Partition of Unity Method for the Stokes Equations

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joint work with Jinru Chen, Yunqing Huang
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Outline

- Partition of Unity (PU) Related Works
- PU Method: The Huang - Xu Approach
- Discretization of Stokes Problem via PU
  - Overlapping grids
  - Non-overlapping non-matching grids
- Remarks and Future Plans
Related Works


\[ |\nabla \phi_i|_\infty \simeq (\text{diam}(\Omega_i))^{-1} \]

\( \phi_i \)-the (PU)-function corresponding to the subdomains \( \Omega_i \).


Huang-Xu (2002) designed a conforming fem for overlapping and nonmatching grids for elliptic BVP using a (PU)-method.

\[ |\nabla \phi_i|_\infty \simeq (\text{minimal overlapping size of } (\Omega_i))^{-1} \]
The Huang-Xu Approach

\( \Omega \) is an open set in \( \mathbb{R}^d \).
Consider an overlapping domain decomposition of \( \Omega \).

\[ \Omega = \bigcup_{i=1}^{p} \Omega_i. \]

Each \( \Omega_i \) is partitioned by a quasi-uniform triangulation \( \mathcal{T}^{h_i} \) of maximal mesh size \( h_i \).
With each triangulation $\mathcal{T}^{h_i}$, associate a finite element subspace $S^{h_i}(\Omega_i) \subset H^r(\Omega_i)$.

Let $u \in H^r(\Omega)$, and let $m_i \geq 1$ denote additional degree of smoothness of $u$ on $\Omega_i$.

Assume optimal approximation properties on subdomains: For any $u \in H^{m_i+r}(\Omega_i)$, there exists $v_h \in S^{h_i}(\Omega_i)$ such that

$$
\sum_{k=0}^{r} h_i^k |u - v_h|_{k, \Omega_i} \lesssim h_i^{m_i+r} \|u\|_{m_i+r, \Omega_i}.
$$
Partition of Unity Functions

The main ingredient is a partition of unity \( \{ \phi_i \} \) associated with the overlapping subdomains \( \{ \Omega_i \} \).

\[
\begin{align*}
0 & \leq \phi_i \leq 1, \\
\sum_i \phi_i & \equiv 1, \\
\text{supp}(\phi_i) & \subset \bar{\Omega}_i, \phi_i \in W^{r,\infty}(\Omega), \\
|\nabla^k \phi_i| & \lesssim d_i^{-k} \quad 1 \leq k \leq r,
\end{align*}
\]

where \( d_i \) is the minimal overlapping size of \( \Omega_i \) with its neighboring subdomains.
Use PU to Create Global Spaces

Use the partition of unity to glue together the subspaces $S^{h_i}(\Omega_i)$.

$V^h(\Omega) = \sum_{i=1}^{p} \phi_i S^{h_i}(\Omega_i) = \left\{ v = \sum_{i=1}^{p} \phi_i v_i \mid v_i \in S^{h_i}(\Omega_i) \right\}$.

Theorem (Huang - Xu): If the overlapping size $d_i \geq c h_i$, then for $0 \leq k \leq r$,

$$\inf_{v_h \in V^h(\Omega)} \| u - v_h \|_{k,\Omega} \lesssim \sum_{i=1}^{p} h_i^{m_i+r-k} \| u \|_{m_i+r-k,\Omega_i},$$

for any $u \in H^r(\Omega) \bigcap_{i=1}^{p} H^{m_i+r}(\Omega_i)$. 

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The Steady-State Stokes Problem

\begin{align*}
-\Delta u - \nabla p &= F \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \\
\int_{\Omega} p &= 0.
\end{align*}

The variational formulation: Find \( u \in V \) and \( p \in P \) s.t.

\begin{align*}
\begin{cases}
a(u, v) + b(v, p) &= (F, v) \quad \text{for all } v \in V, \\
b(u, q) &= 0 \quad \text{for all } q \in P.
\end{cases}
\end{align*}

\((\cdot, \cdot)\)- the \( L^2 \)-inner product, \( V := (H^1_0(\Omega))^2 \), \( P := L^2_0(\Omega) \),

\[ a(u, v) = \sum_{i=1}^{2} \int_{\Omega} \nabla u_i \cdot \nabla v_i \, dx, \quad \text{and} \quad b(v, q) = (q, \nabla \cdot v). \]
(LBB) Condition - Stability

The inf-sup condition

\[ c_0 \|p\| \leq \sup_{v \in V} \frac{(p, \nabla \cdot v)}{\|v\|_{1,\Omega}}, \quad \text{for all } p \in P, \]

holds for a positive constant \( c_0 \). We are interested in building stable pairs \((V_h, P_h)\), \( V_h \subset V \) and \( P_h \subset P \), i.e., pairs \((V_h, P_h)\) which satisfy the discrete inf-sup condition

\[ c_0 \|p\| \leq \sup_{v \in V_h} \frac{(p, \nabla \cdot v)}{\|v\|_{1,\Omega}}, \quad \text{for all } p \in P_h. \]

\[
\begin{cases}
    a(u_h, v) + b(v, p_h) = (F, v) & \text{for all } v \in V_h, \\
    b(u_h, q) = 0 & \text{for all } q \in P_h,
\end{cases}
\]
\( \Omega_1, \Omega_2 \) -overlapping subdomains of \( \Omega = \Omega_1 \cup \Omega_2 \).

Assume that \( \Omega_1 \) and \( \Omega_2 \) are partitioned by quasi-uniform finite element triangulations \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) of maximal mesh sizes \( h_1 \) and \( h_2 \) (which might not match on \( \Omega_0 = \Omega_1 \cap \Omega_2 \)).

Assume that \( \Omega_0 \) is a strip-type domain of width \( d = O(h_1) \).
Let \( \{\phi_1, \phi_2\} \) be a partition of unity subordinate to the covering partition \( \{\Omega_1, \Omega_2\} \) of \( \Omega \), i.e.

\[
\phi_1 + \phi_2 = 1, \quad 0 \leq \phi_i \leq 1, \quad \| \nabla \phi_i \|_{\infty, \Omega} \lesssim 1/d,
\]
and \( \phi_1 \equiv 1 \) on \( \Omega_1 \setminus \Omega_0 \), and \( \phi_1 \equiv 0 \) on \( \Omega_2 \setminus \Omega_0 \).

The local discrete spaces for velocity and pressure:

\[
V_{h_i}(\Omega_i) := \{ \mathbf{v} \in (H^1_0(\Omega_i; \partial \Omega \cap \partial \Omega_i))^2 \mid \mathbf{v}|_K \in P_1 \},
\]

\[
P_{h_i}(\Omega_i) := \{ p \in C^0(\Omega_i) \mid p|_K \in P_1 \},
\]

\[
\hat{P}_{h_i}(\Omega_i) := \{ p \in P_{h_i}(\Omega_i) \mid p = 0 \text{ on } \partial \Omega_i \setminus \partial \Omega \}.
\]
First Example of Mini-Element Pair

Let \( B \) be the space of bubble functions associated with the "union" partition \( \mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_2 \) as follows. For a triangle \( T \), we define the bubble function \( B_T \) supported on \( T \) as the product of the nodal functions associated with the vertices of \( T \).

If \( K = T_1 \cap T_2 \in \mathcal{T}_1 \cup \mathcal{T}_2 \) we define

\[
B_K := B_{T_1} \cdot B_{T_2}.
\]

If \( K = T_i \) for some \( T_i \in \mathcal{T}_i, (i = 1, 2) \), then we just take \( B_K := B_{T_i} \).

Take \( B := \text{Span}\{B_K | K \in \mathcal{T}\} \).
A composite conforming finite element space for velocity can be defined by

\[ \mathbf{V}_h \equiv \mathbf{V}_h(\Omega) := \phi_1 \mathbf{V}_h^1 + \phi_2 \mathbf{V}_h^2 + B^2. \]

The discrete pressure space associated with \( \mathbf{V}_h \) is

\[ P_h := (\hat{P}_h^1(\Omega_1) + \hat{P}_h^2(\Omega_2)) \cap P. \]
Main Result

Assumptions:

(A0) \( h := h_1 \geq h_2 = rh_1 \), for some positive constant \( r \).

(A1) There exists a positive constant \( c \) such that \( |K| \approx ch^2 \) for any \( K \in \mathcal{T} \), where \( |K| \) denotes the Lebesgue measure of \( K \in \mathcal{T} \).

Theorem 1 : The pair \((V_h, P_h)\) defined above is a stable pair.
Proof Outline

Construct $\Pi_1 : \mathbf{V} \rightarrow \mathbf{V}_h$, $\Pi_2 : \mathbf{V} \rightarrow \mathbf{V}_h$ s.t.

\[ |v - \Pi_1 v|_{1,\Omega} \lesssim |v|_{1,\Omega}, \quad \text{for all } v \in \mathbf{V}, \quad (1) \]

\[ |\Pi_2 (I - \Pi_1) v|_{1,\Omega} \lesssim |v|_{1,\Omega}, \quad \text{for all } v \in \mathbf{V}, \quad (2) \]

\[ b(v - \Pi_2 v, q) = 0, \quad \text{for all } q \in P_h, v \in \mathbf{V}. \quad (3) \]

Take $\Pi_h = \Pi_1 + \Pi_2 (I - \Pi_1)$.

For $i = 1, 2$ let $\mathbf{V}_i := (H^1_0(\Omega_i; \partial \Omega_i \cap \partial \Omega))^2$, and let $\Pi^i_1 : \mathbf{V}_i \rightarrow \mathbf{V}_{h_i}$ be Clement operators.

Define $\Pi_1$ by

\[ \Pi_1 v := \phi_1 \Pi^1_1(v_{|\Omega_1}) + \phi_2 \Pi^2_1(v_{|\Omega_2}). \]
The definition of $\Pi_2$

For $v \in V$, $K \in T$, we define

$$\Pi_2 v|_K := \alpha B_K,$$

where $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ is chosen such that

$$\int_K (v - \Pi_2 v) \, dx = 0,$$

i.e.,

$$\alpha = \frac{\int_K v \, dx}{\int_K B_K \, dx}.$$

For $v \in V$ and $q \in P_h$, we have

$$b(v - \Pi_2 v, q) = -(\Pi_2 v - v, \nabla q) = -\sum_{K \in T} \nabla q \int_K (\Pi_2 v - v) = 0.$$
Special Case

When any $K \in T$ is a triangle in either $T_1$ or $T_2$, the condition (A1) is satisfied. For example, any $T_1 \in T_1$, $T_1 \subset \Omega_0$ is a union of triangles of $T_2$.

(A1) $|K| = |T_i| \approx h_i^2$ for any $K \in T$.

Following the proof of Theorem 1, we deduce that the constants involved are also independent of the ratio $r = h_2/h_1$.

**Corollary**: The discrete inf-sup condition holds with a constant $c_0$ independent of $h_2$, $h_1$ and $r = h_2/h_1$. 
The space $V_h$ has good approximation properties,

$$\inf_{v_h \in V_h} \| v - v_h \|_{1,\Omega} \lesssim h_1 \| v \|_{1,\Omega_1} + h_2 \| v \|_{1,\Omega_2}, \quad \text{for all } v \in V.$$ 

If $P_h = \hat{P}_{h_1}(\Omega_1) + \hat{P}_{h_2}(\Omega_2)$ is a linear space containing the locally constant functions, then $P_h$ has the following approximation property:

$$\inf_{p_h \in P_h} \| P - P_h \|_{0,\Omega} \lesssim h \| P \|_{1,\Omega}, \quad \text{for all } p_h \in P_h.$$ 

**Conclusion:** The pair $(V_h, P_h)$ has good approximation properties and is a stable pair.
Consider $P_h$ defined using the PU as

$$P_h := (\phi_1 P_{h_1}(\Omega_1) + \phi_2 P_{h_2}(\Omega_2)) \cap P.$$ 

$P_h$ has better approximation properties. The pressure space is enriched on the overlapping region. To prove stability we enrich the bubble space $B$.

For each $K \in T, K \subset \Omega_0$, we consider $B_1^1, B_2^2$ two bubble functions supported on $K$.

$$B := Span\{B_1^1, B_2^2 | K \in T\} \quad \text{and} \quad V_h := \phi_1 V_{h_1} + \phi_2 V_{h_2} + B^2.$$ 

**Theorem 2**: The new pair $(V_h, P_h)$ is a stable pair.
Non-overlapping non-matching grids

Extend the mesh of one subdomain inside the neighboring subdomain.
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Remarks and future plans

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- The condition \((A1)\) is too restrictive. In practice, we can slightly change the mesh by moving points of the mesh towards other very close points or edges.

- We conjecture that other classical stable pairs (for example, \((\mathbb{P}_2, \mathbb{P}_1)\)) could be glued by a \(\text{PU}\) in order to construct stable pairs with good approximation properties.
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We will further investigate the PU method for mixed finite elements on overlapping grids.