A NEWTON-KRYLOV PRECONDITIONER
FOR FLUID STRUCTURE PROBLEMS IN BLOOD FLOW

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• A platform for the fluid-structure interaction

• Numerical algorithms
  ★ coupling algorithms
  ★ preconditioners

• Difficulties to use in industrial problems
  ★ Application to blood-flows
  ★ Numerical examples

• Conclusions
• Requirements for the fluid-structure platform
  ⋆ support both weak and strong coupling
  ⋆ use existing state of the art solid and fluid solvers

• Numerical algorithms
  ⋆ where did we start
  ⋆ present achievements

• Application to blood flows

• How to validate?
**Requirements for the Fluid-Structure Platform**

- support both **weak** and **strong** coupling
  - target problems arising from **civil engineering** or **blood flows**
  - better understanding of **criteria** for choosing a method in realistic simulations (Ex: practical importance of the **added mass effect**, trade off between **strong** coupling with **large** time steps vs **weak** coupling with **smaller** time steps . . .)

- use existing state of the art **solid** and **fluid** solvers
  - easy to change solvers and choose the best one for a specific application
  - take advantage of latest progress of solid and fluid solvers
  - minimize the changes needed in the software to support coupling
PRESENT ACHIEVEMENTS

- generic approach
  - a coupling program which implements a particular algorithm (the Master)
  - the fluid and solid solvers are data for the Master

- two coupling algorithms
  - a fixed point with relaxation (the weak coupling algorithm is a particular case)
  - a quasi-newton algorithm using an approximate Jacobian based on a simplified model

- an analysis of the simplified model on particular problems (J.F. Gerbeau, P. Causin) (answer to: Weak vs strong coupling)

- almost all the software ingredients required for realistic simulations.
Weak vs. strong coupling for blood flows

Idealized framework to represent the mechanical interaction between the blood and the arterial wall:

- Geometry at rest: cylinder.
- Fluid: incompressible Navier-Stokes equations in Arbitrary Lagrangian Eulerian formulation
- Structure: either by a 1D generalized string model (for 2D simulations) or by a nonlinear shell model in large displacements regime (for 3D simulations).
- An overpressure is applied at the inlet of the fluid for a short duration of time and propagates along the cylinder with finite speed.
Propagation of a pressure wave in a portion of artery (2D). T=6, 10 and 14 ms.
Weak coupling schemes: attractive... but unstable!

(R1) for a given geometry, as soon as the density of the structure is lower than a certain threshold;

(R2) for a given structure density, as soon as the length of the domain is greater than a certain threshold.

(R3) These observations are independent of the time step.

Moreover, strong coupling needs from 30 to 100 fluid-structure evaluations per time step!

For a simplified problem J.F Gerbeau and P. Causin proved that In blood flows, strong coupling is needed!
\( (P) \quad \left\{ \begin{array}{l}
-\Delta u = f \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\end{array} \right. \)

\( (P) \iff \left\{ \begin{array}{l}
-\Delta u_1 = f \quad \text{in } \Omega_1 \\
u_1 = 0 \quad \text{on } \partial \Omega_1 \setminus \Gamma \\
-\Delta u_2 = f \quad \text{in } \Omega_2 \\
u_2 = 0 \quad \text{on } \partial \Omega_2 \setminus \Gamma \\
u_1 = u_2 \quad \text{on } \Gamma \\
\frac{\partial u_1}{\partial n_1} = -\frac{\partial u_2}{\partial n_2} \quad \text{on } \Gamma
\end{array} \right. \)
Dirichlet Neumann Algorithm

assume $\lambda$ is known

compute $u_1$ solution of

\[
\begin{cases}
-\Delta u_1 &= f \quad \text{in } \Omega_1 \\
 u_1 &= 0 \quad \text{on } \partial\Omega_1 \setminus \Gamma \\
 u_1 &= \lambda \quad \text{on } \Gamma
\end{cases}
\]

compute $u_2$ solution of

\[
\begin{cases}
-\Delta u_2 &= f \quad \text{in } \Omega_2 \\
 u_2 &= 0 \quad \text{on } \partial\Omega_2 \\
 \frac{\partial u_2}{\partial n_2} &= -\frac{\partial u_1}{\partial n_1} \quad \text{on } \Gamma
\end{cases}
\]

set $\lambda = \omega \lambda + (1 - \omega) Tr u_2$
Fluid-Structure Interaction (ALE Framework)

\(\Omega(t)\) is a domain of \(\mathbb{R}^d\) \((d = 2 \text{ or } 3)\) occupied by a continuum medium.

\(\Omega_F(t)\) is occupied by a fluid

\(\Omega_S(t)\) is occupied by an elastic solid

\(\Sigma(t) = \overline{\Omega_F(t) \cap \Omega_S(t)}\) is the fluid-structure interface

The domain \(\Omega(t)\) is the current configuration.
**STRONG FORMULATION OF THE FLUID-STRUCTURE PROBLEM**

\[(F)\] \[
\begin{aligned}
\rho_f \frac{\partial u}{\partial t} &+ \rho_f (u - w) \cdot \nabla u - \text{div} \sigma_F & = 0 \\
\text{in } \Omega_F(t) \\
\rho_f \frac{\partial u}{\partial t} &+ \rho_f (u - w) \cdot \nabla u - \text{div} \sigma_F & = 0 \\
\text{in } \Omega_F(t) \\
\end{aligned}
\]

\[
\begin{aligned}
u & = u^d \\
on \Gamma^D_F(t) \\
\sigma_F \cdot n & = 0 \\
on \Gamma^N_F(t) \\
u(x, t) & = \hat{u}_S(\hat{\varphi}_t^{-1}(x), t) \\
on \Sigma(t) \\
\end{aligned}
\]

\[(S)\] \[
\begin{aligned}
\hat{J} \hat{\rho}_S \frac{\partial^2 \hat{d}}{\partial t^2} &- \text{div} \hat{x} \hat{\Pi} & = 0 \\
\text{in } \hat{\Omega}_S \\
\hat{u}_S & = 0 \\
on \hat{\Gamma}^D_S \\
\hat{\Pi} \cdot n_S & = 0 \\
on \hat{\Gamma}^N_S \\
\hat{\Pi} \cdot n_S & = \hat{J} \sigma_F \hat{\mathbf{F}}^{-T} \hat{n}_S \\
on \hat{\Sigma} \\
\end{aligned}
\]

\[(D)\] \[
\begin{aligned}
The \text{domain velocity in } \hat{\Omega}_F \text{ satisfies:} \\
\hat{w} = Tr^{-1}(\hat{u}_S|_{\hat{\Sigma}})
\end{aligned}
\]
• assume \( \hat{d}_{\Sigma}^{n+1} \) is known
• compute the fluid domain deformation \( \hat{d}_{F}^{n+1} \) as “any” lifting of \( \hat{d}_{\Sigma}^{n+1} \) (for example an harmonic lifting)

\[
\hat{d}_{F}^{n+1} = D(\hat{d}_{\Sigma}^{n+1})
\]

• solve the fluid problem (implicit Euler scheme)

\[
(u^{n+1}, p^{n+1}) = \mathcal{F}(\hat{d}_{F}^{n+1})
\]

• compute the fluid force acting on the structure

\[
f_{\Sigma}^{n+1} = \mathcal{R}(u^{n+1}, p^{n+1})
\]

• solve the solid problem (mid-point scheme)

\[
\hat{d}_{\Sigma}^{n+1} = S(f_{\Sigma}^{n+1})
\]
• Initialization: from staggered schemes (Farhat, Piperno) $k = 0$, 

$$\hat{d}_{\Sigma,0} = \hat{d}_{\Sigma} + \frac{3\delta t}{2} \hat{u}^n_S - \frac{\delta t}{2} \hat{u}^{n-1}_S$$

• Coupling algorithms

- **Dirichlet Neumann → Fixed-point algorithm with relaxation**

  $$\hat{d}_{\Sigma}^{n+1} = T (\hat{d}_{\Sigma}^{n+1}) \quad \text{with}$$

  $$T = S \circ R \circ F \circ D$$

- **Newton-Krylov method**: solve $A\hat{d} = 0$ with

  $$A = I - T = I - S \circ R \circ F \circ D$$

  using a **suitable approximation** of the Jacobian
THE DISCRETISED FLUID-STRUCTURE PROBLEM

\[
\begin{bmatrix}
A^F_{II} & 0 & A^F_{IS} \\
0 & A^S_{II} & A^S_{IS} \\
A^F_{SI} & A^S_{SI} & A^F_{SS} + A^S_{SS}
\end{bmatrix}
\begin{bmatrix}
d^F_I \\
d^S_I \\
d^\Sigma
\end{bmatrix}
= \begin{bmatrix}
b^F_I \\
b^S_I \\
b^\Sigma
\end{bmatrix}
\quad (d^F_I = \delta t \, u^F_I)
\]

- (elim 1st line) \( d^F_{I,k+1} = (A^F_{II})^{-1}(b^F_I - A^F_{IS}d^\Sigma) \)
- (elim 2nd line) \( d^S_{I,k+1} = (A^S_{II})^{-1}(b^S_I - A^S_{IS}d^\Sigma) \)
- (replace 3rd line)

\[
A^F_{SI}(A^F_{II})^{-1}(b^F_I - A^F_{IS}d^\Sigma) + A^S_{SI}(A^S_{II})^{-1}(b^S_I - A^S_{IS}d^\Sigma) + (A^F_{SS} + A^S_{SS})d^\Sigma = b^\Sigma
\]
THE INTERFACE PROBLEM

\[
\left( A_{\Sigma\Sigma}^F - A_{\Sigma I}^F (A_{II}^F)^{-1} A_{I\Sigma}^F \right)_{SF} + \left( A_{\Sigma\Sigma}^S - A_{\Sigma I}^S (A_{II}^S)^{-1} A_{I\Sigma}^S \right)_{SS} \right) d_{\Sigma} = b_{\Sigma} - A_{\Sigma I}^F (A_{II}^F)^{-1} b_I^F - A_{\Sigma I}^S (A_{II}^S)^{-1} \tilde{b} 
\]

Introduce the Schur complement for both the solid and fluid problem

- the interface problem writes

\[
(S_F^F + S_S^S) d_{\Sigma} = \tilde{b} 
\]

- the fixed point method (Dirichlet-Neuman) is a gradient method for the interface problem preconditioned by \( S_S^{-1} \)

- the Newton-Krylov method is a GMRES resolution of the interface problem preconditioned by an approximate Jacobian \( \tilde{J}^{-1} \)
The Newton-Krylov Method

Solve the **nonlinear** interface problem:

\[(S^F + S^S)d_\Sigma = \tilde{b}\]

Newton iteration:

\[\tilde{J}\delta d_\Sigma = \left[\tilde{b} - (S^F + S^S)d_{\Sigma,k}\right]\]

\[d_{\Sigma,k+1} = d_{\Sigma,k} + \lambda_k \tilde{J}^{-1}\left[\tilde{b} - (S^F + S^S)d_{\Sigma,k}\right]\]

an approximate Jacobian acts as a **preconditioner** for the interface problem.
A NEW PRECONDITIONER FOR FLUID-STRUCTURE PROBLEMS

(J.F. Gerbeau- M. V)

In order to build a preconditioner we approximate the tangent operator of the nonlinear problem by the following simplified model:

- geometrical variations are neglected ($\tilde{\Omega}_F = \Omega_F(t^n)$)
- the structure is linearized about its current state
- nonlinear and viscous terms are neglected in the fluid:

$$
\begin{cases}
-\Delta p &= 0 & \text{on } \tilde{\Omega}_F \\
\frac{\partial p}{\partial n} &= -\rho_f \frac{\partial u}{\partial t} \cdot n & \text{on } \Sigma
\end{cases}
$$

→ simple, but take into account the added-mass effect!
RESULTS ON THE 3D MODEL PROBLEM

Geometry: cylinder, Fluid: 3D, Structure: shells in large displacements
DIFFICULTIES OF USE IN INDUSTRIAL PROBLEMS

Scientific limitations:

- **pertinence** of the models used
- **efficiency and reliability** of the numerical methods
- how to validate?

Software limitations:

- **large** number of software components: mesh generation, mesh matching, fluid solver, solid solver, coupling algorithm
- limitations due to the **state of the art**.
  
  **Example**: the reliable elements in the discretization of the solid problem are **quadrilaterals**, the reliable **automatic** mesh generator produces **triangles**.
Application for Blood Flows

Requirements:

- A thin structure model in large displacements
- A fluid solver on a moving domain
- Geometries coming from medical imaging

Softwares used

- A fluid software ALE Navier-Stockes for incompressible flows
- A solid software using MITC4 nonlinear shell elements in MODULEF
- A master code which contains all the coupling algorithms
- Mesh generation for the solid and fluid problems
  - Boundary obtained from medical imaging
  - Automatic mesh generation in 3D in tetrahedra (GHS3D)
  - Extraction of a part of the boundary to obtain the structure mesh in triangles!
  - Transform pairs of triangles into quadrangles, to preserve mesh conformity (YAMS, P. Frey).
[Carotide] (P. Frey, J.F. Gerbeau, M. Vidrascu)
[Aneurism] (P. Frey, J.F. Gerbeau, M. Vidrascu)
HOW TO VALIDATE?

To validate a *fluid-structure interaction* solver is a difficult task.

- lack of experiences and numerical tests
- difficulty in obtaining *material characteristics*
- determination of how to properly define *boundary conditions*?
- evaluation of *precision* is required for the whole problem?
- relevance of the models used

How to identify the origin of a problem for a coupled simulation?

Example: problems with DKT shell elements
CONCLUSION

- 3D fluid-structure interaction in blood flows is challenging in particular because of the importance of the added-mass effect which imposes the use of strong-coupled methods.

- We have proposed a preconditioner based on a reduced model which is able to capture this effect and which allows to reasonably approximate the tangent operator in a Newton-Krylov method. This preconditioner improves both efficiency and robustness.

Further developments

- Improve the efficiency by using a domain decomposition method based on the balanced Neumann-Neumann algorithm to solve the elasto-dynamic problem.