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Coupling of the convection diffusion equation and the convection equation

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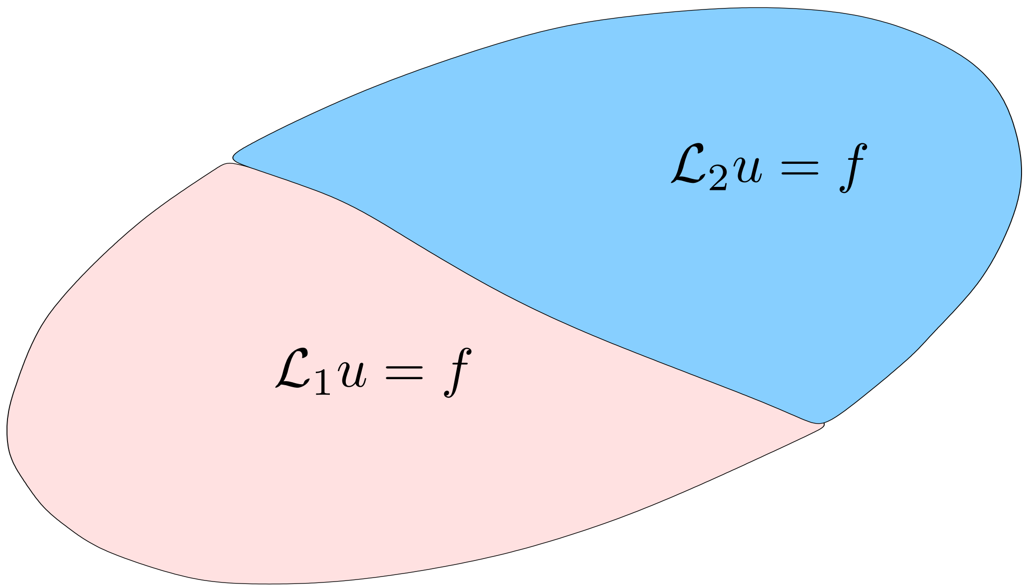
Two motivations for the coupling of equations



$\longrightarrow$ Simulation of multiphysic phenomena

$\longrightarrow$ Simplification of the numerical problem

The equations

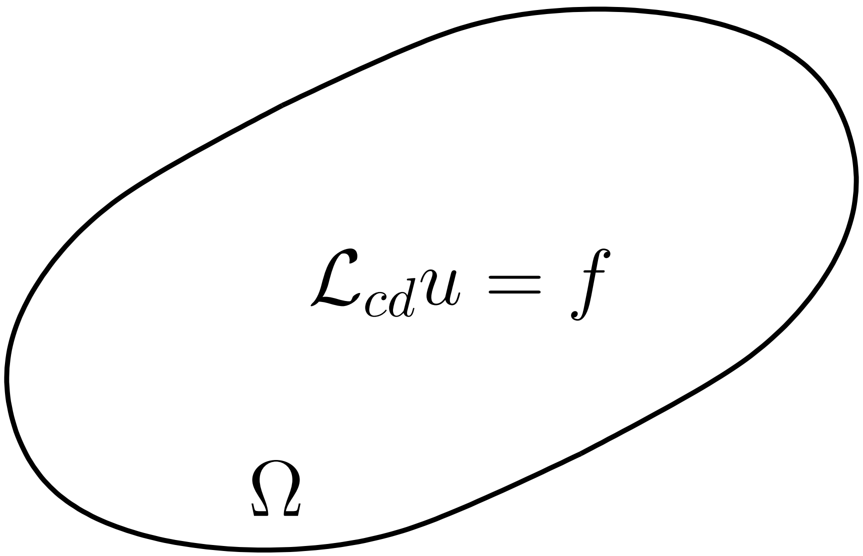
- The convection diffusion equation

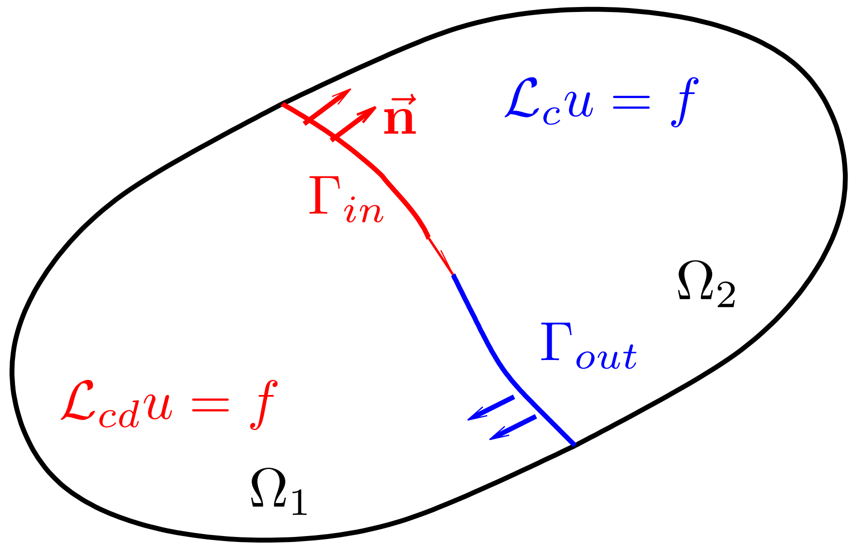
$$\begin{cases} \mathcal{L}_{cd} u_1 \equiv \dfrac{\partial u_1}{\partial t} + \mathbf{a} \cdot \nabla u_1 - \nu \Delta u_1 + c u_1 \text{ in } \Omega_1 \times (0,T) \\ u_1(\cdot,0) = u_0 \text{ in } \Omega_1 \\ + \text{ Boundary conditions on } \partial\Omega \cap \partial\Omega_1 \end{cases}$$

- The convection equation

$$\begin{cases} \mathcal{L}_{c} u_2 \equiv \dfrac{\partial u_2}{\partial t} + \mathbf{a} \cdot \nabla u_2 + c u_2 \text{ in } \Omega_2 \times (0,T) \\ u_2(\cdot,0) = u_0 \text{ in } \Omega_2 \\ + \text{ Boundary condition on } \partial\Omega \cap \partial\Omega_2 \end{cases}$$

Coupling of the convection/convection diffusion equation





$$\Gamma_{in} = \{x \in \Gamma,\ \mathbf{a} \cdot \mathbf{n} > 0\}$$

$$\Gamma_{out} = \{x \in \Gamma,\ \mathbf{a} \cdot \mathbf{n} < 0\}$$

$$\Gamma = \Gamma_{in} \cup \Gamma_{out}$$

Solving the CD equation by a domain decomposition method

- The problem is : find u_1 and u_2 such as

$$\mathcal{L}_{cd}u_1 = f \text{ in } \Omega_1 \qquad \mathcal{L}_{cd}u_2 = f \text{ in } \Omega_2$$

$$u_1 = u_2 \text{ on } \Gamma$$

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \text{ on } \Gamma$$

- We are looking for the fastest algorithm

$$\mathcal{L}_{cd}u_1^{k+1} = f \text{ in } \Omega_1 \qquad \mathcal{L}_{cd}u_2^{k+1} = f \text{ in } \Omega_2$$

$$\mathcal{B}_1(u_1^{k+1}) = \mathcal{B}_1(u_2^k) \text{ on } \Gamma \qquad \mathcal{B}_2(u_2^{k+1}) = \mathcal{B}_2(u_1^k) \text{ on } \Gamma$$

Solving the CD equation by a coupling of equations

- The problem is : find u_1 and u_2 such as

$$\mathcal{L}_{cd} u_1 = f \text{ in } \Omega_1 \qquad \mathcal{L}_c u_2 = f \text{ in } \Omega_2$$

on the interface ?

- What is the corresponding algorithm ?

$$\mathcal{L}_{cd} u_1^{k+1} = f \text{ in } \Omega_1 \qquad \mathcal{L}_c u_2^{k+1} = f \text{ in } \Omega_2$$

$$\mathcal{B}_1(u_1^{k+1}) = \mathcal{B}_1(u_2^k) \text{ on } \Gamma \qquad \mathcal{B}_2(u_2^{k+1}) = \mathcal{B}_2(u_1^k) \text{ on } \Gamma$$

Different ways of defining the coupling for steady problems

- *F. Gastaldi, A. Quarteroni, G. Sacchi Landriani :*

  Definition : Result of a limiting process.

  Key word : Singular perturbations.

  What has been proposed :

  $\longrightarrow$ derivation of the transmission conditions.

  $\longrightarrow$ algorithm.

- *E. Dubach :*

  Definition : Solution that minimizes the error as a function of ν.

  Key word : Absorbing boundary conditions theory.

  What has been proposed :

  $\longrightarrow$ derivation of the transmission conditions.

Conditions of [GQSL]

• Variational arguments prove that the solution of this limiting process satisfies

$$\begin{aligned} -\nu \frac{\partial u_1}{\partial \mathbf{n}} + \mathbf{a}\cdot\mathbf{n}\, u_1 &= \mathbf{a}\cdot\mathbf{n}\, u_2 \quad \text{on } \Gamma = \Gamma_{in} \cup \Gamma_{out} \\ u_1 &= u_2 \quad \text{on } \Gamma_{in} \end{aligned}$$

or equivalently

$$\begin{aligned} -\nu \frac{\partial u_1}{\partial \mathbf{n_1}} + \mathbf{a}\cdot\mathbf{n}\, u_1 &= \mathbf{a}\cdot\mathbf{n}\, u_2 \quad \text{on } \Gamma_{out} \\ u_1 &= u_2 \quad \text{on } \Gamma_{in} \\ -\nu \frac{\partial u_1}{\partial \mathbf{n}} &= 0 \quad \text{on } \Gamma_{in} \end{aligned}$$

New conditions

- Dubach's conditions

$$\begin{cases} u_1 = u_2 & \text{on } \Gamma_{in} \\ \dfrac{\partial u_1}{\partial \mathbf{n}} = \dfrac{\partial u_2}{\partial \mathbf{n}} & \text{on } \Gamma_{in} \\ -\nu \dfrac{\partial u_1}{\partial \mathbf{n}} + \mathbf{a} \cdot \mathbf{n}\, u_1 = \mathbf{a} \cdot \mathbf{n}\, u_2 & \text{on } \Gamma_{out} \end{cases}$$

- Conditions [GJHM]

$$\begin{cases} u_1 = u_2 & \text{on } \Gamma = \Gamma_{in} \cup \Gamma_{out} \\ \dfrac{\partial u_1}{\partial \mathbf{n}} = \dfrac{\partial u_2}{\partial \mathbf{n}} & \text{on } \Gamma_{in} \end{cases}$$

Steady Results in $\mathbb{R}^2$: case $\mathbf{a}\cdot\mathbf{n} > 0$ $(\Gamma \equiv \Gamma_{in})$

	Conditions [GJHM] Dubach's conditions	Conditions [GQSL]
Error in Ω_1 : $\|u - u_1\|_{\Omega_1}$	$\mathcal{O}(\nu^{5/2})$	$\mathcal{O}(\nu^{3/2})$
Error in Ω_2 : $\|u - u_2\|_{\Omega_2}$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$

Continuity of u	*yes*	*yes*
Continuity of the flux of u	*yes*	*no*

Steady Results in $\mathbb{R}^2$: case $\mathbf{a} \cdot \mathbf{n} < 0$ $(\Gamma \equiv \Gamma_{out})$

	Conditions [GJHM]	Conditions [GQSL] Dubach's conditions
Error in Ω_1 : $\|u - u_1\|_{\Omega_1}$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$
Error in Ω_2 : $\|u - u_2\|_{\Omega_2}$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$

Continuity of u	*yes*	*no*
Continuity of the flux of u	*no*	*no*

Algorithm when $\mathbf{a} \cdot \mathbf{n} > 0$ $(\Gamma \equiv \Gamma_{in})$

$$\left\{ \begin{array}{lcll} \mathcal{L}_{cd} u_1^{k+1} & = & f & \text{in } \Omega_1 \times (0,T), \\ u_1^{k+1}(\cdot,0) & = & u_0 & \text{in } \Omega_1, \\ \mathcal{B}_1 u_1^{k+1} & = & \mathcal{B}_1 u_2^k & \text{on } \Gamma \times (0,T), \end{array} \right.$$

$$\left\{ \begin{array}{lcll} \mathcal{L}_{c} u_2^{k+1} & = & f & \text{in } \Omega_2 \times (0,T), \\ u_2^{k+1}(\cdot,0) & = & u_0 & \text{in } \Omega_2, \\ \mathcal{B}_2 u_1^{k+1} & = & \mathcal{B}_2 u_2^k & \text{on } \Gamma \times (0,T), \end{array} \right.$$

$\rightarrow$ How to choose $\mathcal{B}_1$ and $\mathcal{B}_2$?

Algorithm for the error when $\mathbf{a}\cdot\mathbf{n} > 0$ $(\Gamma \equiv \Gamma_{in})$

We introduce $U_1^{k+1} = u_1 - u_1^{k+1}$ and $U_2^{k+1} = u_2 - u_2^{k+1}$

$$\left\{\begin{array}{lcll} \mathcal{L}_{cd} U_1^{k+1} & = & 0 & \text{in } \ \Omega_1 \times (0\,,T), \\ U_1^{k+1}(\cdot,0) & = & 0 & \text{in } \ \Omega_1, \\ \mathcal{B}_1 U_1^{k+1} & = & \mathcal{B}_1 U_2^k & \text{on } \Gamma \times (0\,,T), \end{array}\right.$$

$$\left\{\begin{array}{lcll} \mathcal{L}_{c} U_2^{k+1} & = & 0 & \text{in } \Omega_2 \times (0\,,T), \\ U_2^{k+1}(\cdot,0) & = & 0 & \text{in } \Omega_2, \\ \mathcal{B}_2 U_2^{k+1} & = & \mathcal{B}_2 U_1^k & \text{on } \Gamma \times (0\,,T). \end{array}\right.$$

Algorithm with new conditions when the sign of $\mathbf{a} \cdot \mathbf{n}$ is constant

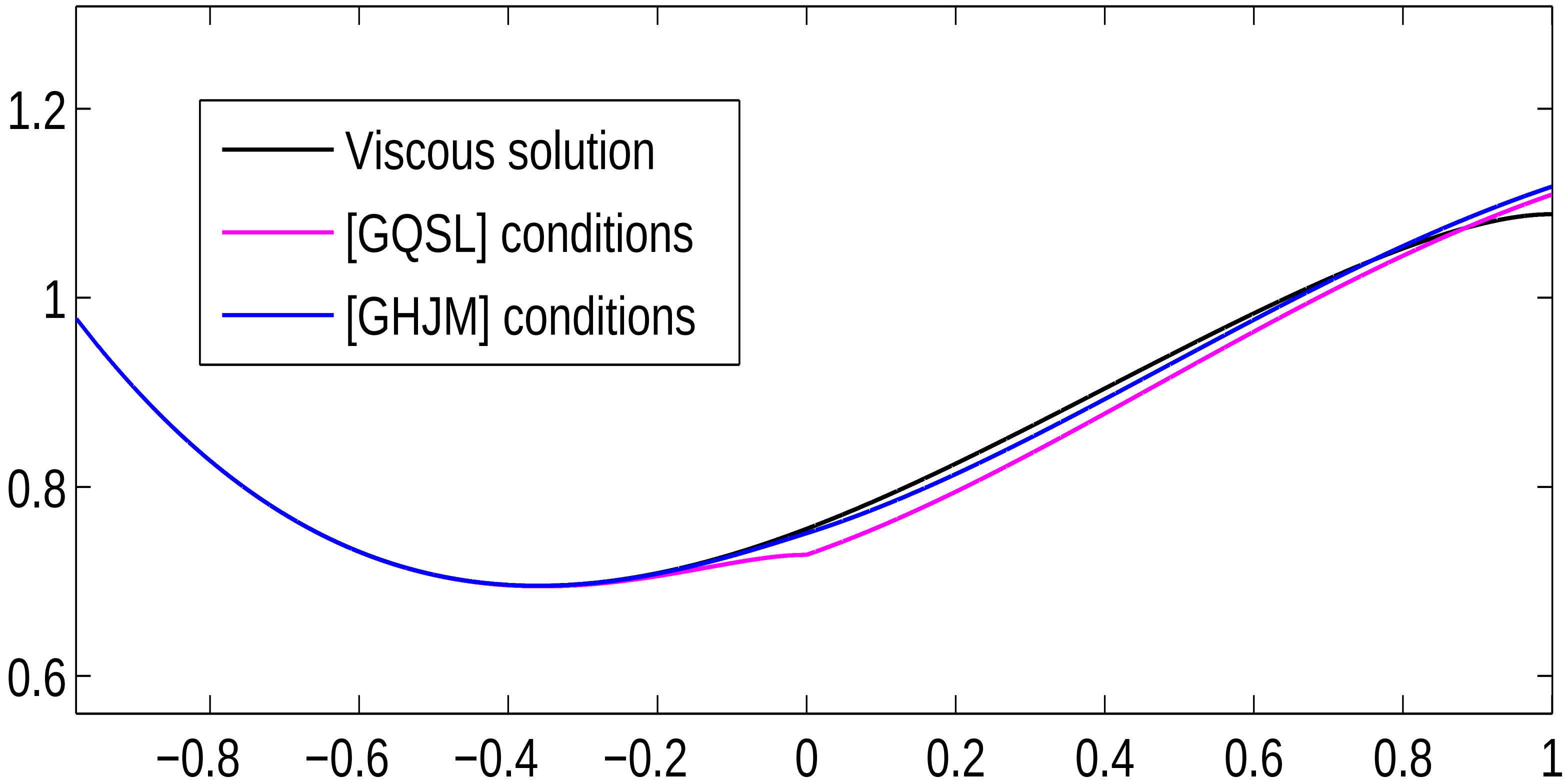
- $\mathbf{a} \cdot \mathbf{n} > 0 : \Gamma \equiv \Gamma_{in}$

$$\begin{cases} \mathcal{L}_{cd} u_1 = f & \text{in } \Omega_1 \times (0, T), \\ u_1(\cdot, 0) = u_0 & \text{in } \Omega_1, \\ \mathcal{L}_c u_1 = f & \text{on } \Gamma \times (0, T), \end{cases} \qquad \begin{cases} \mathcal{L}_c u_2 = f & \text{in } \Omega_2 \times (0, T), \\ u_2(\cdot, 0) = u_0 & \text{in } \Omega_2, \\ u_2 = u_1 & \text{on } \Gamma \times (0, T). \end{cases}$$

- $\mathbf{a} \cdot \mathbf{n} < 0 : \Gamma \equiv \Gamma_{out}$

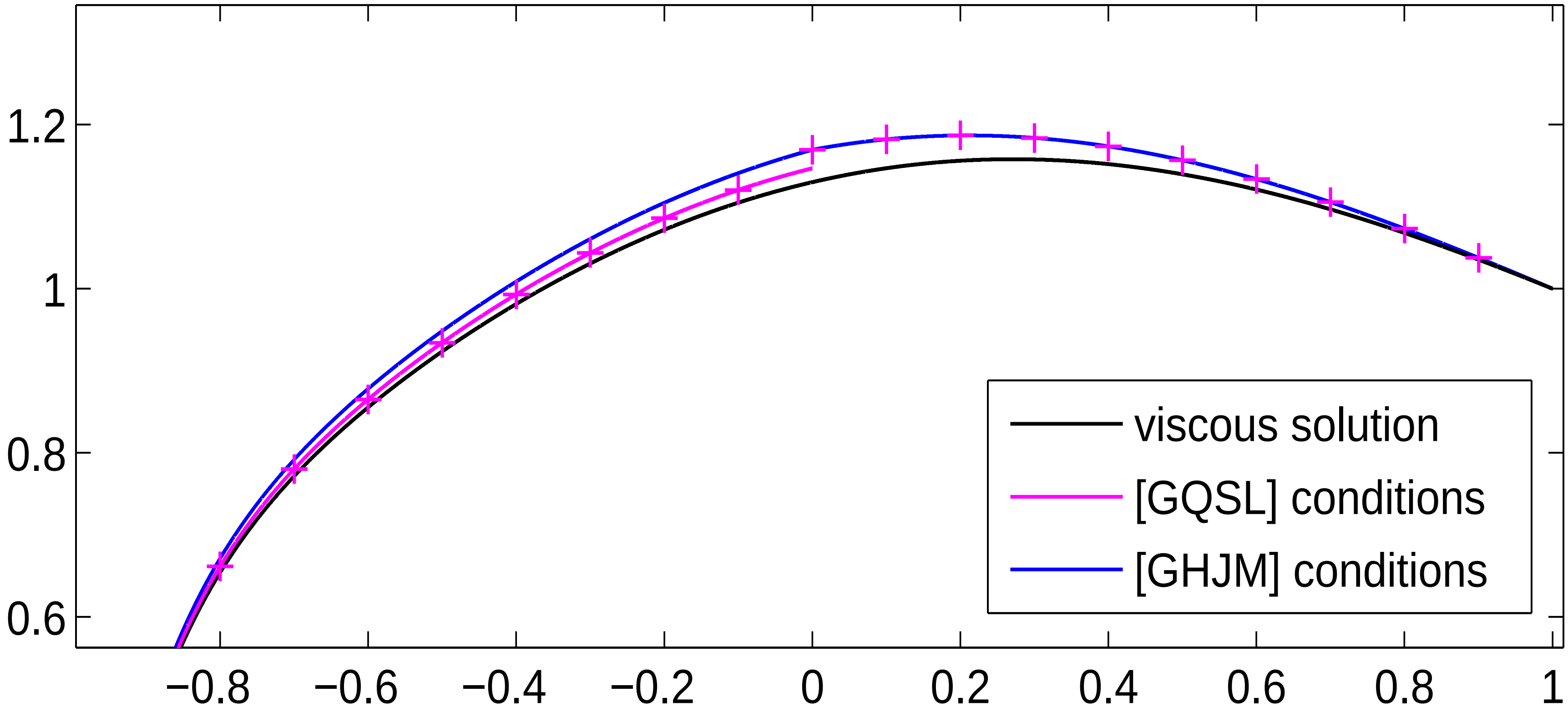
$$\begin{cases} \mathcal{L}_c u_2 = f & \text{in } \Omega_2 \times (0, T), \\ u_2(\cdot, 0) = u_0 & \text{in } \Omega_2, \end{cases} \qquad \begin{cases} \mathcal{L}_{cd} u_1 = f & \text{in } \Omega_1 \times (0, T), \\ u_1(\cdot, 0) = u_0 & \text{in } \Omega_1, \\ u_1 = u_2 & \text{on } \Gamma \times (0, T), \end{cases}$$

Steady case in 1D : $\mathbf{a}\cdot\mathbf{n}$ positive



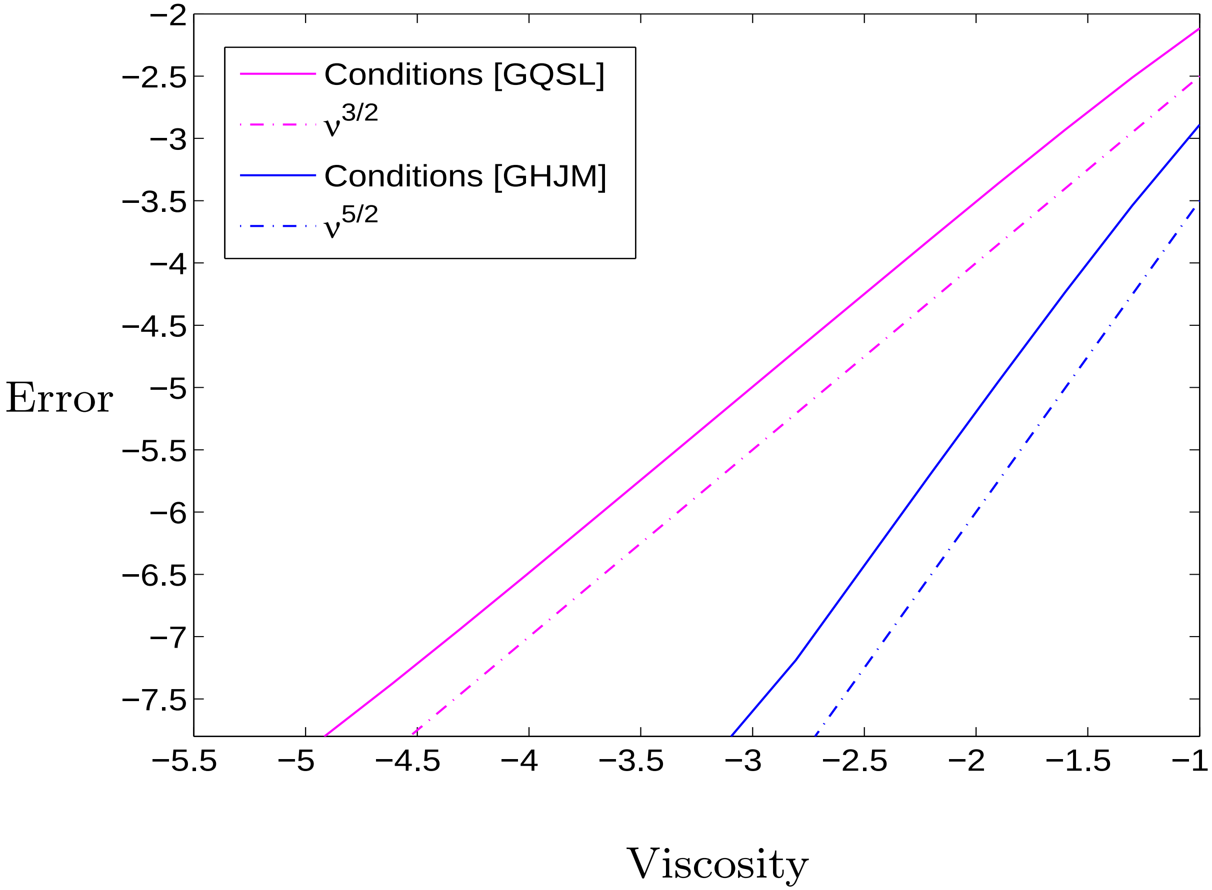
$$f(x) = \sin(x) + \cos(x),\ \nu = 0.1,\ a = 1,\ c = 1.$$

Steady case in 1D : $\mathbf{a} \cdot \mathbf{n}$ is negative

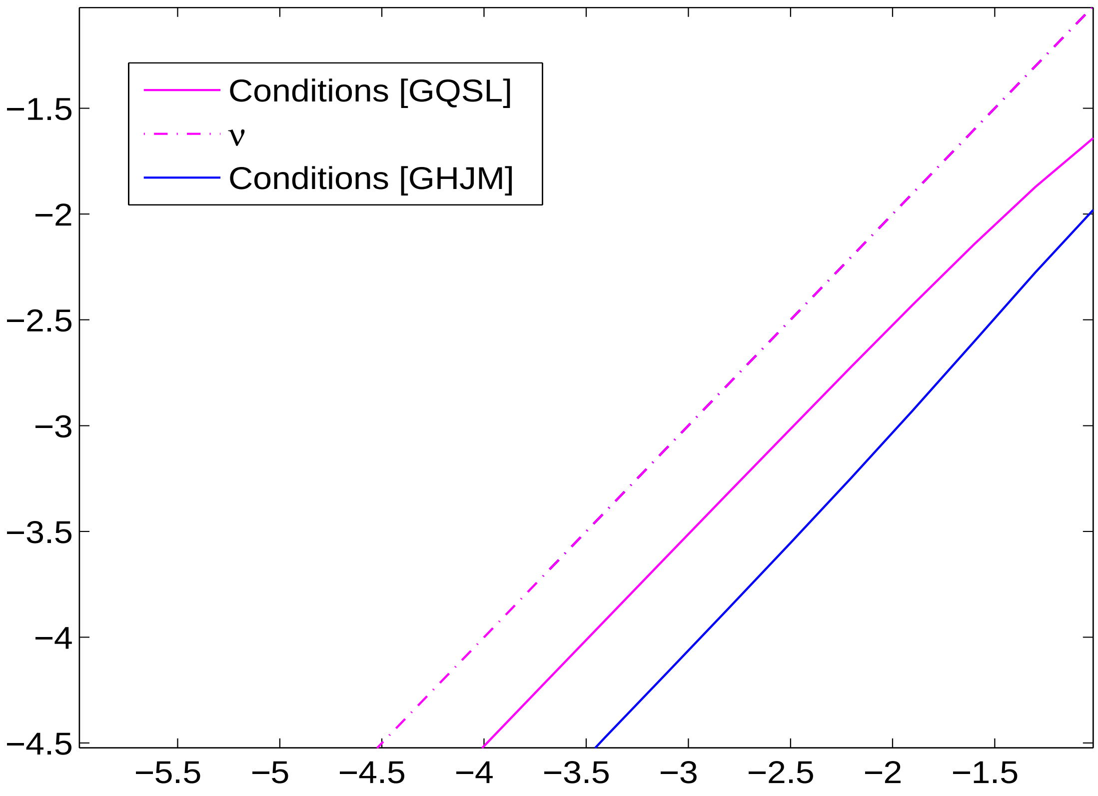


$$f(x) = \sin(x) + \cos(x),\ \nu = 0.1,\ a = -1,\ c = 1.$$

If $\mathbf{a}\cdot\mathbf{n}$ is positive : L^2 error in Ω_1 versus the viscosity



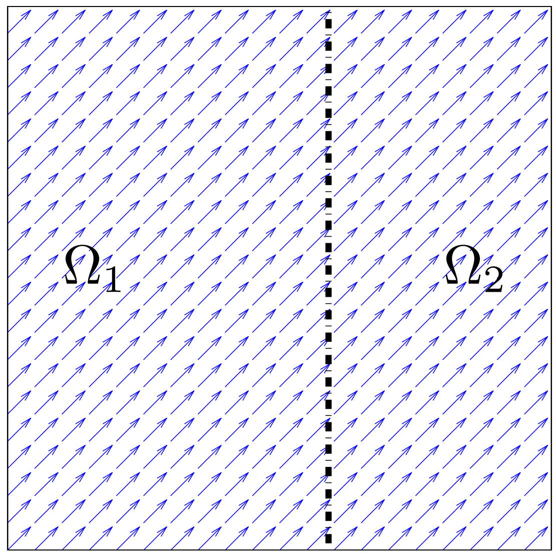
If a · n is positive : L^2 error in Ω_2 versus the viscosity



Unsteady 2D case

$$\frac{\partial u}{\partial y} = 0$$

$u = 0$



$$\frac{\partial u}{\partial x} = 0$$

$$u = \exp(-100 * (x - 0.4)^2)$$

Physical data : $\nu = 0.001$, $a(x, y) = 1$, $b(x, y) = 1$, $f \equiv 0$.

Unsteady 2D case

There was a film here...

General case

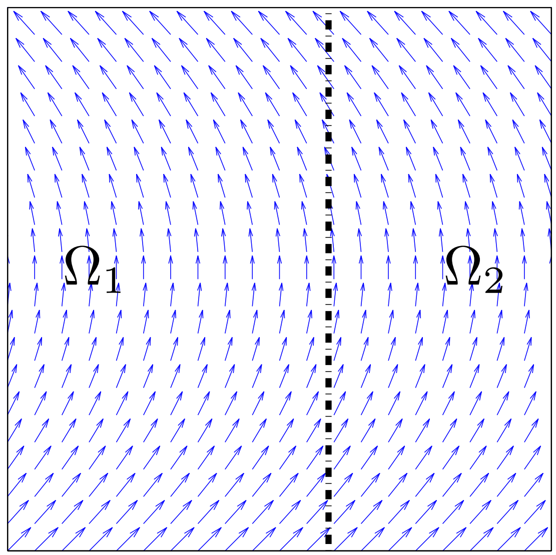
$$\begin{cases} \mathcal{L}_{cd} u_1^{k+1} & = & f & \text{in } \Omega_1 \times (0,T), \\ u_1^{k+1}(\cdot,0) & = & u_0 & \text{in } \Omega_1, \\ \mathcal{L}_c u_1^{k+1} & = & f & \text{on } \Gamma_{in} \times (0,T), \\ u_1^{k+1} & = & u_2^k & \text{on } \Gamma_{out} \times (0,T). \end{cases}$$

$$\begin{cases} \mathcal{L}_c u_2^{k+1} & = & f & \text{in } \Omega_2 \times (0,T), \\ u_2^{k+1}(\cdot,0) & = & u_0 & \text{in } \Omega_2, \\ u_2^{k+1} & = & u_1^k & \text{on } \Gamma_{in} \times (0,T). \end{cases}$$

Unsteady 2D case with rotating velocity

$$\frac{\partial u}{\partial y} = 0$$

$u = 0$



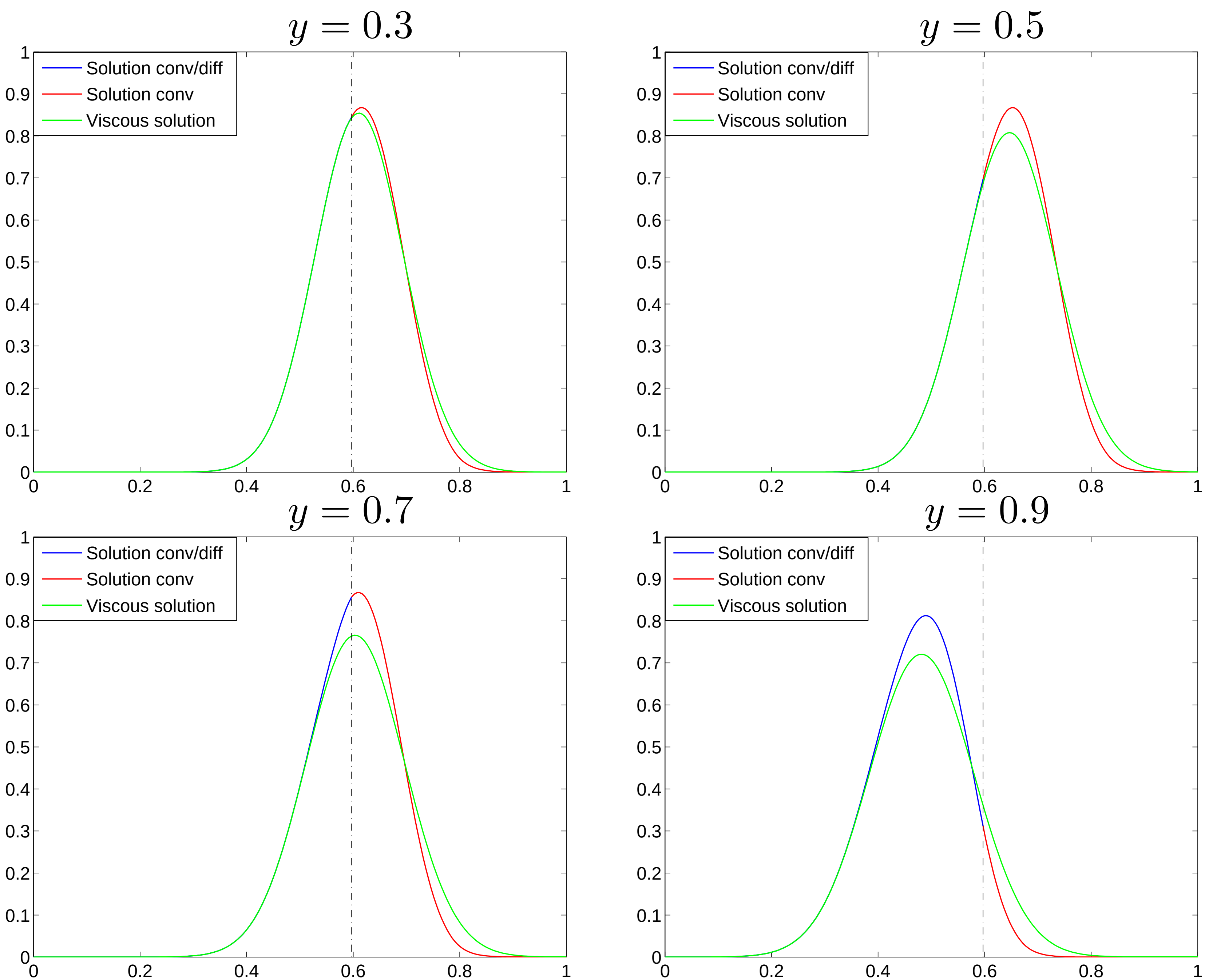
$$\frac{\partial u}{\partial x} = 0$$

$$u = \exp(-100 * (x - 0.4)^2)$$

Physical data : $\nu = 0.001$, $a(x, y) = 0.5 - y$, $b(x, y) = 0.5$, $f \equiv 0$.

Unsteady 2D case

There was a film here...



$y = 0.3$
$y = 0.5$
$y = 0.7$
$y = 0.9$
Solution conv/diff
Solution conv
Viscous solution
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Conclusions and perspectives

Current work :

- Estimates for the unsteady problem
- Proof of the convergence of the algorithm

Future work :

- Coupling algorithm for oceanographic equations