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A New Variant of the Mortar Technique for the Crouzeix-Raviart Finite Element

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Summary. We propose a new variant of the mortar method for the lowest order Crouzeix-Raviart finite element for the approximation of second order elliptic boundary value problems on nonmatching meshes.

1.1 Introduction

The mortar technique (cf. Bernardi et al. (1994); Ben Belgacem (1994)) is the class of domain decomposition method that allows for nonmatching meshes for solving partial differential equations. To ensure that the overall discretization involving the nonmatching meshes makes sense, an optimal coupling between the meshes is required. In a standard mortar technique, this condition is realized by applying the condition of weak continuity on the solution, called the mortar condition, saying that the jump of the solution along the interface between two meshes is orthogonal to some suitable test space. Since its first introduction, the mortar technique has been studied extensively, see Belgacem and Maday (1997); Marcinkowski (1999); Seshaiyer and Suri (1999); Wohlmuth (2000); Braess and Dahmen (2001), and the references therein.

In order to apply the mortar condition, it is necessary to know the function on the interface. For the conforming P1 finite element, it is enough to know the nodal values along the interface. However, for the nonconforming P1 finite element (the lowest order Crouzeix-Raviart finite element), where the degrees of freedom are associated with the edge midpoints, see Fig. 1.1, the function on the interface depends on the nodal values corresponding to interface nodes and some subdomain interior nodes lying closest to the interface, cf. Marcinkowski (1999). The purpose of this paper is to modify the mortar condition, so that the new method will use only the nodal values on the interface. This is a clear advantage compared to the standard method, especially in 3D. The approach can also be seen as the mortar method with an approximate constraint, see Bertoluzza and Falletta (2003) for instance.

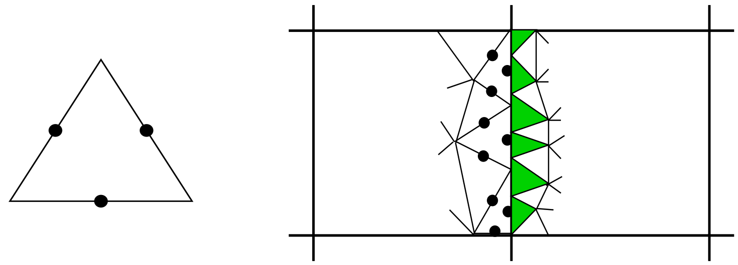

Fig. 1.1. The lowest order Crouzeix-Raviart (CR) finite element (left) and two nonmatching grids (right). CR basis functions associated with the nodes on the mortar side, denoted by dots (in the interior) and semi-dots (on the mortar), have nonzero support on the nonmortar side, denoted by the shaded triangles.

We propose our new mortar variant in Section 1.2, and present its matrix formulation in Section 1.3. An additive Schwarz preconditioner similar to the one in Rahman et al. (2004) for the new mortar variant is formulated in Section 1.4, and finally some numerical results are presented in Section 1.5.

1.2 The new mortar variant

Let $\Omega \subset R^2$ be a simply connected bounded domain, partitioned (conforming) into a collection of nonoverlapping polygonal subdomains, $\Omega_i, i = 1, \ldots, N$, such that $\overline{\Omega} = \bigcup_i \overline{\Omega}_i$. We consider the problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \;\; v \in H_0^1(\Omega), \tag{1.1}$$

where $a(u,v) = \sum_{i=1}^N \int_{\Omega_i} \nabla u \cdot \nabla v \; dx$ and $f(v) = \sum_{i=1}^N \int_{\Omega_i} fv \; dx$. With each subdomain Ω_i, we associate a quasi-uniform triangulation $\mathcal{T}_h(\Omega_i)$ of mesh size h_i. The resulting triangulation can be nonmatching across subdomain interfaces.

Let $X_h(\Omega_i)$ be the nonconforming P1 (Crouzeix-Raviart) finite element space defined on the triangulation $\mathcal{T}_h(\Omega_i)$ of Ω_i, consisting of functions which are piecewise linear in each triangle $\tau \subset \Omega_i$, continuous at the interior edge midpoints of Ω_{ih}^{CR}, and vanishing at the edge midpoints of $\partial\Omega_{ih}^{CR} \cap \partial\Omega$ lying on the boundary $\partial\Omega$. Here, Ω_{ih}^{CR} and $\partial\Omega_{ih}^{CR}$ represent the sets of edge midpoints, i.e., the Crouzeix-Raviart nodal points, of Ω_i and $\partial\Omega_i$, respectively. In the same way, we use Ω_{ih} and $\partial\Omega_{ih}$ (without the superscript CR) to denote the corresponding sets of triangle vertices.

Since the triangulations on Ω_i and Ω_j do not match on their common interface Γ_{ij}, the functions in $X_h(\Omega) = \Pi_i X_h(\Omega_i)$ are discontinuous at the edge midpoints along the interface. In the standard mortar technique, see Marcinkowski (1999), the condition of weak continuity, called the mortar condition, is therefore imposed. In this paper, we introduce a new variant of the mortar condition. Let $\gamma_{m(i)} \subset \partial\Omega_i$ and $\delta_{m(j)} \subset \partial\Omega_i$ be the mortar and the nonmortar side of the interface Γ_{ij}, respectively. Let $u_h \in X_h$, where $u_h = \{u_i\}_{i=1}^N$. A function $u_h \in X_h$ satisfies the mortar condition on $\delta_{m(j)} = \Gamma_{ij} = \gamma_{m(i)}$, if

$$Q_m I_m u_i = Q_m u_j, \tag{1.2}$$

where I_m is an interpolation operator, to be defined in the next paragraph, and Q_m is the L^2-projection operator $Q_m : L^2(\Gamma_{ij}) \to M^{h_j}(\delta_{m(j)})$ defined as $(Q_m u, \psi)_{L^2(\delta_{m(j)})} = (u, \psi)_{L^2(\delta_{m(j)})}, \quad \forall\psi \in M^{h_j}(\delta_{m(j)})$, where $M^{h_j}(\delta_{m(j)}) \subset L^2(\Gamma_{ij})$ is the test space of functions which are piecewise constant on the triangulation of $\delta_{m(j)}$, and $(\cdot,\cdot)_{L^2(\delta_{m(j)})}$ denotes the L^2 inner product on $L^2(\delta_{m(j)})$. We note that, for the standard mortar method, I_m is simply the identity.

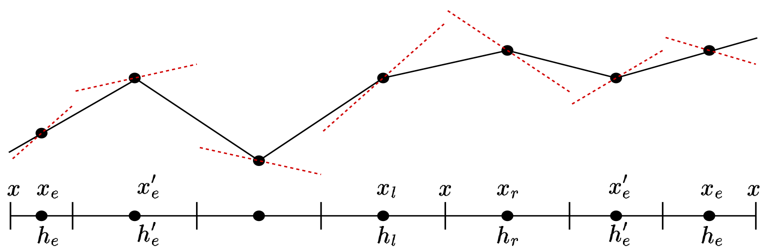

Fig. 1.2. Showing $u|_{\gamma_m}$ by dotted lines, and $I_m u|_{\gamma_m}$ by the solid line.

Let $\mathcal{T}_{\frac{h}{2}}(\gamma_m)$ be the triangulation associated with the mortar γ_m, which is obtained as a result of dividing the edges of $\mathcal{T}_h(\gamma_m)$. Let $W_{\frac{h}{2}}(\gamma_m)$ be the conforming space of piecewise linear continuous functions on the triangulation $\mathcal{T}_{\frac{h}{2}}(\gamma_m)$. The functions of this space are defined by their values at the set $\overline{\gamma}_{m\frac{h}{2}}$ of all edge endpoints of $\mathcal{T}_{\frac{h}{2}}(\gamma_m)$. It is easy to see that $\overline{\gamma}_{m\frac{h}{2}} = \gamma_{mh}^{CR} \cup \overline{\gamma}_{mh}$, where γ_{mh}^{CR} and $\overline{\gamma}_{mh}$ are respectively the sets of edge midpoints and edge endpoints of $\mathcal{T}_h(\gamma_m)$. We now define the operator $I_m : X_h(\gamma_m) \to W_{\frac{h}{2}}(\gamma_m)$ below.

Definition 1. *For $u \in X_h(\gamma_m)$, $I_m u \in W_{\frac{h}{2}}(\gamma_m)$ is defined by the nodal values as*

$$I_m u(x) = \begin{cases} u(x), & x \in \gamma_{mh}^{CR}, \\ \frac{h_r}{h_l+h_r} u(x_l) + \frac{h_l}{h_l+h_r} u(x_r), & x \in \gamma_{mh}, \\ u(x_e) + \frac{h_e}{h_e+h'_e}(u(x_e) - u(x'_e)) & x \in \partial\gamma_{mh}. \end{cases} \tag{1.3}$$

Here, x_l and x_r are the left- and the right neighboring edge midpoints of x, respectively. Correspondingly, h_l and h_r are the left- and the right edge lengths. x_e and h_e are the midpoint and the length of the edge of $\mathcal{T}_h(\gamma_m)$, touching $\partial\gamma_m$. The edge midpoint x'_e and the edge length h'_e correspond to the neighboring edge.

The interpolation is done basically by first joining the edge midpoints with piecewise straight lines, and then stretching the two end straight lines to the end of the mortar γ_m, cf. Fig. 1.2. It is not difficult to see that the operator I_m preserves all linear functions on the mortar.

$V_h \subset X_h$ is a subspace of functions which satisfy the mortar condition for all $\delta_m \subset \mathcal{S}$. Since functions of V_h are not continuous, we use the broken bilinear form $a_h(\cdot,\cdot)$ defined according to $a_h(u,v) = \sum_{i=1}^N a_i(u,v) = \sum_{i=1}^N \sum_{\tau\in\mathcal{T}_h(\Omega_i)} (\nabla u, \nabla v)_{L^2(\tau)}$. The discrete problem takes the following form: Find $u_h^* = \{u_i\}_{i=1}^N \in V_h$ such that

$$a_h(u_h^*, v_h) = f(v_h), \quad \forall v_h \in V_h. \tag{1.4}$$

If the h_i's are of the same order h, then the following error estimate can be shown.

Theorem 1. *For all* $u \in V_h$,

$$\| u^* - u_h^* \|_{L^2(\Omega)} + h|u^* - u_h^*|_{H_h^1(\Omega)} \leq ch^2 \| u^* \|_{H^2(\Omega)} \tag{1.5}$$

1.3 Matrix Formulation

Like in the standard mortar case, each basis function of V^h is associated with an edge midpoint either in the interior of a subdomain or on a mortar, and not on any nonmortar. Let $\varphi_k^{(i)}$ denote a standard nodal basis function of $X_h(\Omega_i)$, associated with an edge midpoint $x_k \in \overline{\Omega}_{ih}^{CR}$. The basis functions of V^h can be defined as follows. If $x_k \in \Omega_{ih}^{CR}$, a subdomain interior node, then ϕ_k is identical with $\varphi_k^{(i)}$. If $x_k \in \gamma_{m(i)h}^{CR}$, a mortar node, then $\phi_k(x) = \varphi_k^{(i)}(x)$ on $\overline{\Omega}_i$, while on $\overline{\delta}_{m(j)}$, where $\gamma_{m(i)} = \delta_{m(j)}$, $\phi_k(x) = Q_m(I_m\varphi_k^{(i)})(x)$ at $x \in \delta_{m(j)h}^{CR}$. ϕ_k is zero at the remaining edge midpoints of $\overline{\Omega}_j$, and zero everywhere on the remaining subdomains. Using the basis functions of V_h, the problem (1.4) can be rewritten in the matrix form as

$$\mathbf{A}\mathbf{u}^* = \mathbf{f}, \tag{1.6}$$

where $\mathbf{u}^*$ is a vector of nodal values of u_h^*, and $\mathbf{A}$ is a matrix generated by the bilinear form $a_h(.,.)$ on $V_h \times V_h$. We shall now see how this matrix can be obtained from the local matrices $\hat{\mathbf{E}}_i$ generated by $a_i(.,.)$ on $X_h(\Omega_i) \times X_h(\Omega_i)$.

Observing that $a_h(.,.) = \sum_{i=1}^N a_i(.,.)$, where $a_i(.,.) = a_h(.,.)|_{\Omega_i}$, we can calculate the elements of $\mathbf{A}$ from their local contributions restricted to individual subdomains Ω_i. In order to calculate the local contribution $a_i(.,.)$, we use only those basis functions that have nonzero supports on $\overline{\Omega}_i$. These basis functions are exactly the ones associated with the nodes of Ω_{ih}^{CR}, $\gamma_{m(i)h}^{CR}$ ($\gamma_{m(i)} \subset \partial\Omega_i$), and the set $\gamma_{m(j)h}^{CR}$ ($\gamma_{m(j)} = \delta_{m(i)} \subset \partial\Omega_i$) of neighboring mortar edge midpoints except those on $\partial\Omega$. Let Λ_i be the set of all these nodes, see Fig. 1.3 for an illustration.

Let $\mathbf{P}_i$ be the restriction matrix which is a permutation of a rectangular identity matrix, such that $\mathbf{P}_i\mathbf{u}$ returns the vector of all coefficients of $\mathbf{u}$, associated with the nodes of Λ_i. $\mathbf{P}_i^T$ is the corresponding extension matrix. Let $\mathbf{E}_i$, associated with the subdomain Ω_i, be the matrix generated by $a_i(.,.)$ on $span\{\phi_k : x_k \in \Lambda_i\} \times span\{\phi_l : x_l \in \Lambda_i\}$. Using these three types of matrices, we can assemble the global matrix as $\mathbf{A} = \sum_{i=1}^{N} \mathbf{P}_i^T \mathbf{E}_i \mathbf{P}_i$.

We note that $\mathbf{E}_i = \{a_i(\phi_k, \phi_l)\}$, where $x_k, x_l \in \Lambda_i$, and $\hat{\mathbf{E}}_i = \{a_i(\varphi_k, \varphi_l)\}$, where $x_k, x_l \in \overline{\Omega}_{ih}^{CR}$. If $x_k, x_l \in \Omega_{ih}^{CR} \cup \gamma_{m(i)h}^{CR}$, then $a_i(\phi_k, \phi_l) = a_i(\varphi_k, \varphi_l)$. If $x_k \in \gamma_{m(j)h}^{CR}$, then the calculation of an element of $\mathbf{E}_i$ involving ϕ_k, requires the values of $Q_m(I_m\varphi_k)(x_o)$ at the nodes $x_o \in \delta_{m(i)h}^{CR}$, since by definition $\phi_k = \sum_{x_o \in \delta_{m(i)h}^{CR}} Q_m(I_m\varphi_k)(x_o)\varphi_o$ in $\overline{\Omega}_i$. In the following, we derive these coefficients $\{Q_m(I_m\varphi_k)(x_o)\}$ from the mortar condition.

For a mortar γ_m, let $\mathbf{I}_m$ be the matrix representation of the interpolation operator $I_m : X_h(\gamma_m) \to W_{\frac{h}{2}}(\gamma_m)$, whose columns correspond to the nodes $\gamma_{mh}^{CR} = \{x_2, x_4, \ldots, x_{2p}\}$ (edge midpoints of $\mathcal{T}_h(\gamma_m)$), and the rows correspond to the nodes $\overline{\gamma}_{m\frac{h}{2}} = \{x_1, x_2, \ldots, x_{2p+1}\}$ (edge midpoints of $\mathcal{T}_{\frac{h}{2}}(\gamma_m)$), along the mortar γ_m. Let $h_i = |x_{2i-1} - x_{2i+1}|$, then

$$
\mathbf{I}_m = \begin{bmatrix}
(1+\frac{h_1}{h_1+h_2}) & \frac{-h_1}{h_1+h_2} & & & & & \\
1 & & & & & & \\
\frac{h_2}{h_1+h_2} & \frac{h_1}{h_1+h_2} & & & & & \\
 & 1 & & & & & \\
 & \frac{h_3}{h_2+h_3} & * & & & & \\
 & & * & & & & \\
 & & * & \frac{h_{i-1}}{h_{i-1}+h_i} & & & \\
 & & & 1 & & & \\
 & & & \frac{h_{i+1}}{h_i+h_{i+1}} & * & & \\
 & & & & * & & \\
 & & & & * & \frac{h_{n-2}}{h_{n-2}+h_{n-1}} & \\
 & & & & & 1 & \\
 & & & & & \frac{h_n}{h_{n-1}+h_n} & \frac{h_{n-1}}{h_{n-1}+h_n} \\
 & & & & & & 1 \\
 & & & & & \frac{-h_n}{h_{n-1}+h_n} & (1+\frac{h_n}{h_{n-1}+h_n})
\end{bmatrix}.
$$

We assume that the subdomain Ω_i has only one nonmortar side $\delta_{m(i)}$, cf. Fig. 1.3, the extension to more than one nonmortar edge is straightforward. Let the master matrix be $\mathbf{M}_{\gamma_{m(j)}} = \{(I_m\varphi_k, \psi_o)_{L^2(\delta_{m(i)})}\}$, and the slave matrix be $\mathbf{S}_{\delta_{m(i)}} = \{(\varphi_l, \psi_o)_{L^2(\delta_{m(i)})}\}$, where $x_k \in \gamma_{m(j)h}^{CR}$ and $x_l, x_o \in \delta_{m(i)h}^{CR}$. Let ξ_n be the basis function of $W_{\frac{h}{2}}$, associated with the edge midpoint $x_n \in \overline{\gamma}_{m\frac{h}{2}}$. Then

$$
\mathbf{O}_{m(i)} = \mathbf{S}_{\delta_{m(i)}}^{-1} \mathbf{M}_{\gamma_{m(j)}} = \mathbf{S}_{\delta_{m(i)}}^{-1} \mathbf{N}_{\gamma_{m(j)}} \mathbf{I}_m
$$

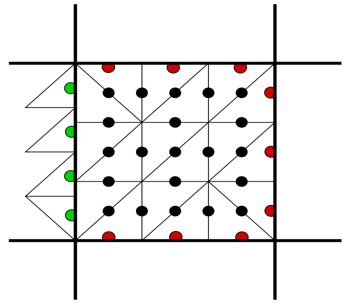

Fig. 1.3. Showing Ω_i with one nonmortar side and the corresponding set Λ_i of edge midpoints shown as dots (in the interior) and semi-dots (on the mortars).

is the matrix representation of the mortar projection $Q_m I_m : X_h(\gamma_{m(j)}) \to M^{h_i}(\delta_{m(i)})$, where $\mathbf{N}_{\gamma_{m(j)}} = \left\{(\xi_n, \psi_o)_{L^2(\delta_{m(i)})}\right\}$ with $x_n \in \overline{\gamma}_{m(j)\frac{h}{2}}$ and $x_o \in \delta^{CR}_{m(i)h}$. The columns of this matrix correspond to the nodes $x_k \in \gamma^{CR}_{m(j)h}$, containing exactly the coefficients $\{Q_m(I_m\varphi_k)(x_o)\}$. We note that $\mathbf{S}_{\delta_{m(i)}}$ is a diagonal matrix containing the lengths of the edges along $\delta_{m(i)}$ as entries.

Now define the matrix $\mathbf{O}_i = diag(\mathbf{I}, \mathbf{O}_{m(i)})$, where $\mathbf{I}$ is the identity matrix corresponding to the nodes of Ω^{CR}_{ih} and $\gamma^{CR}_{m(i)h}$, and $\mathbf{O}_{m(i)}$ is the mortar projection matrix corresponding to the nodes of $\gamma^{CR}_{m(j)h}$. Then it is easy to see that $\mathbf{E}_i = \mathbf{O}_i^T \hat{\mathbf{E}}_i \mathbf{O}_i$. Finally, we have $\mathbf{A} = \sum_{i=1}^N \mathbf{P}_i^T \mathbf{O}_i^T \hat{\mathbf{E}}_i \mathbf{O}_i \mathbf{P}_i$. In the same way, we get $\mathbf{f} = \sum_{i=1}^N \mathbf{P}_i^T \mathbf{O}_i^T \hat{\mathbf{f}}_i$.

1.4 An additive Schwarz method

In this section, we design an additive Schwarz method for the problem (1.4), which is an extension of the algorithm in Rahman et al. (2004) for the standard mortar case, to the new mortar variant. The method is defined using the general framework for additive Schwarz methods (cf. Smith et al. (1996)). We decompose V_h as $V_h = V^{\mathcal{S}} + V^0 + \sum_{i=1}^N V^i$. For $i = 1, \ldots, N$, V^i is the restriction of V_h to Ω_i, with functions vanishing at subdomain boundary edge midpoints $\partial\Omega^{CR}_{ih}$ as well as on the remaining subdomains. $V^{\mathcal{S}}$ is a space of functions given by their values on the skeleton edge midpoints $\mathcal{S}^{CR}_h = \bigcup_{\gamma_m} \gamma^{CR}_{mh}$, $V^{\mathcal{S}} = \{v \in V_h : v(x) = 0,\ x \in \overline{\Omega}^{CR}_h \setminus \mathcal{S}^{CR}_h\}$. The coarse space V^0, a special space having a dimension equal to the number of subdomains, is defined using the function $\chi_i \in X_h(\Omega_i)$ associated with the subdomain Ω_i. χ_i is defined by its nodal values as: $\chi_i(x) = 1/\sum_j \rho_j(x)$ at $x \in \overline{\Omega}^{CR}_{ih}$, where the sum is taken over the subdomains Ω_j to which x belongs, and $\rho_j = 1, \forall j$. Note that the ρ_j's may represent physical parameters with jumps across interfaces, see Rahman et al. (2004). V^0 is given as the span of its basis functions, $\Phi_i, i = 1, \ldots, N$, i.e., $V^0 = span\{\Phi_i : i = 1, \ldots, N\}$, where Φ_i associated with Ω_i, is defined as follows.

$$\Phi_i(x) = \begin{cases} 1, & x \in \Omega_{ih}^{CR}, \\ \rho_i \chi_i(x), & x \in \gamma_{m(i)h}^{CR}, \\ \rho_i Q_m(I_m \chi_j)(x), & x \in \delta_{m(i)h}^{CR},\ \delta_{m(i)} = \gamma_{m(j)}, \\ \rho_i Q_m(I_m \chi_i)(x), & x \in \delta_{m(j)h}^{CR},\ \delta_{m(j)} = \gamma_{m(i)}, \\ \rho_i \chi_j(x), & x \in \gamma_{m(j)h}^{CR},\ \gamma_{m(j)} = \delta_{m(i)}, \\ 0, & x \in \partial\Omega_{ih}^{CR} \cap \partial\Omega, \end{cases} \tag{1.7}$$

and $\Phi_i(x) = 0$ at all other x in $\overline{\Omega}_h^{CR}$. We use exact bilinear forms for all our subproblems. The projection like operators $T^i : V_h \to V^i$ are defined in the standard way, i.e., for $i \in \{\mathcal{S}, 0, \ldots, N\}$ and $u \in V_h$, $T^i u \in V^i$ is the solution of $a_h(T^i u, v) = a_h(u, v),\ v \in V^i$. Let $T = T^{\mathcal{S}} + T^0 + T^1 + \ldots + T^N$. The problem (1.4) is now replaced by the preconditioned system

$$T u_h^* = g, \tag{1.8}$$

where $g = T^{\mathcal{S}} u_h^* + \sum_{i=0}^{N} T^i u_h^*$. Let c and C represent constants independent of the mesh sizes $h = \inf_i h_i$ and $H = \max_i H_i$, then the following theorem holds.

Theorem 2. *For all $u \in V_h$,*

$$c\frac{h}{H} a_h(u, u) \le a_h(Tu, u) \le C a_h(u, u). \tag{1.9}$$

The theorem can be shown in the same way as the proof in Rahman et al. (2004), which uses the general theory for Schwarz methods, cf. Smith et al. (1996). It follows from the theorem that the condition number of the operator T grows as $\frac{H}{h}$.

1.5 Numerical results

For the experiment, we consider our model problem to be defined on a unit square domain, Ω, with the forcing function f so chosen that the exact solution u is equal to $\sin(\pi x)\sin(\pi y)$. The domain Ω is initially divided into $3^2 = 9$ square subdomains (subregions). Each subdomain is then discretized uniformly using, in a checkerboard fashion, either $2m^2$ or $2n^2$ right angle triangles of equal size, where m and n are fixed and $m \neq n$ resulting in a grid which is nonmatching across all interfaces.

A comparison between the standard and the proposed mortar technique for the Crouzeix-Raviart finite element is shown in Table 1.1. The Preconditioned Conjugate Gradients (PCG) method has been used to solve the resulting algebraic systems with their respective additive Schwarz preconditioners. As seen from the table, the numerical results agree with the theory. The proposed method exhibits a similar behavior as that of the standard method.

Table 1.1. Condition number estimates (κ_2), PCG-iteration counts (#iter), and L^2-norm (error$_{L^2}$) and H^1-seminorm (error$_{H^1}$) of the error in each case.

$\{m,n\}$	Standard CR Mortar				Proposed CR Mortar			
	κ_2	#iter	error$_{L^2}$	error$_{H^1}$	κ_2	#iter	error$_{L^2}$	error$_{H^1}$
$\{06, 05\}$	28.85	25	0.002020	0.065293	30.11	23	0.002484	0.078409
$\{12, 10\}$	63.44	35	0.000497	0.032843	60.90	31	0.000667	0.038768
$\{24, 20\}$	134.18	49	0.000123	0.016479	122.55	45	0.000175	0.019321

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