A new algorithm for solving 3D contact problems with Coulomb friction *

Radek Kučera

VŠB-TU Ostrava, Department of Mathematics and Descriptive Geometry,
name@email.address

Summary. The paper deals with solving of contact problems with *Coulomb* friction for a system of 3D elastic bodies. The iterative method of successive approximations is used in order to find a fixed point of certain mapping that defines the solution. In each iterative step, an auxiliary problem with *given* friction is solved that is discretized by the FETI method. Then the duality theory of convex optimization is used in order to obtain the constrained quadratic programming problem that, in contrast to 2D case, is subject to quadratic inequality constraints. The solution is computed (among others) by a novelly developed algorithm of constrained quadratic programming. Numerical experiments demonstrate the performance of the whole computational process.

1 Introduction

The FETI method was proposed by [FR92] for parallel solution of problems described by elliptic partial differential equations. The key idea is elimination of the primal variables so that the original problem is reduced to a small, relatively well conditioned quadratic programming problem (QPP) in terms of the Lagrange multipliers. Then the iterative solver is used to compute the solution.

In context of 2D contact problems with friction, the FETI procedure leads to the sequence of QPPs constrained by simple inequality bounds (see [DHK02] or [HDK02]) so that the fast algorithm with proportioning and gradient projection (see [DS05]) can be used. The situation is not so easy in 3D since the QPPs are subject to two types of constraints. The first one, representing nonnegativity of the normal contact stress, are again simple inequality bounds while the second one, representing an effect of isotropic friction, are quadratic inequalities. In our recent papers [HKD04], [KHD05], we have used a linear approximation of quadratic inequalities transforming them to simple inequality bounds so that the fast algorithm mentioned above can

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be used again. Unfortunately, this procedure increases considerably the size of the QPPs if we require a sufficiently accurate approximation of quadratic inequalities. In order to overcome this drawback, we have developed a new algorithm of quadratic programming that treates directly the quadratic inequalities [K05]. In this contribution, we shall show the performance of the whole computational process on model problems.

2 Formulation of the problems

Let us consider a system of elastic bodies that occupy in the reference configuration bounded domains $\Omega^p \subset \mathbb{R}^3$, $p = 1, 2, \ldots, s$, with sufficiently smooth boundaries Γ^p that are split into three disjoint parts Γ_u^p, Γ_t^p and Γ_c^p so that $\Gamma^p = \overline{\Gamma_u^p} \cup \overline{\Gamma_t^p} \cup \overline{\Gamma_c^p}$. Let us suppose that the zero displacements are prescribed on Γ_u^p and that the surface tractions of density $\mathbf{t}^p \in (L^2(\Gamma_t^p))^3$ act on Γ_t^p. Along Γ_c^p the body Ω^p may get into unilateral contact with some other of the bodies. Finally we suppose that the bodies Ω^p are subject to the volume forces of density $\mathbf{f}^p \in (L^2(\Omega^p))^3$.

To describe non-penetration of the bodies, we shall use linearized non-penetration condition that is defined by a mapping $\boldsymbol{\chi} : \Gamma_c \longrightarrow \Gamma_c$, $\Gamma_c = \bigcup_{p=1}^s \Gamma_c^p$, which assigns to each $\mathbf{x} \in \Gamma_c^p$ some nearby point $\boldsymbol{\chi}(\mathbf{x}) \in \Gamma_c^q$, $p \neq q$. Let $\mathbf{v}^p(\mathbf{x}), \mathbf{v}^q(\boldsymbol{\chi}(\mathbf{x}))$ denote the displacement vectors at $\mathbf{x}, \boldsymbol{\chi}(\mathbf{x})$, respectively. Assuming the small displacements, the *non-penetration condition* reads

$$v_n^p(\mathbf{x}) \equiv (\mathbf{v}^p(\mathbf{x}) - \mathbf{v}^q(\boldsymbol{\chi}(\mathbf{x}))) \cdot \mathbf{n}^p(\mathbf{x}) \leq \delta^p(\mathbf{x}),$$

where $\delta^p(\mathbf{x}) = (\boldsymbol{\chi}(\mathbf{x}) - \mathbf{x}) \cdot \mathbf{n}^p(\mathbf{x})$ is the initial gap and $\mathbf{n}^p(\mathbf{x})$ is the critical direction defined by $\mathbf{n}^p(\mathbf{x}) = (\boldsymbol{\chi}(\mathbf{x}) - \mathbf{x})/\|\boldsymbol{\chi}(\mathbf{x}) - \mathbf{x}\|$ or, if $\boldsymbol{\chi}(\mathbf{x}) = \mathbf{x}$, by the outer unit normal vector to Γ_c^p.

We start with an auxiliary contact problem with given friction. To this end we introduce the space of *virtual displacements* V and its closed convex subset of *kinematically admissible* displacements K by

$$V = \{\mathbf{v} = (\mathbf{v}^1, \ldots, \mathbf{v}^s) \in \prod_{p=1}^s (H^1(\Omega^p))^3 : \mathbf{v}^p = 0 \text{ on } \Gamma_u^p\},$$
$$\mathcal{K} = \{\mathbf{v} \in V : v_n^p(\mathbf{x}) \leq \delta^p(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma_c^p\}.$$

Let us assume that the normal contact stress $T_n \in L^\infty(\Gamma_c)$, $T_n \geq 0$, is known *apriori* so that one can evaluate the slip bound g on Γ_c by $g = FT_n$, where $F = F^p > 0$ is a coefficient of friction on Γ_c^p. Denote $g^p = g|_{\Gamma_c^p}$.

The variational formulation of the contact problem with *given* friction reads as follows:

$$\min \mathcal{J}(\mathbf{v}) \quad \text{subject to} \quad \mathbf{v} \in \mathcal{K}, \tag{1}$$

where

$$\mathcal{J}(\mathbf{v}) = \tfrac{1}{2}a(\mathbf{v},\mathbf{v}) - b(\mathbf{v}) + j(\mathbf{v})$$

is the total potential energy functional with the bilinear form a representing the inner energy of the bodies and with the linear form b representing the work of the applied forces $\mathbf{t}^p$ and $\mathbf{f}^p$, respectively. The sublinear functional j represents the work of friction forces

$$j(\mathbf{v}) = \sum_{p=1}^{s} \int_{\Gamma_c^p} g^p \|\mathbf{v}_t^p\| \, d\Gamma, \tag{2}$$

where $\mathbf{v}_t^p$ is the projection of the displacement $\mathbf{v}^p$ on the plane tangential to the critical direction $\mathbf{n}^p$. Let us introduce unit tangential vectors $\mathbf{t}_1^p, \mathbf{t}_2^p$ such that the triplet $\mathcal{B} = \{\mathbf{n}^p, \mathbf{t}_1^p, \mathbf{t}_2^p\}$ is an orthonormal basis in $\mathbb{R}^3$ for almost all $\mathbf{x} \in \Gamma_c^p$ and denote $v_{t_1}^p = \mathbf{v}^p \cdot \mathbf{t}_1^p$, $v_{t_2}^p = \mathbf{v}^p \cdot \mathbf{t}_2^p$. Then $\mathbf{v}_t^p = (0, v_{t_1}^p, v_{t_2}^p)$ with respect to the basis $\mathcal{B}$ so that the norm appearing in j reduces to the Euclidean norm in $\mathbb{R}^2$. More details about the formulation of contact problems can be found in [HHNL88].

Let us point out that the solution $\mathbf{u} \equiv \mathbf{u}(g)$ of (1) depends on a particular choice of $g \in L^\infty(\Gamma_c)$, $g \geq 0$. We can define a mapping Φ which associates with every g the product $FT_n(\mathbf{u}(g))$, where $T_n(\mathbf{u}(g)) \geq 0$ is the normal contact stress related to $\mathbf{u}(g)$. The classical Coulomb's law of friction corresponds to the fixed point of Φ which is defined by $g = FT_n(\mathbf{u}(g))$. To find it, we can use the *method of successive approximations* which starts from a given $g^{(0)}$ and generates the iterations $g^{(l)}$ by

(MSA)
$$g^{(l+1)} = \Phi(g^{(l)}), \; l = 1, 2, \ldots.$$

This iterative process converges provided Φ is *contractive*, that is guaranteed for sufficiently small F (see [H83]).

3 Domain decomposition and discretization

We divide the bodies Ω^p into tetrahedron finite elements $\mathcal{T}$ with the maximum diameter h and assume that the partitions are regular and consistent with the decompositions of $\partial\Omega^p$ into Γ_u^p, Γ_t^p and Γ_c^p. Moreover, we restrict ourselves to the geometrical conforming situation where the intersection between the boundaries of any two different bodies $\partial\Omega^p \cap \partial\Omega^q$, $p \neq q$, is either empty, a vertex, an entire edge, or an entire face.

Let the domains Ω^p be decomposed into nonoverlaping subdomains $\Omega^{p,i}$, $i = 1, \ldots, n_p$, each of which is the union of finite elements of $\mathcal{T}$. On $\Omega^{p,i}$, we introduce the finite element space $V_h^{p,i}$ by

$$V_h^{p,i} = \{\mathbf{v}^{p,i} \in (C(\Omega^{p,i}))^3 : \mathbf{v}^{p,i}|_{\mathcal{T}} \in (P_1(\mathcal{T}))^3 \text{ for all } \mathcal{T} \subset \Omega^{p,i},$$
$$\mathbf{v}^{p,i}|_{\partial\Omega^{p,i} \cap \Gamma_u^p} = 0\},$$

where $P_m(\mathcal{T})$ denotes the set of all polynomials on $\mathcal{T}$ of degree $\leq m$. Finally, let us introduce the product space $V_h = \prod_{p=1}^{s} \prod_{i=1}^{n_p} V_h^{p,i}$.

Replacing V by V_h and using the *gluing condition* $\mathbf{v}^{p,i}(\mathbf{x}) = \mathbf{v}^{p,j}(\mathbf{x})$ for any $\mathbf{x}$ in the interface $\partial\Omega^{p,i} \cap \partial\Omega^{p,j}$, we can rewrite the approximative contact problem with given friction (1) into the algebraic form

$$\begin{aligned} &\min \frac{1}{2}\mathbf{u}^\top \mathbf{K}\mathbf{u} - \mathbf{u}^\top \mathbf{f} + \sum_{k=1}^{m} g_k \|((\mathbf{T}_1\mathbf{u})_k, (\mathbf{T}_2\mathbf{u})_k)\| \\ &\text{s.t. } \mathbf{N}\mathbf{u} \leq \mathbf{d},\ \mathbf{B}_E\mathbf{u} = 0. \end{aligned} \tag{3}$$

Here, $\mathbf{K}$ denotes the positive semidefinite block diagonal stiffness matrix, $\mathbf{f}$ is the vector of nodal forces, $\mathbf{N}, \mathbf{d}$ describe the discretized non-penetration condition and $\mathbf{B}_E$ describes the gluing condition. The summation term in the minimized functional arises using numerical quadrature in (2), where $\mathbf{T}_1$, $\mathbf{T}_2$ describe projections of displacements at the nodes lying on Γ_c to the tangential planes and g_k are values of slip bound.

Let us point out that the problem (3) is non-differentiable due to $\mathbb{R}^2$-norms appearing in the summation term. Therefore we shall introduce two kinds of Lagrange multipliers $\boldsymbol{\lambda}_t = (\boldsymbol{\lambda}_{t_1}^\top, \boldsymbol{\lambda}_{t_2}^\top)^\top$ and $\boldsymbol{\lambda}_c = (\boldsymbol{\lambda}_I^\top, \boldsymbol{\lambda}_E^\top)^\top$. While the first one removes the non-differentiability, the second one accounts for the constraints in (3). Denote

$$\mathbf{B}_t = \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} \mathbf{N} \\ \mathbf{B}_E \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{d} \\ \mathbf{o} \end{bmatrix}$$

and introduce the Lagrange multiplier sets

$$\Lambda_t(\mathbf{g}) = \{\boldsymbol{\lambda}_t : \|((\boldsymbol{\lambda}_{t1})_k, (\boldsymbol{\lambda}_{t2})_k)\| \leq g_k\} \text{ and } \Lambda_c = \{\boldsymbol{\lambda}_c : (\boldsymbol{\lambda}_I)_k \geq 0\}.$$

It is well-known that (3) is equivalent to the saddle-point problem

$$\text{Find } (\mathbf{u}, \boldsymbol{\lambda}_t, \boldsymbol{\lambda}_c) \quad \text{s.t.} \quad \mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}_t, \boldsymbol{\lambda}_c) = \sup_{\substack{\boldsymbol{\mu}_t \in \Lambda_t(\mathbf{g}) \\ \boldsymbol{\mu}_c \in \Lambda_c}} \inf_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \boldsymbol{\mu}_t, \boldsymbol{\mu}_c), \tag{4}$$

where $\mathcal{L}$ is the Lagrangian to (3) defined by

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}_t, \boldsymbol{\lambda}_c) = \frac{1}{2}\mathbf{u}^\top \mathbf{K}\mathbf{u} - \mathbf{u}^\top \mathbf{f} + \boldsymbol{\lambda}_t^\top \mathbf{B}_t\mathbf{u} + \boldsymbol{\lambda}_c^\top(\mathbf{B}_c\mathbf{u} - \mathbf{c}).$$

After eliminating the primal variables $\mathbf{u}$ from (4), we obtain the minimization problem

$$\begin{aligned} &\min \frac{1}{2}\boldsymbol{\lambda}^\top \mathbf{F}\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top \mathbf{h} \\ &\text{s.t. } \boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_t \\ \boldsymbol{\lambda}_c \end{bmatrix}, \boldsymbol{\lambda}_t \in \Lambda_t(\mathbf{g}), \boldsymbol{\lambda}_c \in \Lambda_c,\ \mathbf{G}\boldsymbol{\lambda} = \mathbf{e} \end{aligned} \tag{5}$$

with

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{tt} & \mathbf{F}_{tc} \\ \mathbf{F}_{tc}^\top & \mathbf{F}_{cc} \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} \mathbf{h}_t \\ \mathbf{h}_c \end{bmatrix}, \quad \mathbf{G} = [\mathbf{G}_t, \mathbf{G}_c],$$

and $\mathbf{F}_{ii} = \mathbf{B}_i\mathbf{K}^\dagger\mathbf{B}_i^\top$, $\mathbf{G}_i = \mathbf{R}^\top\mathbf{B}_i^\top$, $i = t, c$, $\mathbf{F}_{tc} = \mathbf{B}_t\mathbf{K}^\dagger\mathbf{B}_c^\top$, $\mathbf{h}_t = \mathbf{B}_t\mathbf{K}^\dagger\mathbf{f}$, $\mathbf{h}_c = \mathbf{B}_c\mathbf{K}^\dagger\mathbf{f} - \mathbf{c}$, $\mathbf{e} = \mathbf{R}^\top\mathbf{f}$, where $\mathbf{K}^\dagger$ denotes a generalized inverse to $\mathbf{K}$ and $\mathbf{R}$ is the full rank matrix whose columns span the kernel of $\mathbf{K}$.

The problem (5) can be adapted by using the orthogonal projectors as proposed in [FMR94]. To simplify our presentation, we omit description of this modification here.

4 Algorithms

The problem (5) can be solved by using the algorithm based on the augmented Lagrangian

$$L(\boldsymbol{\lambda}, \boldsymbol{\mu}, \rho) = \frac{1}{2}\boldsymbol{\lambda}^\top\mathbf{F}\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top\mathbf{h} + \boldsymbol{\mu}^\top(\mathbf{G}\boldsymbol{\lambda} - \mathbf{e}) + \frac{\rho}{2}(\mathbf{G}\boldsymbol{\lambda} - \mathbf{e})^\top(\mathbf{G}\boldsymbol{\lambda} - \mathbf{e}).$$

Algorithm 1. Set $\boldsymbol{\mu}^{(0)}$, $l := 0$.

repeat

$\boldsymbol{\lambda}^{(l+1)} \doteq \operatorname{argmin} L(\boldsymbol{\lambda}, \boldsymbol{\mu}^{(l)}, \rho)$, s.t. $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_t(\mathbf{g}) \times \boldsymbol{\Lambda}_c$

$\boldsymbol{\mu}^{(l+1)} = \boldsymbol{\mu}^{(l)} + \rho(\mathbf{G}\boldsymbol{\lambda}^{(l+1)} - \mathbf{e})$

Update ρ and increse l by one.

until stopping criterion

Algorithms of this type have been intensively studied recently [DFS03], [D05] with the inner minimization represented by the QPP with simple inequality bounds of $\boldsymbol{\Lambda}_c$. Here, the quadratic inequality constraints of $\boldsymbol{\Lambda}_t(\mathbf{g})$ are imposed furthermore. In order to separate two types of constraints, we can split the inner minimization by the constrained block Gauss-Seidel method. Then the efficient algorithm using projections and adaptive precision control may be used for the first QPP with simple inequality bounds [DS05] while the second QPP constrained by quadratic inequalities can be solved by the algorithm proposed in [K05]. Let us point out that augmented Lagrangian based algorithms accept an inexact solution of the inner minimizations without loss of the accuracy. Therefore it is natural to reduce the number of Gauss-Seidel iterations even onto one.

The method of successive approximations (MSA) for solving the contact problem with Coulomb friction can be implemented so that the Algorithm 1 is used in each iterative step to evaluate the mapping Φ. We shall present a more efficient version of this method, in which the iterative steps of (MSA) and the loop of the Algorithm 1 are connected in one loop. The resulting algorithm can be viewed as the method of successive approximations with an *inexact* solving of the auxiliary problems with given friction.

Algorithm 2. Set $\boldsymbol{\mu}^{(0)}$, $\boldsymbol{\lambda}_t^{(0)}$, $l := 0$.

repeat

$\boldsymbol{\lambda}_c^{(l+1)} \doteq \operatorname{argmin} \{\frac{1}{2}\boldsymbol{\lambda}_c^\top(\mathbf{F}_{cc} + \rho\mathbf{G}_c^\top\mathbf{G}_c)\boldsymbol{\lambda}_c - \boldsymbol{\lambda}_c^\top(\mathbf{h}_c + \mathbf{G}_c^\top(\rho\mathbf{e} + \boldsymbol{\mu}^{(l)}) - (\mathbf{F}_{tc}^\top + \rho\mathbf{G}_c^\top\mathbf{G}_t)\boldsymbol{\lambda}_t^{(l)})\}$, s.t. $\boldsymbol{\lambda}_c \in \boldsymbol{\Lambda}_c$

$\boldsymbol{\lambda}_t^{(l+1)} \doteq \operatorname{argmin} \{\frac{1}{2}\boldsymbol{\lambda}_t^\top(\mathbf{F}_{tt} + \rho\mathbf{G}_t^\top\mathbf{G}_t)\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t^\top(\mathbf{h}_t + \mathbf{G}_t^\top(\rho\mathbf{e} + \boldsymbol{\mu}^{(l)}) - (\mathbf{F}_{tc} + \rho\mathbf{G}_t^\top\mathbf{G}_c)\boldsymbol{\lambda}_c^{(l+1)})\}$, s.t. $\boldsymbol{\lambda}_t \in \boldsymbol{\Lambda}_t(F\boldsymbol{\lambda}_I^{(l+1)})$

$\boldsymbol{\mu}^{(l+1)} = \boldsymbol{\mu}^{(l)} + \rho(\mathbf{G}\boldsymbol{\lambda}^{(l+1)} - \mathbf{e})$

Update ρ and increse l by one.

until stopping criterion

We have used the fact that the Lagrange multiplier $\boldsymbol{\lambda}_I$ represents the normal contact stress so that $\mathbf{g} = F\boldsymbol{\lambda}_I^{(l+1)}$ approximates the slip bound.

5 Numerical experiments and conclusions

Let us consider the model brick $\Omega = \langle 0, 3\rangle \times \langle 0, 1\rangle \times \langle 0, 1\rangle$ made of an elastic isotropic, homogeneous material characterized by Young modulus $E = 21.2 \times 10^{10}$ and Poisson's ratio $\sigma = 0.277$ (steel). The brick is unilaterally supported by the rigid foundation, where the non-penetration condition and the effect of Coulomb friction is considered. The applied surface tractions and the parts of the boundary Γ_u and Γ_c are seen in Figure 1. The volume forces vanish. The brick Ω is artificially decomposed onto three parts as seen in Figure 2 so that the resulting problem has 12 rigid modes.

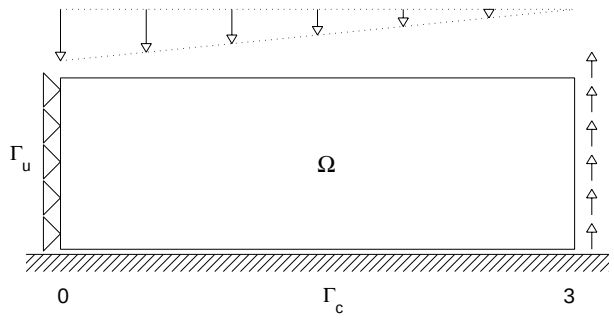

Fig. 1. The cross-section of the brick Ω.

The tables below summarize results of numerical experiments, where F is the coefficient of friction; n denotes the number of primal unknowns (displacements); m denotes the number of dual unknowns (stresses); *Time* is CPU

time in seconds (in Matlab 7, Pentium(R)4, 3GHz, 512MB); *Iter* is the number of outer iterations; n_A^{QPP}, n_A^{QPQ} is the total number of multiplications by the Hessian in the QPP, QPQ solver, respectively, and $n_A = n_A^{QPP} + n_A^{QPQ}$.

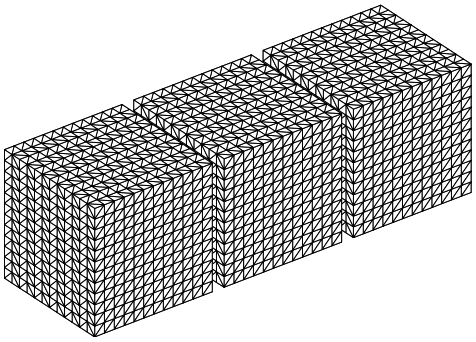

Fig. 2. Discretization and decomposition of the brick Ω.

Table 1. $F = 0.1$

n	m	*Time*	*Iter*	n_A
900	180	1	5	**102**(=63+39)
2646	378	11	5	**180**(=98+82)
5832	648	34	5	**156**(=94+62)
10890	990	67	5	**112**(=50+62)
18252	1404	221	5	**155**(=73+82)

Table 2. $F = 0.3$

n	m	*Time*	*Iter*	n_A
900	180	2	7	**140**(=46+32)
2646	378	12	7	**186**(=54+69)
5832	648	38	7	**169**(=72+50)
10890	990	94	7	**153**(=35+49)
18252	1404	254	7	**176**(=78+54)

Table 1 and Table 2 demonstrate the numerical scalability of the algorithm for various coefficients of friction. Table 3 shows the substantial progress with respect to approximative method used in [HKD04] represented here by *Time2* and *Time4*.

Table 3. $F = 0.3$

n	m	*Time*	*Time2*	*Time4*
900	180	2	15	61
2646	378	12	101	548
5832	648	38	486	2114
10890	990	94	1542	7724
18252	1404	254	5004	20534

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