New Convergence Results for the Parareal Algorithm Applied to ODEs and PDEs

Martin J. Gander
martin.gander@math.unige.ch

Université de Genève

Joint work with Stefan Vandevalle
The Parareal Algorithm

The parareal algorithm for the model problem

\[ u' = f(u), \quad u(0) = u_0 \]

is defined using two propagation operators:

1. \( G(t_1, t_2, u_1) \) is a rough approximation to \( u(t_2) \) with initial condition \( u(t_1) = u_1 \),

2. \( F(t_1, t_2, u_1) \) is a more accurate approximation of the solution \( u(t_2) \) with initial condition \( u(t_1) = u_1 \).

Starting with a coarse approximation \( U_k^0 \) at the coarse time points \( t_1, t_2, \ldots, t_k \), the parareal algorithm performs for \( n = 0, 1, \ldots \) the correction iteration

\[ U_{k+1}^{n+1} = G(t_{k+1}, t_k, U_k^{n+1}) + F(t_{k+1}, t_k, U_k^n) - G(t_{k+1}, t_k, U_k^n). \]
Convergence Results for Linear Problems

For the linear model problem

\[ u' = -au, \quad u(0) = u_0, \quad \Re(a) \geq 0. \]

**Theorem (Superlinear Convergence):** Let \( F(t_{k+1}, t_k, U_k^n) \) denote the exact solution at \( t_{k+1} \) and \( G(t_{k+1}, t_k, U_k^n) = \beta U_k^n \) be a one step method. If the method is in its region of absolute stability, \( |\beta| \leq 1 \), then at iteration \( n \), we have

\[
\max_{1 \leq k \leq \bar{k}} |u(t_k) - U_k^n| \leq \frac{|e^{-a\Delta T} - \beta|^n n! \prod_{j=1}^{n} (\bar{k} - j) \max_{1 \leq k \leq \bar{k}} |u(t_k) - U_k^0|}{n! \prod_{j=1}^{n} (\bar{k} - j) \max_{1 \leq k \leq \bar{k}} |u(t_k) - U_k^0|}.
\]

If the local truncation error is bounded by \( C \Delta T^{p+1} \), then

\[
\max_{1 \leq k \leq \bar{k}} |u(t_k) - U_k^n| \leq \frac{(CT)^n \Delta T^{pn}}{n!} \max_{1 \leq k \leq \bar{k}} |u(t_k) - U_k^0|.
\]
Convergence Results for Linear Problems

Theorem (Linear Convergence): Let $F(t_{k+1}, t_k, U^m_k)$ denote the exact solution at $t_{k+1}$ and $G(t_{k+1}, t_k, U^n_k) = \beta U^n_k$ be a one step method. If $\Delta T$ is such that the method is in its region of absolute stability, then at iteration $n$, we have

$$\sup_{k > 0} |u(t_k) - U^n_k| \leq \left( \frac{|e^{-a\Delta T} - \beta|}{1 - |\beta|} \right)^n \sup_{k > 0} |u(t_k) - U^0_k|.$$  

If the local truncation error is bounded by $C\Delta T^{p+1}$, then for $\Delta T$ small, we have

$$\sup_{k > 0} |u(t_k) - U^n_k| \leq \left( \frac{C\Delta T^p}{\Re(a) + O(\Delta T)} \right)^n \sup_{k > 0} |u(t_k) - U^0_k|.$$  

Note: uniform convergence for all time
Convergence for the Heat Equation

**Corollary:** The parareal algorithm applied to the heat equation \( u_t = \Delta u \) discretized with an L-stable method in time converges superlinearly on bounded time intervals,

\[
\max_{1 \leq k \leq \bar{k}} \| u(t_k) - U^n_k \|_2 \leq \frac{\gamma_s^n}{n!} \prod_{j=1}^{n} (\bar{k} - j) \max_{1 \leq k \leq \bar{k}} \| u(t_k) - U^0_k \|_2,
\]

where the constant \( \gamma_s < 1 \) is universal for each L-stable method. On unbounded time intervals the convergence is linear,

\[
\sup_{k > 0} \| u(t_k) - U^n_k \|_2 \leq \gamma_l^n \sup_{k > 0} \| u(t_k) - U^0_k \|_2,
\]

where again the constant \( \gamma_l < 1 \) is universal for each L-stable method.
## Convergence Constants for the Heat Equation

<table>
<thead>
<tr>
<th>method</th>
<th>order</th>
<th>( \gamma_s )</th>
<th>( \gamma_l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BE</td>
<td>1</td>
<td>0.2036321888</td>
<td>0.2984256075</td>
</tr>
<tr>
<td>SDIRK 3.1</td>
<td>3</td>
<td>0.1717941220</td>
<td>0.2338191487</td>
</tr>
<tr>
<td>SDIRK 3.2</td>
<td>3</td>
<td>0.2073822267</td>
<td>0.1718033767</td>
</tr>
<tr>
<td>Radau IIA</td>
<td>5</td>
<td>0.0634592650</td>
<td>0.0677592165</td>
</tr>
</tbody>
</table>
Convergence for pure Advection Problems

Corollary: The parareal algorithm applied to the advection equation $u_t = u_x$ with backward Euler in Time converges superlinearly on bounded time intervals,

$$\max_{1 \leq k \leq \bar{k}} ||u(t_k) - U^n_k||_2 \leq \frac{\alpha^n_s}{n!} \prod_{j=1}^{n} (k - j) \max_{1 \leq k \leq \bar{k}} ||u(t_k) - U^0_k||_2,$$

where the constant $\alpha_s$ is universal, $\alpha_s = 1.224353426$.

Remarks:

- There is no convergence result for unbounded time intervals in the case of pure advection.

- The number of iterations in parareal is limited: as soon as more than $\bar{k}$ iterations are needed, the method looses all interest, since it takes more time than the sequential one.
Speedup

We define the speedup of the parareal algorithm by $\frac{k}{n}$, where $k$ is the number of processors (number of coarse intervals), and $n$ is the number of iterations to achieve a given precision $\varepsilon$.

To quantify the speedup of the parareal algorithm, we need to study for $n < k$ the function

$$f(\gamma, k, n) := \frac{\gamma^n}{n!} \prod_{j=1}^{n} (k - j) \leq \frac{\gamma^n}{n!} \frac{k^n}{e^n (k - n)(k - n)}.$$ 

Goal: for a given $\gamma$ from an L-stable method, and a desired precision $\varepsilon$, find $k$, such that the speedup $k/n$ is maximized.
Maximum Speedup for the Heat Equation

For $\varepsilon \in \{1/10, 1/100, \ldots, 1/100000\}$, the speedups as a function of the number of coarse intervals $\bar{k}$:

Forward Euler

\[ \log_2(\bar{k}) \]

Radau IIA

\[ \log_2(\bar{k}) \]
Speedup for the Advection Equation?

We have

\[ \lim_{n \to k} f(\gamma, k, n) \leq \frac{k^k}{k!e^k} =: f(\gamma, k), \]

and hence for a given \( k \), speedup is possible if

\[ f(\gamma, k) < 1. \]

The limiting case \( f(\gamma, k) = 1 \) defines the function

\[ \gamma = \gamma(k) = \frac{e(n!)^{\frac{1}{n}}}{n}. \]

and for \( \gamma \)-values above this curve, speedup is not possible.
Possible Speedup Depending on $\gamma$

Curves $\bar{f}(\gamma, \bar{k}) = \varepsilon$ for $\varepsilon = \{1, 1/2, \ldots, 1/512\}$.

Note: the limit for $\bar{k}_i$ large is 1 for all $\varepsilon$. 
Nonlinear Convergence Result

For the model problem

\[ u' = f(u), \quad u(0) = u_0. \]

**Theorem (G, Hairer 2004):** Let \( F(t_{k+1}, t_k, U^n_k) \) denote the exact solution at \( t_{k+1} \) and \( G(t_{k+1}, t_k, U^n_k) \) be a one step method with local truncation error bounded by \( C_1 \Delta T^{p+1} \). If

\[ |G(x) - G(y)| \leq (1 + C_2 \Delta T)|x - y|, \]

then

\[
\max_{1 \leq k \leq n} |u(t_k) - U^n_k| \leq \frac{C_1 \Delta T^{n(p+1)}}{n!} (1 + C_2 \Delta T)^{k-1-n} \prod_{j=1}^{n} (k-j) \max_{1 \leq k \leq k} |u(t_k) - U^n_k|. 
\]
Further Estimates

The new term in the nonlinear convergence result can be estimated by

\[ (1 + C_2 \Delta T)^{\bar{k}-1-n} \leq e^{C_2 \Delta T(\bar{k}-1-n)} = e^{C_2(T-(n+1)\Delta T)} \]

and since \( \prod_{j=1}^{n} (\bar{k} - j) \leq \bar{k}^n \), we obtain

\[
\max_{1 \leq k \leq \bar{k}} |u(t_k) - U_k^n| \leq \frac{(C_1 T)^n}{n!} \Delta T^{pn} \max_{1 \leq k \leq \bar{k}} |u(t_k) - U_k^0|.
\]

only difference with linear case
Results for the Lorenz Equations

Using the classical 4-th order Runge-Kutta method for $T = 5$ with 90 coarse intervals and 10 fine steps per interval.
Future work

- Study of the roundoff phenomenon in the case of the Lorenz equations.

- Can one reformulate the parareal algorithm such that it integrates backward and forward to circumvent the roundoff phenomenon?

- Use this to compute highly accurate solutions for the Lorenz equations in parallel using shadowing.

- Study of a multilevel version of the parareal algorithm.