A FETI-DP formulation for the three dimensional elasticity problem with mortar methods

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The Neumann-Dirichlet preconditioner

- FETI-DP methods for 2D elliptic problems with mortar matching (Kim and Lee (2002))

\[ \kappa(M_{ND}^{-1} F_{DP}) \leq C \max_{i=1,\ldots,N} \left\{ (1 + \log(H_i/h_i))^2 \right\} \]

\[ M_{ND}^{-1} = \sum_{i=1}^{N} B^{(i)} D^{(i)} S^{(i)} D^{(i)} B^{(i)\text{T}}, \]

\( n \) : dofs on nonmortar, \( m \) : dofs on remaining parts

\[ D^{(i)} = \begin{pmatrix} D_{mm}^{(i)} & 0 \\ 0 & D_{nn}^{(i)} \end{pmatrix}, \]

\[ D_{mm}^{(i)} = 0, \quad D_{nn}^{(i)} = \left( B_n^{(i)\text{T}} B_n^{(i)} \right)^{-1}. \]
• Efficient one for the problems with discontinuous coefficients
• Extended to 3D elliptic, and 2D Stokes problem
• Well connected to the BDDC method

$$ (\text{BDDC method}) \quad M_{BDDC}^{-1} \quad \cdots \quad M_{ND}^{-1} \quad (\text{FETI-DP method}) $$

$$(E_D, \ P_D : \text{average and jump operators})$$

$$ E_D + P_D = I, $$

$$ E_D^2 = E_D, \quad P_D^2 = P_D, $$

$$ E_D P_D = P_D E_D = 0 $$
Elasticity problem

Partition of domain $\Omega \in \mathbb{R}^3$

$$\partial \Omega = \partial \Omega_D \cup \partial \Omega_N \ (D: \text{Dirchlet B.C.} \ N: \text{natural B.C.})$$

$$\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega_i} \ (\text{geometrically conforming})$$

Find $u \in [H_D^1(\Omega)]^3$ such that

$$\sum_{i=1}^N \left( \int_{\Omega_i} G_i \varepsilon(u) : \varepsilon(v) \, dx + \int_{\Omega_i} G_i \beta_i \nabla \cdot u \nabla \cdot v \, dx \right) = \langle F, v \rangle \quad \forall v \in [H_D^1(\Omega)]^3,$$

(1)

$$\langle F, v \rangle = \int_{\Omega} f \cdot v \, dx + \int_{\partial \Omega_N} g \cdot v \, ds$$

$$\varepsilon(u)_{lk} := \frac{1}{2} \left( \frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} \right) \ (\text{strain tensor})$$

$$G_i = E_i / (1 + \nu_i), \quad \beta_i = \nu_i / (1 - 2\nu_i), \quad E_i > 0, \ \nu_i \in (0, 1/2].$$

We consider the case of **Compressible Elasticity** ($\nu_i \leq \gamma < 1/2$).
• Finite Elements

\[ T_i : \text{ a quasi-uniform triangulation in } \Omega_i, \]
\[ X_i : P^1\text{-conforming finite elements based on } T_i, \]
\[ \{T_i\}_{i=1}^N : \text{ nonmatching across interfaces}, \]
\[ X = \{v : v|_{\Omega_i} \in X_i, \quad i = 1, \ldots, N\} \text{ Nonconforming} \]
• Mortar Matching Condition

\[ W_{ij} : \text{finite elements on } F^{ij} \text{ equipped with interior nodes of the nonmortar} \]

\[ M_{ij} : \text{dual/standard Lagrange multiplier space} \]

\[ \Omega_i \text{ (nonmortar) , } \Omega_j \text{ (mortar) if } G_i \leq G_j, \]

\[ \int_{F^{ij}} (v_i - v_j) \cdot \lambda \, ds = 0, \quad \forall \lambda \in M_{ij}, \forall F^{ij}. \]
Primal variables

- Redundant constraints
  1. Point-wise matching condition (matching grids)
     \(3D\) Elliptic : Face or Edge average constraints
     (Klawonn, Widlund, Dryja (2002))
     \[
     \int_{F_{ij}} (v_i - v_j) \, ds = 0, \quad \int_{E_{lk}} (v_i - v_j) \, ds = 0.
     \]

     \(3D\) Elasticity : Edge average or momentum constraints
     (Klawonn, Widlund (2004))
     \[
     \int_{E_{lk}} (v_i - v_j) \, ds = 0, \quad \int_{E_{lk}} (v_i - v_j) \cdot r \, ds = 0. \quad (r : \text{rotation})
     \]

  2. Mortar matching condition (nonmatching grids)
     \(3D\) Elliptic : Face average constraints (Kim (2004))
     Note: Edge constraints are not redundant to mortar matching
Why we need more redundant constraints than three face average constraints for the elasticity problem?

1. Ker(ε) has six rigid body motions $\{r_i\}_{i=1}^6$ as its basis:

   $r_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $r_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $r_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $r_4 = \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}$, $r_5 = \begin{pmatrix} -x_3 \\ 0 \\ x_1 \end{pmatrix}$, $r_6 = \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix}$

2. Vertex constraints are not strong enough to get a good condition number bound
• Projected momentum constraints

1. On the interface $F^{ij} = \partial \Omega_i \cap \partial \Omega_j$, we consider

$$r_4 = \frac{1}{H} \begin{pmatrix} x_2 - \hat{x}_2 \\ -x_1 + \hat{x}_1 \\ 0 \end{pmatrix}, \ r_5 = \frac{1}{H} \begin{pmatrix} -x_3 + \hat{x}_3 \\ 0 \\ x_1 - \hat{x}_1 \end{pmatrix}, \ r_6 = \frac{1}{H} \begin{pmatrix} 0 \\ x_3 - \hat{x}_3 \\ -x_2 + \hat{x}_2 \end{pmatrix},$$

where $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in F^{ij}$ and $H = \text{diam}(F^{ij})$.

2. Projection $Q : [L^2(F^{ij})]^3 \to M_{ij}$

$$\int_{F^{ij}} (Q(w) - w) \cdot \phi \, ds = 0 \quad \forall \phi \in W_{ij}.$$

3. We get three projected momentum constraints:

$$\int_{F^{ij}} (v_i - v_j) \cdot Q(r_l) \, ds = 0, \quad l = 4, \ldots, 6.$$
• Primal variables

Adding more redundant constraints (primal variables) to the FETI-DP formulation causes bigger coarse problem that is becoming the bottleneck of the computation.

Figure 1: Set of primal variables

Primal faces : face average and projected momentum
Primal vertices : pointwise matching
Figure 2: Acceptable face path between $\Omega_i$ and $\Omega_j$ (red: primal, blue: not-primal)

**Definition 1 (Acceptable face path)**

For a pair of subdomains $(\Omega_i, \Omega_j)$ with a common face $F$ and $G_i \leq G_j$, an acceptable face path $\{\Omega_i, \Omega_{k1}, \cdots, \Omega_{kn}, \Omega_j\}$ is a path from $\Omega_i$ to $\Omega_j$ through **primal faces** with the coefficients $G_{kl}$ satisfying

$$TOL \ast \{(1 + \log(H_i/h_i))^{-1}(1 + \log(H_{kl}/h_{kl}))^2\} \ast G_{kl} \geq G_i.$$
**FETI-DP formulation**

- Finite elements with primal constraints
  Finite elements over the **interfaces** \((W)\) and the **nonmortar interfaces** \((W_n)\)
  
  \[
  W = \prod_{i=1}^{N} W_i, \quad (W_i = X_i|_{\partial \Omega_i}), \quad W_n = \prod_{ij \text{ nonmortar}} W_{ij}
  \]

  \(\tilde{W} := \{ w \in W : w \text{ satisfies vertex constraints at the primal vertices}
  \]
  \(\text{and face constraints across the primal faces } \}\),

  \(\tilde{W}_n := \{ w_n \in W_n : w_n \text{ has the zero averages and momentums on primal faces } \}\).

  \[
  \int_{F_{ij}} w_n \, ds = 0,
  \]

  \[
  \int_{F_{ij}} w_n \cdot Q(r_l) \, ds = 0, \quad l = 4, \ldots, 6.
  \]

  Note: \( w \) (zero extension of \( w_n \) into \( W \)) \( \in \tilde{W} \).
• Notation (c: primal vertices, r: the remaining part)

\[ w^{(i)}(\in W_i) = \begin{pmatrix} w_r^{(i)} \\ w_c^{(i)} \end{pmatrix}, \quad S^{(i)} = \begin{pmatrix} S_{rr}^{(i)} & S_{rc}^{(i)} \\ S_{cr}^{(i)} & S_{cc}^{(i)} \end{pmatrix}, \quad B^{(i)} = \begin{pmatrix} B_r^{(i)} & B_c^{(i)} \end{pmatrix} \]

• Restriction

\[ R_c^{(i)} : V_c \text{ (Set of primal vertices)} \rightarrow V_c|_{\Omega_i} \]

• Assemble unknowns

\[ w_r = \begin{pmatrix} w_r^{(1)} \\ \vdots \\ w_r^{(N)} \end{pmatrix}, \quad w_c \text{ with } R_c^{(i)} w_c = w_c^{(i)}, \forall i. \]

• Assemble mortar matching matrix

\[ B_r = \begin{pmatrix} B_r^{(1)} & \ldots & B_r^{(N)} \end{pmatrix}, \quad B_c = \sum_{i=1}^{N} B_c^{(i)} R_c^{(i)}. \]
• Assemble stiffness matrix

\[
S_{rr} = \begin{pmatrix}
S_{rr}^{(1)} & \cdots & S_{rr}^{(N)} \\
0 & \ddots & 0 \\
S_{rr}^{(N)} & \cdots & S_{rr}^{(1)}
\end{pmatrix}, \quad S_{rc} = \begin{pmatrix}
S_{rc}^{(1)} R_c^{(1)} \\
\vdots \\
S_{rc}^{(N)} R_c^{(N)}
\end{pmatrix}, \quad S_{cc} = \sum_{i=1}^{N} R_c^{(i)^t} S_{cc}^{(i)} R_c^{(i)}.
\]

\(c\) : unknowns at primal vertices

\(r\) : unknowns at the remaining part
- mortar matching condition

\[ B_r w_r + B_c w_c = 0. \]  \hfill (2)

- redundant constraints

\[ R^t(B_r w_r + B_c w_c) = 0. \]  \hfill (3)

- mixed problem

\[
\begin{pmatrix}
S_{rr} & S_{rc} & B^t_r R & B_r \\
S_{cr} & S_{cc} & B^t_c R & B_c \\
R^t B_r & R^t B_c & 0 & 0 \\
B_r & B_c & 0 & 0
\end{pmatrix}
\begin{pmatrix}
w_r \\
w_c \\
\mu \\
\lambda
\end{pmatrix}
= \begin{pmatrix}
g_r \\
g_c \\
0 \\
0
\end{pmatrix}.
\]

- FETI-DP equation

\[ F_{DP} \lambda = d. \]
- **Preconditioner** $M_{ND}^{-1}$

  $$\langle M_{ND} \lambda, \lambda \rangle = \max_{w_n \in \tilde{W}_n} \frac{\langle Bw, \lambda \rangle^2}{\langle Sw, w \rangle}, \quad (4)$$

  $$S = \text{diag}_{i=1,\ldots,N} \left( S(i) \right), \quad B = \left( B^{(1)} \ldots B^{(N)} \right)$$

  $w$: zero extension of $w_n$. (Note that $w \in \tilde{W}$)

- **Lower bound**

  $$\langle F_{DP} \lambda, \lambda \rangle = \max_{w \in \tilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\langle Sw, w \rangle}. \quad (5)$$

  Therefore we have the lower bound

  $$\langle M_{ND} \lambda, \lambda \rangle \leq \max_{w \in \tilde{W}} \frac{\langle Bw, \lambda \rangle^2}{\langle Sw, w \rangle} = \langle F_{DP} \lambda, \lambda \rangle. \quad (6)$$

  We will show the upper bound

  $$\langle F_{DP} \lambda, \lambda \rangle \leq C \langle M_{ND} \lambda, \lambda \rangle.$$. 
Condition number analysis

Assumption on mesh size $G_i \leq G_j$

$$\frac{h_j}{h_i} \leq \left( \frac{G_j}{G_i} \right)^\gamma, \quad 0 \leq \gamma \leq 1.$$  

Mortar projection $\pi_{ij} : [L^2(F^{ij})]^3 \to W^{ij}$

$$\int_{F^{ij}} (\pi_{ij}(w) - w) \cdot \psi \, ds = 0, \quad \forall \psi \in M^{ij}.$$

- Primal face $F$

Lemma 1 For $F(= \partial \Omega_i \cap \partial \Omega_j)$, a primal face with $G_i \leq G_j$, and $w \in \tilde{W}$, we have

$$G_i \| \pi_{ij}(w_i - w_j) \|_{H^{1/2}_0(F)}^2 \leq C \left\{ \left( 1 + \log \frac{H_i}{h_i} \right)^2 |w_i|_{S_i}^2 + \left( 1 + \log \frac{H_j}{h_j} \right)^2 |w_j|_{S_j}^2 \right\},$$

where $|w_i|_{S_i}^2 = \langle S^{(i)} w_i, w_i \rangle$. 

• **Not Primal face** $F$

Assumptions on mesh sizes and the acceptable face path

$$\frac{h_j}{h_i} \leq \left( \frac{G_j}{G_i} \right) ^{\gamma}, \quad 0 \leq \gamma \leq 1$$

$$\text{TOL} \ast \left\{ (1 + \log(H_i/h_i))^{-1} (1 + \log(H_{k_l}/h_{k_l})) \right\} \ast G_{k_l} \geq G_i$$

**Lemma 2** For $F(= \partial \Omega_i \cap \Omega_j)$, not primal face with $G_i \leq G_j$, we assume that there is an acceptable face path $\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_j\}$. Then we have

$$G_i \| \pi_{ij}(w_i - w_j) \|^2_{H^{1/2}_0(F)} \leq C(\text{TOL}) \sum_{l \in N_{ij}} \left( 1 + \log \frac{H_l}{h_l} \right) ^2 |w_l|_{S_l}^2,$$

where $N_{ij} = \{i, k_1, \cdots, k_n, j\}$.
• Upper bound

Lemma 3 We have

\[
\langle F_{DP} \lambda, \lambda \rangle \leq C \max_{i=1,\ldots,N} \left\{ \left( 1 + \log \frac{H_i}{h_i} \right)^2 \right\} \langle M_{ND} \lambda, \lambda \rangle,
\]

where \( C = C(TOL, L) \) with \( L \) (maximum face path length).

• Condition number bound

Theorem 2 For the elasticity problem with discontinuous material coefficients, we obtain

\[
\kappa(M_{ND}^{-1} F_{DP}) \leq C(TOL, L) \max_{i=1,\ldots,N} \left\{ \left( 1 + \log \frac{H_i}{h_i} \right)^2 \right\}.
\]
Algorithm for selecting primal faces

Definition 3 (Essentially primal face) A face $F = \partial \Omega_i \cap \partial \Omega_j$ is essentially primal, if there is no acceptable face path for $(\Omega_i, \Omega_j)$ with all faces except the face $F$ chosen as primal faces.

Figure 3: Essentially primal face $F = \partial \Omega_i \cap \partial \Omega_j$ when $TOL = 10, \frac{H_i}{h_i} = 4$

$$TOL \ast \left\{ \left( 1 + \log \frac{H_i}{h_i} \right)^{-1} \left( 1 + \log \frac{H_{k_i}}{h_{k_i}} \right)^2 \right\} \ast G_{k_i} \geq G_i$$
Algorithm \((TOL, L, \{G_l, H_l, h_l\})\) given

- **Step 1.** Determine essentially primal faces and add them into the primal face set \(P\).
- **Step 2.** Determine not-primal faces based on the set \(P\).
- **Step 3.** Order the undetermined faces decreasingly according to the ratio of coefficients across the faces \(F = \partial \Omega_i \cap \partial \Omega_j\) according to the number of neighbors what the subdomains \(\Omega_i\) and \(\Omega_j\) have.
- **Step 4.** Do until every undetermined face \(F\) determined
  - Add undetermined face \(F\) into primal face set \(P\)
  - Then determine not primal faces based on the updated primal face set \(P\)

End
Test example

\( \Omega = [0, 1]^3 \) partitioned into \( N^3 \) cubical subdomains

\( L = 6, \ TOL = 10 \)

In case constant coefficient \( G_i = 1.0 \) for all \( i \)

In case random coefficient \( 1, 10, 10^2, 10^3 \) are randomly distributed

<table>
<thead>
<tr>
<th>( N^3 )</th>
<th>Total</th>
<th>Optimal ( (N^3 - 1) )</th>
<th>Primal (const)</th>
<th>Primal (random)</th>
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<td>12</td>
<td>7</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>( 4^3 )</td>
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<td>63</td>
<td>68</td>
<td>89</td>
</tr>
<tr>
<td>( 6^3 )</td>
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<td>246</td>
<td>322</td>
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<tr>
<td>( 8^3 )</td>
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<td>511</td>
<td>646</td>
<td>804</td>
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<tr>
<td>( 10^3 )</td>
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<td>999</td>
<td>1300</td>
<td>1598</td>
</tr>
</tbody>
</table>

Table 1: Number of primal faces for the constant coefficient and randomly distributed coefficients
Conclusion

- For the compressible elasticity problem with discontinuous material coefficients by introducing average and momentum constraints

\[ \kappa(M_{ND}^{-1}F_{DP}) \leq C(TOL, L) \max_{i=1,\ldots,N} \left\{ (1 + \log \frac{H_i}{h_i})^2 \right\} \]

- Reduced the number of primal variables with the concept of primal and not-primal faces

- Geometrically nonconforming subdomain partition by introducing Change of Basis formulation

- Three-level BDDC method in order to solve the huge coarse problem efficiently

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