Primal and Dual Schur complement solvers for engineering problems

A family picture

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Pure mathematicians sometimes are satisfied with showing that the non-existence of a solution implies a logical contradiction, while engineers might consider a numerical result as the only reasonable goal.

Such one sided views seem to reflect human limitations rather than objective values.

In itself mathematics is an indivisible organism uniting theoretical contemplation and active application.

...”

R. Courant

in Variational Methods for the solution of problems of equilibrium and vibrations
Introduction

Non-overlapping Domain decomposition

- Basic ideas of Schur complement methods from end 80ties
- Many extensions, modifications, improvements in the 90ties
- Mathematical results exists for academic problems (uniform domains, smooth interfaces ...)
- Since end 90ties, DD methods based on Schur complement used in certain engineering applications with success
- Significant improvement in robustness is still needed for DD methods to be used as general solver in complex engineering models

Note: references
- Domain decomposition: interface conditions
- How the FETI family grew
- Important extensions of Schur complement techniques
- Challenges for industrial applications
Domain Decomp. : Interface conditions

\[ K^{(s)} u^{(s)} = f^{(s)} + g^{(s)} \]

**Interface equilibrium**

\[ \sum_{s=1}^{N_s} L^{(s)T} g^{(s)} = 0 \]

**Boolean assembling matrix**

**Interface compatibility**

\[ \sum_{s=1}^{N_s} B^{(s)} u^{(s)} = 0 \]

**Signed Boolean matrix**
In block diagonal notations:

\[
\begin{cases}
    Ku = f + g \\
    L^T g = 0 \\
    Bu = 0
\end{cases}
\]

\[
K = \begin{bmatrix}
    K^{(1)} & 0 \\
    & \ddots \\
    0 & K^{(N_s)}
\end{bmatrix}, \quad u = \begin{bmatrix}
    u^{(1)} \\
    \vdots \\
    u^{(N_s)}
\end{bmatrix}
\]

\[
L^T = \begin{bmatrix}
    L^{(1)^T} & \cdots & L^{(N_s)^T}
\end{bmatrix}, \quad B = \begin{bmatrix}
    B^{(1)} & \cdots & B^{(N_s)}
\end{bmatrix}
\]
In block diagonal notations:

\[
\begin{align*}
Ku &= f + g \quad (1) \\
L^T g &= 0 \quad (2) \\
Bu &= 0 \quad (3)
\end{align*}
\]

In parallel computation, the local problems (1) are satisfied exactly while the interface unknowns are iteratively found to satisfy (2) and (3)
Primal methods

\[
\begin{align*}
Ku &= f + g \\
L^Tg &= 0 \\
Bu &= 0
\end{align*}
\]

Take \( u \) unique on interface: \( u = Lu_g \)

Find \( u_g \) such that

\[
\begin{align*}
KLu_g &= f + g \\
L^Tg &= 0
\end{align*}
\]

Iterate on \( u_{interface} \) until \( L^Tg = L^T(KLu_g - f) = 0 \)
Dual methods

\[
\begin{cases}
Ku = f + g \\
L^T g = 0 \\
Bu = 0
\end{cases}
\]

Take \( g \) equal on opposite on interface: \( g = -B^T \lambda \)

Find \( \lambda \) such that

\[
\begin{cases}
Ku + B^T \lambda = f \\
Bu = 0
\end{cases}
\]

Iterate on \( \lambda \) until

\[
Bu = B \left( K^{-1} (f - B^T \lambda) \right) = 0
\]
Some nodes connected on interface through unique dof, other through internal forces
2 special methods iterating on interface equilibrium and compatibility

3-field formulations

- Uses intermediate interface displacements
- Can be obtained from variational principles
- Leads to symmetric formulation
- Useful for smoothing the interface (non-matching/contact)
2 special methods iterating on interface equilibrium and compatibility

Mixed conditions (Robin)
2 special methods iterating on interface equilibrium and compatibility

\[
\begin{align*}
(K^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & A^{(1)} \end{bmatrix}) u^{(1)} &= f^{(1)} + \begin{bmatrix} 0 \\ \lambda^{(1)} \end{bmatrix} \\
(K^{(2)} + \begin{bmatrix} A^{(2)} & 0 \\ 0 & 0 \end{bmatrix}) u^{(2)} &= f^{(2)} + \begin{bmatrix} \lambda^{(2)} \\ 0 \end{bmatrix}
\end{align*}
\]

Equivalent to initial problem if

\[
\begin{align*}
\lambda^{(1)} &= g^{(1)} + A^{(1)} u^{(1)} \\
\lambda^{(2)} &= g^{(2)} + A^{(2)} u^{(2)}
\end{align*}
\]

\[
g^{(1)} - g^{(2)} = (\lambda^{(1)} - A^{(1)} u^{(1)}) - (\lambda^{(2)} - A^{(2)} u^{(2)}) = 0
\]

\[
\begin{bmatrix} u^{(1)}_{\text{interf}} \end{bmatrix} + \begin{bmatrix} u^{(2)}_{\text{interf}} \end{bmatrix} = 0
\]

- Regularizes the local problems of needed (Helmoltz)
- Fast convergence if \( A^{(s)} \) approximation of Schur complement
Summary

3-field formulation

Mixed conditions (Robin)

Interface Comp.+Equil.

Dual

Primal

Domain Decomp. : Interface conditions
- Domain decomposition: interface conditions
- How the FETI family grew
- Important extensions of Schur complement techniques
- Challenges for industrial applications
**Original FETI (Dual Schur complement)**

\[
\begin{bmatrix}
K & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
u \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
f \\
0
\end{bmatrix}
\]

\[u = K^{-1} \left(f - B^T \lambda\right)\]  
(assuming non-floating domains)

\[Bu = 0\]

\[F_I \lambda = d\]

\[F_I = BK^{-1}B^T\]

\[d = BK^{-1}f^T\]

Dual interface problem \[\rightarrow\] C.G.

Dirichlet / lumped preconditioner
How the Dual Schur Family grew ...

Original FETI

If floating domains

\[
\begin{bmatrix}
K & B^T R & B^T \\
R^T B & o & o \\
B & o & o \\
\end{bmatrix}
\begin{bmatrix}
u \\
\alpha \\
\lambda \\
\end{bmatrix}
= \begin{bmatrix} f \\
o \\
o \end{bmatrix}
\]

- At every iteration on \( \lambda \), compatibility satisfied on average
- Defines a natural coarse grid that ensures scalability
- Solved by general inverses & projection
FETI 2 level

When using FETI on 4th order problems (plates/shells), the interface forces on the interface converge badly (Kirchhoff corner forces)

→ need to impose strong compatibility on corners

\[
\begin{bmatrix}
K & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
u \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
f \\
0
\end{bmatrix}
\]

FETI-2 level (local problem is itself a FETI problem)

Two-level primal method exist also for corners

How the Dual Schur Family grew ...
FETI Dual Primal

If corner links sufficient to fix the subdomains
“average” compatibility not required for regularity:

\[
\begin{bmatrix}
K & B^T R & B^T C & B^T \\
R^T B & 0 & 0 & 0 \\
C^T B & 0 & 0 & 0 \\
B & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
\alpha \\
\mu \\
\lambda
\end{bmatrix}
= \begin{bmatrix} f \\
0 \\
0 \\
0 \end{bmatrix}
\]

Corner compatibility can be enforced by assembly on the corners:

\[
\begin{bmatrix}
L_c^T K L_c & B^T \\
\theta & 0 \\
B & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
\alpha \\
u_c \\
\lambda
\end{bmatrix}
= \begin{bmatrix} f \\
f_c \\
\theta \\
0 \end{bmatrix}
\]

How the Dual Schur Family grew …
FETI Dual Primal

\[
\begin{bmatrix}
L_c^T K L_c & B^T \\
B & o \\
o & o
\end{bmatrix}
\begin{bmatrix}
u \\
u_c \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
f_r \\
f_c \\
o
\end{bmatrix}
\]

Compatibility enforced in **Primal** on some points at every iteration in **Dual iteratively** elsewhere

→ **FETI-DP**

How the Dual Schur Family grew ...
FETI Dual Primal

- Less connecting variables at corners
- “average” compatibility not enforced

FETI-2

FETI-DP

Smaller coarse grid
Smaller cost per iteration
Slower convergence

- only point-wise compatibility enforced in inner problem
- no “smooth” coarse grid

FETI-DP scalable in 2D
NOT scalable in 3D
Dual – Primal Schur Complement methods

\[
\begin{bmatrix}
L_c^T K L_c & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
u \\
u_c \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
f_r \\
f_c \\
o
\end{bmatrix} \quad \text{FETI-DP}
\]

If simultaneous iteration on some primal and dual interface dofs:

\[
\begin{bmatrix}
L_p^T K L_p & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
u \\
u_p \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
f_r \\
f_p \\
o
\end{bmatrix}
\]

Hybrid Primal-Dual Schur Complement methods

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Interface conditions

Local solvers

Iterative interface solvers (GMRES, CG)

Coarse grids (natural/auxiliary)

Preconditioners (optimal/simplified/scaling)
• Domain decomposition: interface conditions

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• Challenges for industrial applications
Important extensions

Multiple r.h.s. (multiple load cases, dynamics)

Multiple l.h.s. (changing operators: non-linear)
Heterogeneous coefficients

Contact

Plates and shells (4th order problems)

Non-Boolean interface constraints (non-matching interfaces)

Account for LMPC (multi-point constraints) on system

Treat incompressibility

Several application fields (structural, electromagnetics, acoustics …)

Multiphysics (vibroacoustics, poro-elasticity, aeroelasticity)
Quasi-axisymmetric

Quasi-cyclic

Important extensions
- Domain decomposition: interface conditions
- How the FETI family grew
- Important extensions of Schur complement techniques
- Challenges for industrial applications
Aspect ratio and heterogeneity of subdomains

1,000,000 d.o.f
Highly heterogeneous
Bad aspect ratios

Reentry vehicle (SANDIA)

Further challenges
Simple tire test

- 3 subdomains
- Very heterogeneous along interface
- Homogeneous across interface
- bad aspect ratio of subdomains
- nearly incompressible

Need for more robust solvers!

Further challenges
Multiphysics

Time problems (good coarse grids, dynamic substructuring)

Parallel in time (parareel)

Multibody analysis

\[ f(u) + B(u)^T \lambda = 0 \]
\[ \phi(u) = 0 \]

linearized

\[ Kq + B(u)^T \lambda = 0 \]
\[ B(u)q = 0 \]

(Multiple constraint matrices B.)
Lots of room for new ideas!!