Robin Interface Conditions
for
an Overlapping Schwarz Algorithm

Jung-Han Kimn

Department of Mathematics
and
The Center for Computation and Technology
Louisiana State University
kimn@math.lsu.edu
Plan of the Talk and Notes

1. Plan
   - Robin Interface Condition
   - Discontinuous Overlapping Schwarz Method (OSM-D)
     - Convergence Theory of OSM-D
     - Numerical Results

2. Notes
   - We consider Poisson’s equation with a Robin boundary condition on the original boundary.
   - We emphasis the use of Schwarz methods as solver than preconditioners
Model problem

\[- \Delta u = f \quad \text{in} \quad \Omega, \quad (1)\]

\[u + \alpha \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial \Omega,\]

\(\Omega: \text{Lipschitz and } \alpha > 0.\)

Find \(u \in H^1(\Omega)\) such that,

\[a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + \frac{1}{\alpha} \int_{\partial \Omega} u v
\]

\[= \int_{\Omega} f v + \frac{1}{\alpha} \int_{\partial \Omega} g v = F(v), \quad \forall v \in H^1(\Omega). \quad (2)\]

A Convergence Analysis based on Lagrange Multipliers.
\[ -\Delta u = f \quad \text{in} \quad \Omega, \]
\[ u = g \quad \text{on} \quad \partial\Omega \]

Converges in the maximum norm at a **geometric** rate.
Robin Iteration Method of P. L. Lions

\[- \Delta u = f \quad \text{in} \quad \Omega, \]
\[u = g \quad \text{on} \quad \partial\Omega\]  \hfill (4)

- The convergence of nonoverlapping domain decomposition methods by using energy estimate.

\[A_j = \frac{1}{4\tilde{\alpha}} \int_{\Gamma_j} (e_j + \tilde{\alpha} \frac{\partial e_j}{\partial n_j})^2 dS, \quad B_j = \frac{1}{4\tilde{\alpha}} \int_{\Gamma_j} (e_j - \tilde{\alpha} \frac{\partial e_j}{\partial n_j})^2 dS,\]

\[E_j = |e_j|_{H^1(\Omega_i)} \quad E_1^{n+1} + B_1^{n+1} = A_1^{n+1}, \quad E_2^{n+1} + B_2^{n+1} = A_2^{n+1},\]
\[A_1^{n+1} = B_2^n, \quad A_2^{n+1} = B_1^{n+1}.\]

\[\sum_{n=1}^{\infty} (E_1^n + E_2^n) = B^0. \quad \rightarrow \quad \lim_{n \to \infty} (E_1^n + E_2^n) = 0.\]
Classical Algorithm

- Splitting $a(\cdot, \cdot)$ into $\hat{a}_j(\cdot, \cdot)$ and $\hat{a}_j^c(\cdot, \cdot)$, where

$$
\hat{a}_j(u, v) = \int_{\Omega_j} \nabla u \cdot \nabla v + \frac{1}{\alpha} \int_{\Theta_j} u v,
$$

$$
\hat{a}_j(u, v) = F(v) - \hat{a}_j^c(u, v).
$$

- Compute $u_{n+j/p}$ by using the old values in the $\hat{a}_j^c(\cdot, \cdot)$ and solving a problem defined by the $\hat{a}_j(\cdot, \cdot)$ for new values.

- The Dirichlet interface condition,

$$
u_{n+j/p} = u_{n+(j-1)/p} \quad \text{on} \quad \Gamma_j, \ j = 1, \cdots, p,$$

- The continuity of the iterates is preserved by keeping the old values on $\Omega_j^c$. 
A Robin Interface Condition

A Robin interface condition on $\Gamma_j$,

$$u_{n+j/p,j} + \tilde{\alpha} \frac{\partial u_{n+j/p,j}}{\partial n_j} = u_{n+(j-1)/p,j}^c - \tilde{\alpha} \frac{\partial u_{n+(j-1)/p,j}^c}{\partial n_{j}^c}$$

on $\Gamma_j$. \hfill (5)

$\tilde{\alpha}$ is a Robin parameter for the artificial interface and not necessarily equal to $\alpha$.

The continuity is not enforced because a natural boundary condition is used and the new values on the artificial interface are generally not equal to the old ones.

Two alternatives:

- The new values replace the old ones. $\rightarrow$ OSM-C (Continuous)
- The new and old values are kept. $\rightarrow$ OSM-D (Discontinuous)
\{x | 0 \leq x \leq 1\} \text{ and } \{x | \delta_2 = 1/2 - \delta \leq x \leq 1/2 + \delta = \delta_1\}

- \quad f(0) - \alpha f'(0) = 0 \rightarrow f_1(x) = a(x + \alpha)
- \quad f(1) + \alpha f'(1) = 0 \rightarrow f_2(x) = c(x - 1 - \alpha).
- \quad (\delta_1 + \alpha)a + \tilde{\alpha}a = (\delta_1 - 1 - \alpha)c + \tilde{\alpha}c.
- \quad a = (1 - \frac{1+2\alpha}{\delta_1+\alpha+\alpha})c \text{ with } |1 - \frac{1+2\alpha}{\delta_1+\alpha+\alpha}| < 1.
Splitting:

\[ a_j(u, v) = F(v) - a_j^c(u, v) \]

\[ a_j(u, v) = \int_{\Omega_j} \nabla u \cdot \nabla v + \frac{1}{\alpha} \int_{\Theta_j} u v + \frac{1}{\tilde{\alpha}} \int_{\Gamma_j} u v. \] (6)

In the \( j \)-th fractional step, the values at all the nodes of the closure of \( \Omega_j \) is updated, while all other nodal values unchanged.

A solution on \( \Omega_j \) which extends continuously to \( \Omega \setminus \Omega_j \).
Discontinuity and new Notations

- Derived by modifying OSM-C and allowing discontinuities across the artificial interfaces $\Gamma_j$.

- The multiple values (discontinuities) on the artificial interfaces produce a jump across $\partial \Omega_j$ in the $j$-th fractional step.

- The **atomic subdomains** $\{\Omega^q\}$ is the smallest collection of the disjoint open sets which generates the subdomains $\{\Omega_j\}$ as unions.

- The local bilinear forms $a_j(\cdot, \cdot)$ are redefined as sums of atomic bilinear forms $a_{\Omega^q}(\cdot, \cdot)$ for $\Omega^q \subset \Omega_j$. 
Atomic Subdomains

Figure 1: Three atomic subdomains and two overlapping subdomains

The OSM-D Algorithm

Using the same local bilinear forms $a_j(\cdot, \cdot)$ of OSM-C.

\[
a_{\Omega^1}(u^1, v) = \int_{\Omega^1} \nabla u^1 \cdot \nabla v + \frac{1}{\alpha} \int_{\Theta^1} u^1 v - \frac{1}{\tilde{\alpha}} \int_{\Gamma^2} u^1 v,
\]

\[
a_{\Omega^{12}}(u^{12}, v) = \int_{\Omega^{12}} \nabla u^{12} \cdot \nabla v + \frac{1}{\alpha} \int_{\Theta^{12}} u^{12} v + \frac{1}{\tilde{\alpha}} \int_{\Gamma_1 \cup \Gamma_2} u^{12} v,
\]

\[
a_{\Omega^2}(u^2, v) = \int_{\Omega^2} \nabla u^2 \cdot \nabla v + \frac{1}{\alpha} \int_{\Theta^2} u^2 v - \frac{1}{\tilde{\alpha}} \int_{\Gamma^1} u^2 v.
\]

Only the second is positive definite for any choice of positive $\alpha$ and $\tilde{\alpha}$.

\[
a_1(\tilde{u}_1, v) = a_{\Omega^1}(u^1, v) + a_{\Omega^{12}}(u^{12}, v),
\]

\[
a_2(\tilde{u}_2, v) = a_{\Omega^{12}}(u^{12}, v) + a_{\Omega^1}(u^2, v).
\]

$a_{\Omega^{12}}$ participates in all of the iteration.
The approximate solution $\tilde{u}_{n+i/2}$ in the $i$-th fractional step of the two overlapping subdomain case is updated only in $\Omega_i \cup \partial \Omega_i$, $i = 1, 2$,

$$
\tilde{u}_{n+i/2} = \begin{cases} 
  u_{i,n+i/2} & \text{on } \Omega_i \cup \partial \Omega_i \\
  u_{n+(i-1)/2}^j & \text{on } \Omega_j \cup \partial \Omega_j, \quad j = 3 - i, i = 1, 2.
\end{cases}
$$

The residual corresponding to $\tilde{u}_{n+i/2}$ of the $i$-th fractional step

$$
a(\tilde{u}_{n+i/2}, v) - F(v) = \frac{1}{\tilde{\alpha}} \int_{\Gamma_i} [\tilde{u}_{n+i/2}] v,
$$

where $j = 3 - i$, $i = 1, 2$ and $[\cdot]$ denotes the jump across the interface.

Compute the residual from the jumps across the interfaces.
Finding the three functions $u^1$, $u^2$, and $u^{12}$ that are stationary points.

$$J^q(v^q) = \frac{1}{2} a_{\Omega q}(v^q, v^q) - \int_{\Omega q} f v^q - \frac{1}{\alpha} \int_{\Theta q} g v^q, \quad q = 1, 2, 12,$$

(9)

that satisfy the continuity conditions across the two interfaces,

$$u^1 = u^{12} \quad \text{on} \quad \Gamma_2 \quad u^{12} = u^2 \quad \text{on} \quad \Gamma_1. \quad (10)$$

Equivalent to finding the saddle point of the Lagrangian

$$J^*(v^1, v^2, v^{12}, \mu_1, \mu_2) = \sum_q J^q(v^q) + \sum_{i=1}^2 \int_{\Gamma_i} \mu_i (v^{12} - v^j),$$

where $j = 3 - i$, $i = 1, 2$, $q = 1, 2, 12$. 

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The problem results in a discrete problem and the following algebraic system:

\[
\begin{align*}
A^1 u^1 &= f^1 - (R^1)^T \lambda_2, \quad A^2 u^2 = f^2 - (R^2)^T \lambda_1, \\
A^{12} u^{12} &= f^{12} + (R_{\Gamma_1}^{12,2})^T \lambda_1 + (R_{\Gamma_2}^{12,1})^T \lambda_2 \\
R^1 u^1 &= R_{\Gamma_2}^{12,1} u^{12}, \quad R^2 u^2 = R_{\Gamma_1}^{12,2} u^{12}.
\end{align*}
\] (11)

\[
\begin{pmatrix}
A^1 & 0 & 0 & (R^1)^T & 0 \\
0 & A^{12} & 0 & -(R_{\Gamma_2}^{12,1})^T & -(R_{\Gamma_1}^{12,2})^T \\
0 & 0 & A^2 & 0 & (R^2)^T \\
R^1 & -R_{\Gamma_2}^{12,1} & 0 & 0 & 0 \\
0 & -R_{\Gamma_1}^{12,2} & R^2 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
u^1 \\
u^{12} \\
u^2 \\
\lambda_2 \\
\lambda_1
\end{pmatrix}
= 
\begin{pmatrix}
f^1 \\
f^{12} \\
f^2 \\
0 \\
0
\end{pmatrix}.
\]
For given \( u_{n}^{1} \) and \( u_{n}^{12} \),

\[
\lambda_{1,n} = R^{2}(f^{2} - A^{2}u_{n}^{2}).
\]

Given \( \lambda_{1,n} \),

\[
u_{1,n+1/2} = (A_{1})^{-1}(f_{1} + (R_{1})^{T}\lambda_{1,n}).
\]

The second fractional step:

\[
\lambda_{2,n+1/2} = R^{1}(f^{1} - A^{1}u_{n+1/2}^{1}).
\]

\[
u_{2,n+1} = (A_{2})^{-1}(f_{2} + (R_{2})^{T}\lambda_{2,n+1/2}).
\]

The \( \lambda_{1} \) and \( \lambda_{2} \) play a main role in communicating and updating the primal variables.
Convergence Theorem

If $\Omega$ is a Lipschitz domain, then, for a given Robin parameter $\alpha$ on the original boundary and for any sufficient large $\tilde{\alpha}$, the Robin parameter of the artificial interfaces, there exist a uniform geometric convergence factor $\mu < 1$. This factor depends only on the geometry and mesh size $h$ and is of the form

$$
\mu = \left( \frac{(1 - \rho) \gamma + \sqrt{\gamma^2 (1 - \rho)^2 + 4\rho}}{2} \right)^2,
$$

where

$$
\rho = \frac{(C_F + 1) \max (1, \alpha) C_T}{\tilde{\alpha}} < 1.
$$

Here $C_T$ and $C_F$ are constants which depend only on $\Omega_1$ and $\Omega^1$. Additionally, $\gamma$ is the constant of the strengthened Cauchy-Schwarz inequality of two subspaces and is bounded by,

$$
\gamma \leq \sqrt{\frac{C(1 + \log(\frac{1}{h}))^2 - 1}{C(1 + \log(\frac{1}{h}))^2 + 1}}.
$$

(12)
Proof of Convergence Theorem

- Splitting the error vector on $\Omega_1^{12}$ into two parts depending on two Robin interface conditions.
- Construction two new quantities which will be compared.
- Comparison between two quantities on the general geometry.
- Certain sufficient conditions for the geometric convergence.
- The relation between mesh size and convergence factor in terms of strengthened Cauchy-Schwarz inequalities.
Splitting of the Error Vector

\[ A^{12} e^{12}_{n+\frac{1}{2}} = (R_{\Gamma_1}^{12,2})^T \lambda_{1,n} + (R_{\Gamma_2}^{12,1})^T \lambda_{2,n+\frac{1}{2}} \]

Since \( A^{12} \) is invertible, decompose \( e^{12}_{n+\frac{1}{2}} \) as,

\[ e^{12}_{n+\frac{1}{2}} = e^{12}_{1,n+\frac{1}{2}} + e^{12}_{2,n+\frac{1}{2}}, \text{ with} \]

\[ A^{12} e^{12}_{1,n+\frac{1}{2}} = (R_{\Gamma_1}^{12,2})^T \lambda_{1,n}, \quad A^{12} e^{12}_{2,n+\frac{1}{2}} = (R_{\Gamma_2}^{12,1})^T \lambda_{2,n+\frac{1}{2}}. \]

\[ \| e^{12}_{i,n+\frac{1}{2}} \|_{A^{12}}^2 = (\lambda_{i,n+i-\frac{1}{2}})^T C_i \lambda_{i,n+i-\frac{1}{2}} = \| \lambda_{i,n+i-\frac{1}{2}} \|_{C_i}^2, \quad i = 1, 2. \text{ with} \]

\[ C_i = R_{\Gamma_i}^{12,j} (A^{12})^{-1} (R_{\Gamma_i}^{12,j})^T, \quad j = 3 - i, \quad i = 1, 2. \]

\[ \| \lambda_{2,n+\frac{1}{2}} \|_{C_2} \leq \mu \| \lambda_{1,n} \|_{C_1} \rightarrow \| e^{12}_{2,n+\frac{1}{2}} \|_{A^{12}} \leq \mu \| e^{12}_{1,n+\frac{1}{2}} \|_{A^{12}} \]

However, the two quantities \( \| e^{12}_{i,n+\frac{1}{2}} \|_{A^{12}}, \quad i = 1, 2 \), cannot be compared directly.
Main Issue: Find other comparable quantities.

Idea: Add a common quantity

\[(u, v)_{A^{12}} = \int_{\Omega^{12}} \nabla u \cdot \nabla v + \frac{1}{\alpha} \int_{\Theta^{12}} u v + \frac{1}{\tilde{\alpha}} \int_{\Gamma_1 \cup \Gamma_2} u v = a_{\Omega^{12}}(u, v).\]

\(e^{12}_{i,n+\frac{1}{2}}\) is a solution with zero Robin boundary data except on \(\Gamma_i\)

\[Q_i = \frac{1}{\tilde{\alpha}} \int_{\Gamma_i} e^{12}_{n+\frac{1}{2}} \lambda_{i,n+\frac{i-1}{2}} = (e^{12}_{n+\frac{1}{2}}, e^{12}_{i,n+\frac{1}{2}})_{A^{12}}\]

\[= ||e^{12}_{i,n+\frac{1}{2}}||^2_{A^{12}} + (e^{12}_{1,n+\frac{1}{2}}, e^{12}_{2,n+\frac{1}{2}})_{A^{12}}, \quad i = 1, 2.\]
Comparison between $Q_1$ and $Q_2$

- Main issue: Find a uniform factor $\rho < 1$ such that

$$\rho Q_1 \geq Q_2 \quad (13)$$

$$Q_1 \geq \int_{\Omega_1} \nabla e_{1,n+\frac{1}{2}} \cdot \nabla e_{1,n+\frac{1}{2}} + \frac{1}{\alpha} \int_{\Theta_1} |e_{1,n+\frac{1}{2}}|^2, \quad \frac{1}{\tilde{\alpha}} \int_{\Gamma_2} |e_{n+\frac{1}{2}}^{12}|^2 \geq Q_2.$$ 

- Friedrichs’ Inequality and the trace theorem with $F(\alpha) = (\max(1, \alpha))$

$$\rho Q_1 \geq \rho (C_F+1)^{-1} (F(\alpha))^{-1} C_T^{-1} \int_{\Gamma_2} |e_{n+\frac{1}{2}}^{12}|^2 = \frac{1}{\tilde{\alpha}} \int_{\Gamma_2} |e_{n+\frac{1}{2}}^{12}|^2 \geq Q_2$$

with

$$\rho = \frac{(C_F + 1) F(\alpha) C_T}{\tilde{\alpha}} < 1.$$


\[ \exists \mu \text{ such that } \left\| e_{2,n+\frac{1}{2}}^{12} \right\|_{A^{12}}^2 \leq \mu \left\| e_{1,n+\frac{1}{2}}^{12} \right\|_{A^{12}}^2 \quad \mu < 1 \]

\[ \exists \gamma \text{ such that } \left| (v_1, v_2)_{A^{12}} \right| \leq \gamma \| v_1 \|_{A^{12}} \| v_2 \|_{A^{12}}, \forall v_1 \in V(\Gamma_1), \forall v_2 \in V(\Gamma_2) \]

\[ X = \left\| e_{2,n+\frac{1}{2}}^{12} \right\|_{A^{12}}^2, \quad Y = \left\| e_{1,n+\frac{1}{2}}^{12} \right\|_{A^{12}}^2, \quad Z = \left( e_{1,n+\frac{1}{2}}^{12}, e_{2,n+\frac{1}{2}}^{12} \right)_{A^{12}}, \]

\[ X + (1 - \rho)Z \leq \rho Y \]

If \( Z \geq 0 \), then we have \( X \leq \rho Y \).

If \( Z < 0 \), \( 0 < -Z \leq \gamma \sqrt{XY} \rightarrow \)

\[ 0 < \sqrt{\frac{X}{Y}} \leq b, \quad \text{with} \quad b = \frac{(1 - \rho)\gamma + \sqrt{\gamma^2(1 - \rho)^2 + 4\rho}}{2} < 1. \]

\[ \mu = b^2 = \left( \frac{(1 - \rho)\gamma + \sqrt{\gamma^2(1 - \rho)^2 + 4\rho}}{2} \right)^2 \]
\[ \| e_{1,n+\frac{1}{2}}^{12} \|_{A^{12}}^2 + \| e_{2,n+\frac{1}{2}}^{12} \|_{A^{12}}^2 \leq C_0^2 \| e_{n+\frac{1}{2}}^{12} \|_{A^{12}}^2 \]

\[ \gamma^2 = \cos^2(V(\Gamma_1), V(\Gamma_2)) \leq \frac{C_0^2 - 1}{C_0^2 + 1} < 1 \]

It is enough to show, \[ \| e_{1,n+\frac{1}{2}}^{12} \|_{A^{12}}^2 \leq C^2 \| e_{n+\frac{1}{2}}^{12} \|_{A^{12}}^2 \]

\[ \| \cdot \|_{A^{12}}^2 \text{ and } \| \cdot \|_{H^{-\frac{1}{2}}(\partial \Omega^{12})}^2 \]

are equivalent

\[ \| \tilde{w} \|_{H^{-\frac{1}{2}}(\partial \Omega^{12})}^2 \leq C_{\Gamma_1} \| w \|_{H^{-\frac{1}{2}}(\partial \Omega^{12})}^2, \]

where \( \tilde{w} = w|_{\Gamma_1} \) and \( \tilde{w} = 0 \) on the rest of \( \partial \Omega \)

\[ C_{\Gamma_1} = C(1 + \log(\frac{1}{h}))^2 \]
Table 1: Number of iterations for a residual reduction of $10^{-6}$ for $\alpha$, the number of grid points ($N$), the number of subdomains ($\#(D) \times \#(D)$), and overlapping size (ovlp).

<table>
<thead>
<tr>
<th>$N$</th>
<th>Multiplicative algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(101)^2$</td>
</tr>
<tr>
<td>$#(D) \times #(D)$</td>
<td>$10 \times 10$</td>
</tr>
<tr>
<td>ovlp</td>
<td>1</td>
</tr>
<tr>
<td>$\alpha=0.01$</td>
<td>188</td>
</tr>
<tr>
<td>$\alpha=1$</td>
<td>(**</td>
</tr>
<tr>
<td>$\alpha=100$</td>
<td>(**</td>
</tr>
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## Numerical Results for OSM-D

<table>
<thead>
<tr>
<th>$N$</th>
<th>$(101)^2$</th>
<th>$(101)^2$</th>
<th>$(201)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(D) \times (D)$</td>
<td>$10 \times 10$</td>
<td>$20 \times 20$</td>
<td>$10 \times 10$</td>
</tr>
<tr>
<td>$\alpha, \tilde{\alpha} \backslash$ ovlp</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$\alpha=0.01, \tilde{\alpha}=100$</td>
<td>104</td>
<td>61</td>
<td>166</td>
</tr>
<tr>
<td>$\alpha=0.01, \tilde{\alpha}=1$</td>
<td>40</td>
<td>28</td>
<td>72</td>
</tr>
<tr>
<td>$\alpha=0.01, \tilde{\alpha}=0.01$</td>
<td>91</td>
<td>57</td>
<td>150</td>
</tr>
<tr>
<td>$\alpha=1, \tilde{\alpha}=100$</td>
<td>238</td>
<td>139</td>
<td>462</td>
</tr>
<tr>
<td>$\alpha=1, \tilde{\alpha}=1$</td>
<td>28</td>
<td>22</td>
<td>52</td>
</tr>
<tr>
<td>$\alpha=1, \tilde{\alpha}=0.01$</td>
<td>901</td>
<td>560</td>
<td>(**)</td>
</tr>
<tr>
<td>$\alpha=100, \tilde{\alpha}=100$</td>
<td>47</td>
<td>27</td>
<td>89</td>
</tr>
<tr>
<td>$\alpha=100, \tilde{\alpha}=1$</td>
<td>405</td>
<td>446</td>
<td>751</td>
</tr>
<tr>
<td>$\alpha=100, \tilde{\alpha}=0.01$</td>
<td>(**)</td>
<td>(**)</td>
<td>(**)</td>
</tr>
</tbody>
</table>
Other Extensions

  → A better rate on $\Omega^1$ and $\Omega^2$ for the two overlapping subdomains. (Sum of weighted values is finite.)

- A domain decomposition method for **convection-diffusion equation** with an approximate **factorization of the operator** on several overlapping infinite strips. (F. Nataf (1996))
  → Geometric convergence of OSM-D for several overlapping (finite) strips.

- **Conformal Mapping** (Gaier (1983) and Papamichael (1992))
  → Convergence and Geometric Convergence on overlapping Quadrilaterals.

- **Algebraic Theory** of Multiplicative Schwarz Methods of Benzi, Frommer, Nabben and Szyld (2001)
  → Convergence with $\frac{h}{\alpha} \geq 3/2$ ($h$= mesh size).