \int_{\Gamma_q} u\tfrac{\partial w}{\partial n}dx + \int_{\Gamma_I}\left\{\dot{\widehat{u}}\left[\tfrac{\partial w}{\partial n}\right] - [w]\,\dot{\widehat{\tfrac{\partial u}{\partial n}}}\right\}dx = \\ &\int_\Omega w\bar{b}dx + \int_{\Gamma_u}\bar{u}\tfrac{\partial w}{\partial n}dx - \int_{\Gamma_q} w\bar{q}_n dx + \int_{\Gamma_I}\left\{\dot{\widehat{w}}j^1_\Sigma - j^0_\Sigma\dot{\widehat{\tfrac{\partial w}{\partial n}}}\right\}dx\end{aligned} \tag{22}$$

Both of these formulations are equivalent, because it can be shown that $\langle \mathcal{L}^*_H u, w\rangle \equiv \langle \mathcal{L}_H u, w\rangle = \langle \mathcal{L}_H w, u\rangle$. Furthermore, they are equivalent to the variational condition $\delta\Pi_H(u) = 0$, if $\Pi_H(u)$ is defined to be

$$\begin{aligned}&2\Pi_H(u) \equiv \\ &\int_\Omega u\Delta u dx + \int_{\Gamma_u} u\tfrac{\partial u}{\partial n}dx - \int_{\Gamma_q} u\tfrac{\partial u}{\partial n}dx + \int_{\Gamma_I}\left\{\dot{\widehat{u}}\left[\tfrac{\partial u}{\partial n}\right] - [u]\,\dot{\widehat{\tfrac{\partial u}{\partial n}}}\right\}dx - \\ &2\left\{\int_\Omega w\bar{b}dx + \int_{\Gamma_u}\bar{u}\tfrac{\partial w}{\partial n}dx - \int_{\Gamma_q} w\bar{q}_n dx + \int_{\Gamma_I}\left\{\dot{\widehat{w}}j^1_\Sigma - j^0_\Sigma\dot{\widehat{\tfrac{\partial w}{\partial n}}}\right\}dx\right\}\end{aligned} \tag{23}$$

When u is varied subjected to the restriction $u \in D_P$, so that $\Delta w = 0$, Eqs. (21) and (22) can also be written as

$$\begin{aligned}&\int_{\Gamma_u}(u-\bar{u})\tfrac{\partial w}{\partial n}dx - \int_{\Gamma_q} w\left(\tfrac{\partial u}{\partial n} - \bar{q}_n\right)dx \\ &+ \int_{\Gamma_I}\left\{\dot{\widehat{w}}\left(\left[\tfrac{\partial u}{\partial n}\right] - j^1_\Sigma\right) - \left([u] - j^0_\Sigma\right)\dot{\widehat{\tfrac{\partial w}{\partial n}}}\right\}dx = 0\end{aligned} \tag{24}$$

and

$$\begin{aligned}&\int_{\Gamma_u} w\frac{\partial u}{\partial n}dx - \int_{\Gamma_q} u\frac{\partial w}{\partial n}dx + \int_{\Gamma_I}\left\{\dot{\widehat{u}}\left[\frac{\partial w}{\partial n}\right] - [w]\,\dot{\widehat{\frac{\partial u}{\partial n}}}\right\}dx = \\ &\int_{\Omega} w\bar{b}dx + \int_{\Gamma_u}\bar{u}\frac{\partial w}{\partial n}dx - \int_{\Gamma_q} w\bar{q}_n dx + \int_{\Gamma_I}\left\{\dot{\widehat{w}}j_{\Sigma}^{1} - j_{\Sigma}^{0}\dot{\widehat{\frac{\partial w}{\partial n}}}\right\}dx\end{aligned} \tag{25}$$

respectively. Eq. (24) exhibits the Eqs. (4) and (20), as the Euler equations of *the variational principle in terms of the data* of the BVPJ. However, the use of Eq. (25) is different.

Let $\tilde{u}$, $\tilde{q}_n$, $\tilde{u}_a$ and $\tilde{q}_a$ be functions defined, the first two, on Γ_q and Γ_u respectively, and on Γ_I, the last two. Assume that they satisfy

$$\begin{aligned}&\int_{\Gamma_u} w\tilde{q}_n dx - \int_{\Gamma_q}\tilde{u}\frac{\partial w}{\partial n}dx + \int_{\Gamma_I}\left\{\tilde{u}_a\left[\frac{\partial w}{\partial n}\right] - [w]\,\tilde{q}_a\right\}dx = \\ &\int_{\Omega} w\bar{b}dx + \int_{\Gamma_u}\bar{u}\frac{\partial w}{\partial n}dx - \int_{\Gamma_q} w\bar{q}_n dx + \int_{\Gamma_I}\left\{\dot{\widehat{w}}j_{\Sigma}^{1} - j_{\Sigma}^{0}\dot{\widehat{\frac{\partial w}{\partial n}}}\right\}dx\end{aligned} \tag{26}$$

for every $w \in D_H$, then subtracting Eq. (26) from Eq. (25), it is seen that

$$\begin{aligned}&\int_{\Gamma_u} w\left(\frac{\partial u}{\partial n} - \tilde{q}_n\right)dx - \int_{\Gamma_q}(u - \tilde{u})\frac{\partial w}{\partial n}dx \\ &+\int_{\Gamma_I}\left\{\left(\dot{\widehat{u}} - \tilde{u}_a\right)\left[\frac{\partial w}{\partial n}\right] - [w]\left(\dot{\widehat{\frac{\partial u}{\partial n}}} - \tilde{q}_a\right)\right\}dx = 0\end{aligned} \tag{27}$$

Eq. (26) is a variational principle whose Euler equations, in view of Eq. (27), are $\tilde{q}_n = \frac{\partial u}{\partial n}$ on Γ_u, $\tilde{u} = u$ on Γ_q and $\tilde{u}_a = \dot{\widehat{u}}$, $\tilde{q}_a = \dot{\widehat{\frac{\partial u}{\partial n}}}$ on Γ_I.

When a truncated *T-complete system of homogeneous solutions*, $\mathcal{E} \equiv \left\{w^1, ..., w^N\right\} \subset D_H$, is used to generate a subspace $\hat{D}_H \subset D_H$, and D_H is replaced by $\hat{D}_H$, then these equations are only approximately satisfied. In particular, $\tilde{u}_a$ and $\tilde{q}_a$ are approximations of the averages, across Γ_I, of the function and its normal derivative, respectively. The following systems of *'frames'* are introduced: $\tilde{\mathcal{E}}_u \equiv \left\{\tilde{w}_u^1, ..., \tilde{w}_u^{\tilde{N}_u}\right\}$, $\tilde{\mathcal{E}}_q \equiv \left\{\tilde{w}_q^1, ..., \tilde{w}_q^{\tilde{N}_q}\right\}$, $\tilde{\mathcal{E}}_{ua} \equiv \left\{\tilde{w}_{ua}^1, ..., \tilde{w}_{ua}^{\tilde{N}_{ua}}\right\}$ and $\tilde{\mathcal{E}}_{qa} \equiv \left\{\tilde{w}_{qa}^1, ..., \tilde{w}_{qa}^{\tilde{N}_{qa}}\right\}$. The first two are defined on Γ_q and Γ_u respectively, and the last two on Γ_I. Then the functions are taken to be linear combinations of these bases with suitable coefficients, which are determined by application of the weak formulation of Eq. (26). Of course a necessary condition for this to be possible is that $N = \tilde{N}_u + \tilde{N}_q + \tilde{N}_{ua} + \tilde{N}_{qa}$. The total number of degrees of freedom is N and global matrix associated the system of equations (26) is $N \times N$.

6 Comparisons and Conclusions

Clearly Trefftz-Jirousek and Trefftz-Herrera formulations are closely related. However, this latter approach generalizes Jirousek's since the boundary value problem considered in Section 4 is a particular case of the more general BVPJ treated in Section 5; namely, Trefftz-Jirousek method deals with the particular case of this BVPJ when $j^0_\Sigma = j^1_\Sigma = 0$. Also, according to Eq. (19) in Trefftz-Herrera formulation the number of degrees of freedom is reduced a 33%, in comparison with Trefftz-Jirousek formulation. Indeed, in this latter approach one deals with $\frac{3}{2}N \times \frac{3}{2}N$ global matrices, while these are only $N \times N$ in the former. A more thorough discussion of these points will be presented in a paper now being prepared.

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