Schwarz Preconditioning for High Order Simplicial Finite Elements

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Summary. This paper analyzes two-level Schwarz methods for matrices arising from the p-version finite element method on triangular and tetrahedral meshes. The coarse level consists of the lowest order finite element space. On the fine level, we investigate several decompositions with large or small overlap leading to optimal or close to optimal condition numbers. The analysis is confirmed by numerical experiments for a model problem.

The first and the last author acknowledge their support by the Austrian Science Foundation FWF within project grant Start Y-192, "hpFEM : Fast Solvers and Adaptivity"

1 Introduction

High order finite element methods can lead to very high accuracy and are thus attracting increasing attention in many fields of computational science and engineering. The monographs [SB91, BS94, Sch98, KS99, SDR04] give a broad overview of theoretical and practical aspects of high order methods.

As the problem size increases (due to small mesh-size h and high polynomial order p), the solution of the arising linear system of equations becomes more and more the time-dominating part. Here, iterative solvers can reduce the total simulation time. We consider preconditioners based on domain decomposition methods [DW90, GO95, SBG96, TW04, Qua99]. The concept is to consider each high order element as an individual sub-domain. Such methods were studied in [Man90, BCM91, Pav94, Ain96a, Ain96b, Cas97, Bic97, GC98, SC01, Mel02, EM04]. We assume that the local problems can be solved directly. On tensor product elements, one can apply optimal preconditioners for the local sub-problems as in [KJ99, BSS04, BS04].

In the current work, we study overlapping Schwarz preconditioners with large or small overlap. The condition numbers are bounded uniformly in the mesh size h and the polynomial order p. To our knowledge, this is a new result for tetrahedral meshes. We construct explicitly the decomposition of a global function into a coarse grid part and local contributions associated with the vertices, edges, faces, and elements of the mesh. In this paper, we sketch the analysis for the two dimensional version, and give the result for the 3D case. All proofs are given in the longer version [SMP05].

The rest of the paper is organized as follows: In Section 2 we state the problem and formulate the main results. We sketch the 2D case in Section 3 and extend the result for 3D in Section 4. Finally, in Section 5 we give numerical results for several versions of the analyzed preconditioners.

2 Definitions and Main Result

We consider the Poisson equation on the polyhedral domain Ω with homogeneous Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$, and Neumann boundary conditions on the remaining part Γ_N. With the sub-space $V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, the bilinear-form $A(\cdot,\cdot) : V \times V \to \mathbb{R}$ and the linear-form $f(\cdot) : V \to \mathbb{R}$ defined as

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad f(v) = \int_\Omega f v \, dx,$$

the weak formulation reads

$$\text{find } u \in V \text{ such that} \qquad A(u,v) = f(v) \qquad \forall\, v \in V. \tag{1}$$

We assume that the domain Ω is sub-divided into straight-sided triangular or tetrahedral elements. In general, constants in the estimates depend on the shape of the elements, but they do not depend on the local mesh-size. We define the set of vertices $\mathcal{V} = \{V\}$, the set of edges $\mathcal{E} = \{E\}$, the set of faces (3D only) $\mathcal{F} = \{F\}$, the set of elements $\mathcal{T} = \{T\}$. We define the sets $\mathcal{V}_f, \mathcal{E}_f, \mathcal{F}_f$ of *free* vertices, edges, and faces not completely contained in the Dirichlet boundary. The high order finite element space is

$$V_p = \{v \in V : v|_T \in P^p \;\forall\, T \in \mathcal{T}\},$$

where P^p is the space of polynomials up to total order p. As usual, we choose a basis consisting of lowest order affine-linear functions associated with the vertices, and of edge-based, face-based, and cell-based bubble functions. The Galerkin projection onto V_p leads to a large system of linear equations, which shall be solved with the preconditioned conjugate gradient iteration.

This paper is concerned with the analysis of additive Schwarz preconditioning. The basic method is defined by the following space splitting. In Section 5 we will consider several cheaper versions resulting from our analysis. The coarse sub-space is the global lowest order space

$$V_0 := \{v \in V : v|_T \in P^1 \; \forall T \in \mathcal{T}\}.$$

For each inner vertex we define the vertex patch $\omega_V = \bigcup_{T \in \mathcal{T}: V \in T} T$ and the vertex sub-space

$$V_V = \{v \in V_p : v = 0 \text{ in } \Omega \setminus \omega_V\}.$$

For vertices V not on the Neumann boundary, this definition coincides to $V_p \cap H_0^1(\omega_V)$. The additive Schwarz preconditioning operator is $C^{-1} : V_p^* \to V_p$ defined by

$$C^{-1} d = w_0 + \sum_{V \in \mathcal{V}} w_V$$

with $w_0 \in V_0$ such that

$$A(w_0, v) = \langle d, v \rangle \qquad \forall \, v \in V_0,$$

and $w_V \in V_V$ defined such that

$$A(w_V, v) = \langle d, v \rangle \qquad \forall \, v \in V_V.$$

This method is very simple to implement for the p-version method using a hierarchical basis. The low-order block requires the inversion of the sub-matrix according to the vertex basis functions. The high order blocks are block-Jacobi steps, where the blocks contain all vertex, edge, face, and cell unknowns associated with mesh entities containing the vertex V. The main result of this paper is to prove optimal results for the spectral bounds:

Theorem 1. *The constants λ_1 and λ_2 of the spectral bounds*

$$\lambda_1 \langle Cu, u \rangle \leq A(u,u) \leq \lambda_2 \langle Cu, u \rangle \qquad \forall \, u \in V_p$$

are independent of the mesh-size h and the polynomial order p.

The proof is based on the additive Schwarz theory, which allows to express the C-form by means of the space decomposition:

$$\langle Cu, u \rangle = \inf_{\substack{u = u_0 + \sum_V u_V \\ u_0 \in V_0, u_V \in V_V}} \|u_0\|_A^2 + \sum \|u_V\|_A^2.$$

The constant λ_2 follows immediately from a finite number of overlapping sub-spaces. In the core part of this paper, we construct an explicit and stable decomposition of u into sub-space functions. Section 3 introduces the decomposition for the case of triangles, in Section 4 we prove the results for tetrahedra.

3 Sub-space splitting for triangles

The strategy of the proof is the following: First, we subtract a coarse grid function to eliminate the h-dependency. By stepwise elimination, the remaining function is then split into sums of vertex-based, edge-based and inner functions. For each partial sum, we give the stability estimate. This stronger result contains Theorem 1, since we can choose corresponding vertices for the edge and inner contributions (see also Section 5).

3.1 Coarse grid contribution

In the first step, we subtract a coarse grid function:

Lemma 1. *For any $u \in V_p$ there exists a decomposition*

$$u = u_0 + u_1 \tag{2}$$

such that $u_0 \in V_0$ and

$$\|u_0\|_A^2 + \|\nabla u_1\|_{L_2}^2 + \|h^{-1} u_1\|_{L_2}^2 \preceq \|u\|_A^2.$$

Proof. We choose $u_0 = \Pi_h u$, where Π_h is the Clément-operator [Cle75]. The norm bounds are exactly the continuity and approximation properties of this operator.

From now on, u_1 denotes the second term in the decomposition (2).

3.2 Vertex contributions

In the second step, we subtract functions u_V to eliminate vertex values. Since vertex interpolation is not bounded in H^1, we cannot use it. Thus, we construct a new averaging operator mapping into a larger space.

In the following, let V be a vertex not on the Dirichlet boundary Γ_D, and let φ_V be the piece-wise linear basis function associated with this vertex. Furthermore, for $s \in [0, 1]$ we define the level sets

$$\gamma_V(s) := \{y \in \omega_V : \varphi_V(y) = s\},$$

and write $\gamma_V(x) := \gamma_V(\varphi_V(x))$ for $x \in \omega_V$. For internal vertices V, the level set $\gamma_V(0)$ coincides with the boundary $\partial\omega_V$ (cf. Figure 1). The space of functions being constant on these sets reads

$$S_V := \{w \in L_2(\omega_V) : w|_{\gamma_V(s)} = \text{const},\ s \in [0, 1]\ a.e.\};$$

its finite dimensional counterpart is

$$S_{V,p} := S_V \cap V_p = \text{span}\{1, \varphi_V, ..., \varphi_V^p\}.$$

We introduce the *spider averaging operator*

$$\big(\Pi^V v\big)(x) := \frac{1}{|\gamma_V(x)|} \int_{\gamma_V(x)} v(y)\,dy, \qquad \text{for } v \in L_2(\omega_V).$$

To satisfy homogeneous boundary conditions, we add a correction term as follows (see Figure 2)

$$\big(\Pi_0^V v\big)(x) := \big(\Pi^V v\big)(x) - (\Pi^V v)|_{\gamma_V(0)}(1 - \varphi_V(x)).$$

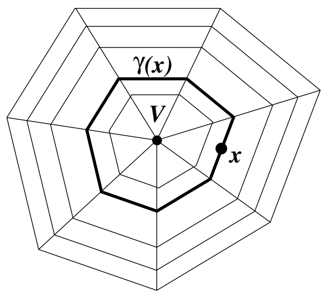

Fig. 1. The level sets $\gamma_V(x)$

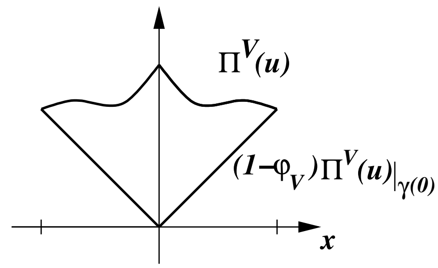

Fig. 2. Construction of Π_0^V

Lemma 2. *The averaging operators fulfill the following algebraic properties*

(i)

$$\Pi^V V_p = S_{V,p},$$

(ii)

$$\Pi_0^V V_p = S_{V,p} \cap V_V,$$

(iii) if u is continuous at V, then

$$(\Pi^V u)(V) = \Pi_0^V u(V) = u(V).$$

The proof follows immediately from the definitions.

We denote the distance to the vertex V, and the minimal distance to any vertex in $\mathcal{V}$ by

$$r_V(x) := |x - V| \qquad \text{and} \qquad r_{\mathcal{V}}(x) := \min_{V \in \mathcal{V}} r_V(x).$$

Lemma 3. *The averaging operators satisfy the following norm estimates*

(i)

$$\|\nabla\, \Pi^V u\|_{L_2(\omega_V)} \preceq \|\nabla u\|_{L_2(\omega_V)}$$

(ii)

$$\|r_V^{-1}\{u - \Pi^V u\}\|_{L_2(\omega_V)} \preceq \|\nabla u\|_{L_2(\omega_V)}$$

(iii)

$$\|\nabla\{\varphi_V u - \Pi_0^V u\}\|_{L_2(\omega_V)} \preceq \|\nabla u\|_{L_2(\omega_V)}$$

(iv)

$$\|r_{\mathcal{V}}^{-1}\{\varphi_V u - \Pi_0^V u\}\|_{L_2(\omega_V)} \preceq \|\nabla u\|_{L_2(\omega_V)}$$

The proof is given in [SMP05].

The *global spider vertex operator* is

$$\Pi_{\mathcal{V}} := \sum_{V \in \mathcal{V}_f} \Pi_0^V.$$

Obviously, $u - \Pi_{\mathcal{V}} u$ vanishes in any vertex $V \in \mathcal{V}_f$. These well-defined zero vertex values are reflected by the following norm definition:

$$|\!|\!| \cdot |\!|\!|^2 := \|\nabla \cdot\|_{L_2(\Omega)}^2 + \|\frac{1}{r_{\mathcal{V}}} \cdot\|_{L_2(\Omega)}^2 \tag{3}$$

Theorem 2. *Let u_1 be as in Lemma 1. Then, the decomposition*

$$u_1 = \sum_{V \in \mathcal{V}_f} \Pi_0^V u_1 + u_2 \tag{4}$$

is stable in the sense of

$$\sum_{V \in \mathcal{V}_f} \|\Pi_0^V u_1\|_A^2 + |\!|\!| u_2 |\!|\!|^2 \preceq \|u\|_A^2. \tag{5}$$

The proof is given in [SMP05]. For the rest of this section, u_2 denotes the second term in the decomposition (4).

3.3 Edge contributions

As seen in the last subsection, the remaining function u_2 vanishes in all vertices. We now introduce an edge-based interpolation operator to carry the decomposition further, such that the remaining function, u_3, contributes only to the inner basis functions of each element.

Therefore we need a lifting operator which extends edge functions to the whole triangle preserving the polynomial order. Such operators were introduced in Babuška et al. [BCM91], and later simplified and extended for 3D by Muñoz-Sola [Mun97]. The lifting on the reference element T^R with vertices $(-1,0)$, $(1,0)$, $(0,1)$ and edges $E_1^R := (-1,1) \times \{0\}$, E_2^R, E_3^R reads:

$$(\mathcal{R}_1 w)(x_1, x_2) := \frac{1}{2x_2} \int_{x_1 - x_2}^{x_1 + x_2} w(s)ds,$$

for $w \in L_1([-1,1])$. The modification by Muñoz-Sola preserving zero boundary values on the edges E_2^R and E_3^R is

$$(\mathcal{R}w)(x_1,x_2) := (1-x_1-x_2)\,(1+x_1-x_2)\left(\mathcal{R}_1\frac{w}{1-x_1^2}\right)(x_1,x_2).$$

For an arbitrary triangle $T = F_T(T^R)$ containing the edge $E = F_T(E_1^R)$, its transformed version reads $\mathcal{R}_T w := \mathcal{R}\big[w \circ F_T\big] \circ F_T^{-1}$. The Sobolev space $H_{00}^{1/2}(E)$ on an edge $E = [V_{E,1}, V_{E,2}]$ is defined by its corresponding norm

$$\|w\|^2_{H_{00}^{1/2}(E)} := \|w\|^2_{H^{1/2}(E)} + \int\limits_E \frac{1}{r_{V_E}} w^2\,ds,$$

with $r_{V_E} := \min\{r_{V_{E,1}}, r_{V_{E,2}}\}$.

We call $\omega_E := \omega_{V_{E,1}} \cap \omega_{V_{E,2}}$ the edge patch. We define an *edge-based interpolation operator* as follows:

$$\begin{aligned} \Pi_0^E : \{v \in V_p : v = 0 \text{ in } \mathcal{V}\} &\to H_0^1(\omega_E) \cap V_p, \\ (\Pi_0^E u)|_T &:= \mathcal{R}_T \operatorname{tr}_E u. \end{aligned} \tag{6}$$

Lemma 4. *The edge-based interpolation operator Π_0^E defined in (6) is bounded in the $|\!|\!| \cdot |\!|\!|$-norm:*

$$\|\nabla \Pi_0^E u\|_{L_2(\omega_E)} \preceq |\!|\!| u |\!|\!|_{\omega_E}$$

The proof follows from [BCM91] and [Mun97], and properties of the norm $|\!|\!| \cdot |\!|\!|$.

Theorem 3. *Let u_2 be as in Theorem 2. Then, the decomposition*

$$u_2 = \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2 + u_3 \tag{7}$$

satisfies $u_3 = 0$ on $\bigcup_{E \in \mathcal{E}_f} E$ and is bounded in the sense of

$$\sum_{E \in \mathcal{E}_f} \|\nabla \Pi_0^E u_2\|^2_{L_2} + \|\nabla u_3\|^2_{L_2} \preceq |\!|\!| u_2 |\!|\!|^2. \tag{8}$$

3.4 Main result

Proof of Theorem 1 for the case of triangles Summarizing the last subsections, we have

$$u_1 = u - \Pi_h u, \qquad u_2 = u_1 - \sum_{V \in \mathcal{V}_f} \Pi_0^V u_1, \qquad u_3 = u_2 - \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2,$$

and the decomposition

$$u = \Pi_h u + \sum_{V \in \mathcal{V}_f} \Pi_0^V u_1 + \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2 + \sum_{T \in \mathcal{T}} u_3|_T. \tag{9}$$

is stable in the $\| \cdot \|_A$-norm.

For any edge E or triangle T, we can find a vertex V, such that the corresponding summand is in V_V. Since for each vertex only finitely many terms appear, we can use the triangle inequality and finally arrive at the missing spectral bound

$$\langle Cu, u \rangle = \inf_{\substack{u = u_0 + \sum_V u_V \\ u_0 \in V_0, u_V \in V_V}} \|u_0\|_A^2 + \sum_V \|u_V\|_A^2 \preceq \langle Au, u \rangle .$$

4 Sub-space splitting for tetrahedra

Most of the proof for the 3D case follows the strategy introduced in Section 3, so we use the definitions thereof. The only principal difference is the edge interpolation operator, which has to be treated in more detail.

We define the level surfaces of the vertex hat basis functions

$$\Gamma_V(x) := \Gamma_V(\varphi_V(x)) := \{y : \varphi_V(y) = \varphi_V(x)\}.$$

As in 2D, we first subtract the coarse grid function

$$u_1 = u - \Pi_h u,$$

and secondly the multi-dimensional vertex interpolant to obtain

$$u_2 = u_1 - \Pi_{\mathcal{V}} u_1,$$

where the definitions of Π^V, Π_0^V, $\Pi_{\mathcal{V}}$ are the same as in Section 3, only the level set lines γ_V are replaced by the level surfaces Γ_V. With the same arguments, one easily shows that

$$\sum_{v \in \mathcal{V}_f} \|\Pi_0^V u_1\|_A^2 + \|\nabla u_2\|_{L_2}^2 + \|r_{\mathcal{V}}^{-1} u_2\|_{L_2}^2 \preceq \|u\|_A^2. \tag{10}$$

We define the level line corresponding to a point x in the edge-patch ω_E as

$$\gamma_E(x) := \{y : \varphi_{V_{E,1}}(y) = \varphi_{V_{E,1}}(x) \text{ and } \varphi_{V_{E,2}}(y) = \varphi_{V_{E,2}}(x)\}$$

The edge averaging operator into S_E reads

$$\big(\Pi^E v\big)(x) := \frac{1}{|\gamma_E(x)|} \int_{\gamma_E(x)} v(y)\, dy.$$

In [SMP05], the edge interpolation operator is modified to preserve zero boundary conditions on the whole edge patch ω_E. This resulting operator is called Π_0^E. We define $\mathcal{E}_f$ as the set of are all free edges, i. e. those which do not lie completely on the Dirichlet boundary. We continue the decomposition with

$$u_3 = u_2 - \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2.$$

It fulfills the stability estimate

$$\sum_{E \in \mathcal{E}_f} \|\Pi_0^E u_2\|_A^2 + \|\nabla u_3\|^2 + \|r_{\mathcal{E}}^{-1} u_3\|^2 \preceq \|\nabla u_2\|^2 + \|r_{\mathcal{V}}^{-1} u_2\|^2. \qquad (11)$$

Moreover, $u_3 = 0$ on $\bigcup_{E \in \mathcal{E}_f} E$. Finally, we set

$$u_4 = u_3 - \sum_{F \in \mathcal{F}_f} \Pi_0^F u_3,$$

where the face interpolation operator Π_0^F is defined similar as the edge interpolation operator in 2D.

Proof of Theorem 1 for the case of tetrahedra. The decomposition

$$u = \Pi_h u + \sum_{V \in \mathcal{V}_f} \Pi_0^V u_1 + \sum_{E \in \mathcal{E}_f} \Pi_0^E u_2 + \sum_{F \in \mathcal{F}_f} \Pi_0^F u_3 + \sum_{T \in \mathcal{T}} u_4|_T \qquad (12)$$

is stable in the $\|\cdot\|_A$-norm.

5 Numerical results

In this section, we show numerical experiments on model problems to verify the theory elaborated in the last sections and to get the absolute condition numbers hidden in the generic constants. Furthermore, we study two more preconditioners.

We consider the $H^1(\Omega)$ inner product

$$A(u, v) = (\nabla u, \nabla v)_{L_2} + (u, v)_{L_2}$$

on the unit cube $\Omega = (0, 1)^3$, which is subdivided into an unstructured mesh consisting of 69 tetrahedra. We vary the polynomial order p from 2 up to 10. The condition numbers of the preconditioned systems are computed by the Lanczos method.

Example 1: The preconditioner is defined by the space-decomposition with big overlap of Theorem 1:

$$V = V_0 + \sum_{V \in \mathcal{V}} V_V$$

The condition number is proven to be independent of h and p. The computed numbers are drawn in Figure 3, labeled 'overlapping V'. The inner unknowns have been eliminated by static condensation. The memory requirement of this preconditioner is considerable: For $p = 10$, the memory needed to store the local Cholesky-factors is about 4.4 times larger than the memory required for the global matrix.

In Section 2 we introduced the space splitting into the coarse space V_0 and the vertex subspaces V_V. However, our proof of Theorem 1 involves the finer splitting of a function u into a coarse function, functions in the spider spaces S_V, edge-, face-based and inner functions. Other additive Schwarz preconditioners with uniform condition numbers are induced by this finer splitting.

Example 2: Now, we decompose the space into the coarse space, the p-dimensional spider-vertex spaces $S_{V,0} = \text{span}\{\varphi_V, \ldots, \varphi_V^p\}$, and the overlapping sub-spaces V_E on the edge patches:

$$V = V_0 + \sum_{V \in \mathcal{V}} S_{V,0} + \sum_{E \in \mathcal{E}} V_E$$

The condition number is proven to be uniform in h and p. The computed values are drawn in Figure 3, labeled 'overlapping E, spider V'. Storing the local factors is now about 80 percent of the memory for the global matrix.

Example 3: The interpolation into the spider-vertex space $S_{V,0}$ has two continuity properties: It is bounded in the energy norm, and the interpolation rest satisfies an error estimate in a weighted L_2-norm, see Lemma 3 and equation (10). Now, we reduce the p-dimensional vertex spaces to the spaces spanned by the low energy vertex functions $\varphi_V^{l.e.}$ defined as solutions of

$$\min_{v \in S_{V,0},\, v(V)=1} \|v\|_A^2.$$

These low energy functions can be approximately expressed by the standard vertex functions via $\varphi_V^{l.e.} = f(\varphi_V)$, where the polynomial f solves a weighted 1D problem and can be given explicitly in terms of Jacobi polynomials, see the upcoming report [BPP05]. The interpolation to the low energy vertex space is uniformly bounded, too. But, the approximation estimate in the weighted L_2-norm depends on p. The preconditioner is now generated by

$$V = V_0 + \sum_{V \in \mathcal{V}} \text{span}\{\varphi_V^{l.e.}\} + \sum_{E \in \mathcal{E}} V_E.$$

The computed values are drawn in Figure 3, labeled 'overlapping E, low energy V', and show a moderate growth in p. Low energy vertex basis functions obtained by orthogonalization on the reference element have also been analyzed in [Bic97, SC01].

Example 4: We also tested the preconditioner without additional vertex spaces, i.e.,

$$V = V_0 + \sum_{E \in \mathcal{E}} V_E.$$

Since vertex values must be interpolated by the lowest order functions, the condition number is no longer bounded uniformly in p. The rapidly growing condition numbers are drawn in Figure 4.

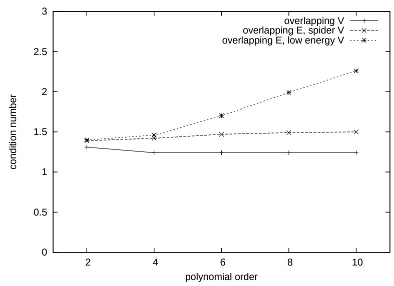

Fig. 3. Overlapping blocks

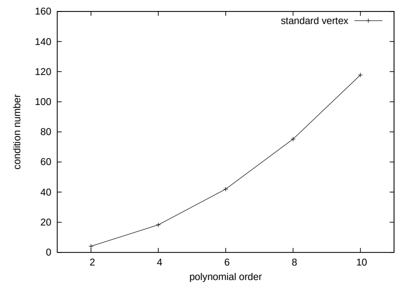

Fig. 4. Standard vertex

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