A Three-Scale Finite Element Method for Elliptic Equations with Rapidly Oscillating Periodic Coefficients

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1 Introduction

On several real world problems the scale ϵ is so smaller than Ω that even with very heavy computer efforts it is impossible to take $h < \epsilon$, h being the scale (mesh-size) of the discrete method used to approximate the solution of

$$L_\epsilon u_\epsilon = -\frac{\partial}{\partial x_i}(a_{ij}(x/\epsilon)\frac{\partial}{\partial x_j}u_\epsilon = f \quad \text{in} \quad \Omega, \quad u_\epsilon = 0 \quad \text{on} \quad \partial\Omega. \tag{1}$$

where the matrix $a(y) = (a_{ij}(y))$ is symmetric positive definite, whose entries are periodic functions of y with periodic cell Y. More specifically we assume $a_{ij} \in C^{1,\beta}(\Re^2)$, $\beta > 0$. It is also assumed that there exists positive constants γ_a and β_a such that $\gamma_a\|\xi\|^2 \leq a_{ij}(y)\xi_i\xi_j \leq \beta_a\|\xi\|^2$ for all $\xi \in \Re^2$ and $y \in \overline{Y}$. Recently new numerical methods have been proposed for approximating the solution u_ϵ with meshes sizes $h > \epsilon$ (or $h >> \epsilon$) but capturing the oscillations presented by the the solution u_ϵ; see for example [HW97,EHW00,SM02,EE03,S03,AB04,]. In [VS05a] we developed a numerical scheme for this problem for the case the domain Ω is rectangular, and quasi-optimal error rate estimates were obtained. That method, opposed to the methods [HW97,EHW00,S03] is strongly based on asymptotic expansions of u_ϵ. We construct a first order asymptotic expansion for u_ϵ, and then we numerically approximate each term separately.

In this paper, we modify the method in [VS05a] for the case where Ω is a convex polygonal regional with rational normals. In this case, a better treatment for the normal derivative of u_0 is required. We propose an approximation based on hybrid finite element for the flux and we obtain optimal error rate estimates for the L_2 norm and H^1 broken semi-norm.

2 Notation

We assume that $Y = [0,1]\times[0,1]$ and Ω is bounded convex polygonal region in $\Re^2$, whose boundary $\partial\Omega = \cup\Gamma^k,\ k = 1,...,m$ where each Γ^k is a line segment with minimal outward normal denoted by $N_k = (p_k, q_k)^t$, where p_k and q_k are integers and relative primes. This hypothesis is required to guarantee periodicity of $a(x/\epsilon)$ on Γ_k [MV97].

Let $D \subset \Re^2$ be an open set. We use the standard notation $\|\cdot\|_{s,D}$, $\|\cdot\|_{s,p,D}$ for $H^s(D)$ and $W^s_p(D)$ norms, $|\cdot|_{s,D}$, $|\cdot|_{s,p,D}$ their semi-norms. and $\|\cdot\|_{s,h,D}$ for the broken norms related to a regular partition $\mathcal{T}_h(D) = K_1, K_2, ..., K_m$ of D. Throughout this paper, when we do not make reference to the domain D it is assumed that $D = \Omega$. It is continually used the Einstein summation convention, i.e. repeated indices indicate summation, except when the indice k is used . In what follows c denotes a generic constant independent of ϵ, h, and functions being evaluated.

3 Theoretical Approximation

3.1 The Asymptotic Expansion

The solution u_ϵ can be approximated by an asymptotic expansion. This approximation can be found using equation (1) and the ansatz

$$u_\epsilon(x) = u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon) + \cdots,$$

where the functions $u_j(x,y)$ are Y periodic in y. These terms are defined below; for more details see [BLP80,OSY92,MV97] .

Let χ^j be the Y periodic solution with zero average on Y of

$$\nabla_y \cdot a(y)\nabla_y \chi^j = \nabla_y \cdot a(y)\nabla_y y_j = \frac{\partial}{\partial y_i} a_{ij}(y). \tag{2}$$

We have that $\chi^j \in C^{2,\beta}(\Re^2)$ when $a_{ij} \in C^{1,\beta}(\Re^2)$. Define the matrix:

$$A_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i)\frac{\partial}{\partial y_m}(y_j - \chi^j)dy. \tag{3}$$

It is easy to see that the matrix A is symmetric positive definite. Define $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ as the solution of

$$-\nabla . A\nabla u_0 = f \ \text{ in } \ \Omega, \qquad u_0 = 0 \ \text{ on } \ \partial\Omega, \tag{4}$$

and let $u_1(x, \frac{x}{\epsilon}) = -\chi^j\left(\frac{x}{\epsilon}\right)\frac{\partial u_0}{\partial x_j}(x)$. Note that $u_0+\epsilon u_1$ does not satisfy the zero Dirichlet boundary condition on $\partial\Omega$. In order to correct this, the boundary corrector term $\theta_\epsilon \in H^1(\Omega)$ is introduced as the solution of

$$-\nabla \cdot a(x/\epsilon)\nabla\theta_\epsilon = 0 \ \text{ in } \ \Omega, \qquad \theta_\epsilon = -u_1(x, \frac{x}{\epsilon}) \ \text{ on } \ \partial\Omega. \tag{5}$$

Therefore we obtain $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon \in H^1_0(\Omega)$.

3.2 Boundary Corrector Approximation

Note that the coefficients $a_{ij}(x/\epsilon)$ and the boundary values $-u_1(x, \frac{x}{\epsilon})$ of the Equation (5) are highly oscillatory, hence it is not a trivial problem to obtain a good discretization for θ_ϵ. We propose an analytical approximation for θ_ϵ, denoted by ϕ_ϵ that satisfies the oscillating boundary condition and is more suitable for numerical approximation.

Note that $u_0 = 0$ along $\partial\Omega$ implies $\nabla u_\epsilon|_{\Gamma_k} = \eta_k \partial_{\eta_k} u_0$. We then decompose $\theta_\epsilon = \tilde{\theta}_\epsilon + \bar{\theta}_\epsilon$ where

$$-\nabla \cdot a(x/\epsilon)\nabla\tilde{\theta}_\epsilon = 0 \quad \text{in} \quad \Omega, \qquad \tilde{\theta}_\epsilon = -u_1 - \chi^* \partial_\eta u_0 \quad \text{on} \quad \partial\Omega, \tag{6}$$

and

$$-\nabla \cdot a(x/\epsilon)\nabla\bar{\theta}_\epsilon = 0 \quad \text{in} \quad \Omega, \qquad \bar{\theta}_\epsilon = \chi^* \partial_\eta u_0 \quad \text{on} \quad \partial\Omega, \tag{7}$$

where $\chi^*|_{\Gamma_k} = \chi_k^*$ are properly chosen constants . In Remark 1 we show that the problems (6) and (7) are well posed. The approximation ϕ_ϵ for θ_ϵ is defined later as $\tilde{\phi}_\epsilon + \bar{\phi}_\epsilon$, where $\tilde{\phi}_\epsilon \approx \tilde{\theta}_\epsilon$ and $\bar{\phi}_\epsilon \approx \bar{\theta}_\epsilon$.

Next we define constants χ_k^* for which the approximation $\tilde{\phi}_\epsilon$ decays exponentially to zero away from the boundary and is suitable for numerical approximation.

Let $\tau_k = (\eta^k)^\perp$ be the $\pi/2$ rotation counterclockwise of η^k. We introduce the following normal and tangential coordinate system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = -\begin{pmatrix} {\eta^k}^T y \\ {\tau_k}^T y \end{pmatrix} \tag{8}$$

We observe that a function periodic in y with period 1 is periodic in y' with period $T_k = (p_k^2 + q_k^2)^{1/2}$. Associated to each side Γ_k of $\partial\Omega$, let $G_k = \{y \in R^2;\ y_1' \le 0;\ \text{and}\ 0 \le y_2' \le T_k\}$; and v_k the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y + \delta_\epsilon \eta^k)\nabla_y v_k = 0 \text{ in } G_k,\\
&v_k(y) = \chi^j(y + \delta_\epsilon \eta^k)\eta_j^k \quad \text{on}\{y \in G_k, y_1' = 0\}\\
&v_k|_{y_2'=0} = v_k|_{y_2'=T_k}, \quad \text{for} \quad -\infty < y_1' < 0,\\
&\text{and} \quad \frac{\partial v_k}{\partial y_i}\exp(-\gamma y_1') \in L^2(G_k), \quad i = 1, 2,
\end{aligned}$$

where $\delta_\epsilon = T_k\,(s_k/\,(\epsilon T_k) - \lfloor s_k/\,(\epsilon T_k)\rfloor)$, and s_k is such that $\Gamma_k \subset \{x \in \Re^2;\ x \cdot \eta_k = s_k\}$; ($\lfloor\cdot\rfloor$ denotes the integer part).

Let

$$\begin{aligned}
\chi_k^* = \frac{1}{(A\eta^k, \eta^k)T_k} &\left(\int_0^{T_k} \left[\chi^l a_{ij} \left(\delta_{jm} - \frac{\partial \chi^m}{\partial y_j} \right) \eta_i^k \eta_m^k \eta_l^k \right] \Bigg|_{y_1'=\delta_\epsilon} dy_2' \right.\\
&\left. + \int_{G_k} (a(y + \delta_\epsilon \eta^k)\nabla_y v_k \cdot \nabla_y v_k) dy \right),
\end{aligned}$$

It can be shown [MV97] that v_k decays exponentially to zero for $y_1' \to -\infty$, i.e. $(v_k - \chi_k^*)\exp(-\gamma y_1') \in L^2(G_k)$.

We note by Remark 1 that $(u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0)|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$. Thus we can split $\tilde{\theta}_\epsilon = \sum_{k\in\{1,\dots,N\}} \tilde{\theta}^k_\epsilon$, where

$$L_\epsilon \tilde{\theta}^k_\epsilon = 0 \;\text{ in }\; \Omega, \qquad \tilde{\theta}^k_\epsilon = \begin{cases} -u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0 & \text{on } \;\Gamma_k, \\ 0 & \text{on } \;\partial\Omega \setminus \Gamma_k. \end{cases}$$

We approximate $\tilde{\theta}^k_\epsilon$ by $\tilde{\phi}^k_\epsilon$ given by

$$\tilde{\phi}^k_\epsilon(x_1, x_2) = \varphi_k(x_1) \left(v_k \left(\frac{x - s_k \eta_k}{\epsilon} \right) - \chi^*_k \right) \nabla u_0 \cdot \eta_k. \tag{9}$$

In order to simplify the definition of the function $\varphi_k(x)$ let us assume $\Gamma_k = \{x \in \Re^2;\; x_1 = 0, 0 \le x_2 \le c\}$ and that x_1^+ is the inner normal direction. Let Γ_{k-1},Γ_{k+1} be the edges with vertices at the point $(0, c)$, $(0, 0)$ respectively and let $\alpha_k > 0$ and $\alpha_{k+1} < 0$ be the angles between x_1 axis and Γ_{k-1} and Γ_{k+1} respectively. Then we define

$$\varphi_k(x) = \begin{cases} 1 & \text{if } \;0 \le x_1 \le \delta;\; 0 \le x_2 \le c \\ 1 - (x_2 - c)/(x_1 \tan\alpha_k) & \text{if } \;0 \le x_1 \le \delta;\; x_2 > c \\ 1 + x_2/(x_1 \tan\alpha_{k+1}) & \text{if } \;0 \le x_1 \le \delta;\; x_2 < 0 \\ \text{smooth} & \text{if } \;\delta \le x_1 \le 2\delta \\ 0 & \text{if } \;x_1 \ge 2\delta \end{cases}$$

Hence $\tilde{\phi}_\epsilon = \sum_{k\in\{1,\dots,N\}} \tilde{\phi}^k_\epsilon$ approximate $\tilde{\theta}_\epsilon$, and $\tilde{\phi}_\epsilon = \tilde{\theta}_\epsilon$ on the boundary of Ω.

The boundary condition imposed on Equation (7) does not depend on ϵ. An effective approximation for $\bar{\theta}_\epsilon$ is given by $\bar{\phi} \in H^1(\Omega)$ the solution of

$$-\nabla \cdot A \nabla \bar{\phi} = 0 \;\text{ in }\; \Omega, \qquad \bar{\phi} = \chi^* \partial_\eta u_0 \;\text{ on }\; \partial\Omega.$$

We define our theoretical approximation for u_ϵ as $u_0 + \epsilon u_1 + \epsilon \phi_\epsilon$, where $\phi_\epsilon = \tilde{\phi}_\epsilon + \bar{\phi}$. Note that $\phi_\epsilon|_{\partial\Omega} = \theta_\epsilon|_{\partial\Omega}$, therefore $u_0 + \epsilon u_1 + \epsilon \phi_\epsilon = 0$ on $\partial\Omega$. In [VS05b] we prove the following error bounds

Theorem 1. *Assume that $a_{ij} \in C^{1,\beta}(\Re^2)$ and $u_0 \in H^2(\Omega)$, ($u_0 \in H^3(\Omega)$). Then there exists a constant c, such that*

$$\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon \phi_\epsilon\|_1 \le c\epsilon \|u_0\|_2$$
$$(\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon \phi_\epsilon\|_0 \le \epsilon^{3/2} \|u_0\|_3).$$

Remark 1. Since u_0 satisfies zero Dirichlet boundary condition on $\partial\Omega$ and $u_0 \in H^2(\Omega)$, we have $\frac{\partial u_0}{\partial \eta^k} \in H^{1/2}_{00}(\Gamma_k)$ and $\|\chi^* \partial_\eta u_0\|_{H^{1/2}(\partial\Omega)} \le c(\chi^*) \|u_0\|_2$. Note also that $u_1(x, \frac{x}{\epsilon}) = -\chi^j \left(\frac{x}{\epsilon}\right) \frac{\partial u_0}{\partial x_j}(x)$, since $\chi^j \in C^{2,\beta}(\Re^2)$ we get $u_1|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$.

4 Finite Element Approximation

We now describe how to numerically approximate the terms u_0, u_1, $\tilde{\phi}_\epsilon$ and $\bar{\phi}$.

- Solve the cell problem (2) with a second order accurate conforming finite element in a partition $\mathcal{T}_{\hat{h}}(Y)$. Call these solutions $\chi^j_{\hat{h}}$.
- Define $A^{\hat{h}}_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i_{\hat{h}})\frac{\partial}{\partial y_m}(y_j - \chi^j_{\hat{h}})dy$.
- Let $V^h(\Omega)$ be a conforming second order accurate finite element in a mesh $\mathcal{T}_h(\Omega)$, and $V^h_0(\Omega) = V^h(\Omega) \cap H^1_0(\Omega)$. Define $u^{h,\hat{h}}_0 \in V^h_0$ the solution of

$$\int_\Omega (A^{\hat{h}}\nabla u^{h,\hat{h}}_0, \nabla v^h)dx = \int_\Omega f v^h dx, \quad \forall v^h \in V^h_0.$$

- Define $u^{h,\hat{h}}_1$ as $u^{h,\hat{h}}_1(x) = -\chi^j_{\hat{h}}\left(\frac{x}{\epsilon}\right)\frac{\partial u^{h,\hat{h}}_0}{\partial x_j}(x)$. Note that this leads to a non-conforming approximation for u_1 in the partition $\mathcal{T}_h(\Omega)$.
- Define Y^h_k the trace of V^h at Γ_k. And let $\lambda^h_k \in \Gamma^h_k$, $\lambda^h_k = 0$ at $\partial\Gamma_k$ satisfying

$$\int_\Omega A^{\hat{h}}_{ij}\partial_i u^h_0 \partial_j \phi dx = \int_\Omega f\phi dx + \int_{\Gamma_k}\lambda^h_k \phi d\sigma. \tag{10}$$

$\forall\ \phi \in V^h; \phi|_{\partial\Omega\setminus\Gamma_k} = 0$ so approximate $\partial_\eta u_0$ by $\mu^{h,\hat{h}}$ where

$$\mu^{h,\hat{h}}|_{\Gamma_k} = \lambda^h_k / A^{\hat{h}}_{ll}, \ \begin{cases} l = 1 \text{ if } k = 1,3. \\ l = 2 \text{ if } k = 2,4. \end{cases}$$

- Let p be a positive integer and $G^p_k = \{y \in R^2; y'_1 \le 0,\ |y'_1| \le p;\ \text{and } 0 \le y'_2 \le T_k\}$. Define $\tilde{v}_k \in H^1(G^p_k)$ the solution of

$$\begin{aligned} &-\nabla_y \cdot a(y + \delta_\epsilon \eta^k)\nabla_y \tilde{v}_k = 0 \ \text{ in } \ G^p_k, \\ &\tilde{v}_k(y) = \chi^1_{\hat{h}}(y + \delta_\epsilon \eta^k), \text{on}\{y \in G_k, y'_1 = 0\}, \\ &\partial_\eta \tilde{v}_k = 0, \ \text{ on } \{y \in G^p_k;\ |y'_1| = p\}, \\ &\text{and } v_k|_{y'_2=0} = v_k|_{y'_2=T_k}, \ \text{ for } \ |y'_1| < p. \end{aligned}$$

Let $v^{\hat{h},p}_k$ be a numerical approximation of $\tilde{v}_k$ using a second order accurate conforming finite element on a mesh $\mathcal{T}_{\hat{h}}(G^p_e)$.
- Define

$$\chi^{*,\hat{h},p}_k = \frac{1}{(A^{\hat{h}}\eta^k,\eta^k)T_k}\left(\int_0^{T_k}\left[\chi^l_{\hat{h}} a_{ij}\left(\delta_{jm} - \frac{\partial \chi^m_{\hat{h}}}{\partial y_j}\right)\eta^k_i\eta^k_m\eta^k_l\right]\Big|_{y'_1=\delta_\epsilon} dy'_2 \right. \\ \left. + \int_{G_k}(a(y+\delta_\epsilon\eta^k)\nabla_y v^{\hat{h},p}_k \cdot \nabla_y v^{\hat{h},p}_k)dy\right),$$

- Given $g : \Gamma_k \to \Re$, let $E_k(g) \in V^h(\Omega)$ be the extension by zero of g to Ω.

- Observe that in Equation. (9) the term $v_k\left((x - s_k\eta_k)/\epsilon\right)$ appears. Since the approximation $v_k^{\hat{h},p}$ is defined in G_k^p, we can calculate $v_k^{\hat{h},p}\left((x - s_k\eta_k)/\epsilon\right)$ only if $|x_1' - s_k| \le \epsilon p$. Since the functions $v_k - \chi_k^*$ decays exponentially to zero in the $-\eta_k$ direction its is natural to consider the following approximation

$$\tilde{\phi}_\epsilon^{e,h,\hat{h},p}(x_1, x_2) = \begin{cases} \varphi_k \left(v_k^{\hat{h},p}\left(\frac{x - s_k\eta_k}{\epsilon}\right) \frac{\partial u_0^{h,\hat{h}}}{\partial x_1} - \chi_k^{*,\hat{h},p} E_k(\mu^{h,\hat{h}}) \right) & \text{if} \;\; |x_1' - s_k| < \epsilon p, \\ 0 & \text{if} \;\; |x_1' - s_k| \ge \epsilon p, \end{cases}$$

  and $\tilde{\phi}_\epsilon^{h,\hat{h},p} = \sum_{k\in\{1,\ldots,N\}} \tilde{\phi}_\epsilon^{k,h,\hat{h},p}$.
- Let $\bar{\phi}^{h,\hat{h},p}$ be a second order accurate finite element approximation in a mesh of size h for the following equation

$$-\nabla A^{\hat{h}} \nabla \psi = 0, \qquad \psi = \chi^{*,\hat{h},p} \mu^{h,\hat{h}} \;\; \text{on} \;\; \partial\Omega. \tag{11}$$

  Remark 2. By construction $\mu^{h,\hat{h}} = 0$ at the corners of Ω, therefore $\chi^* \mu^{h,\hat{h}} \in H^{1/2}(\partial\Omega)$. This implies that Eq.(11) is well posed. In addition $\chi^* \mu^{h,\hat{h}} \in V^h|_{\partial\Omega}$ hence we can look for a numerical solution of Eq.(11) at V^h.

- Approximate θ_ϵ by $\theta_\epsilon^{h,\hat{h},p} := \phi^{r,h,\hat{h},p} + \theta^{*,h,\hat{h},p}$ and finally construct the numerical solution for Eq. (1), $u_\epsilon^{h,\hat{h},p} = u_0^{h,\hat{h}} + \epsilon u_1^{h,\hat{h}} + \epsilon\theta_\epsilon^{h,\hat{h},p}$.

5 Error Analysis

When $p \to \infty$ and $\hat{h} \to 0$ we prove in [VS05b] the following estimates.

Theorem 2. *Assume that $a_{ij} \in C^{1,\beta}(\Re^2)$ and $u_0 \in W^{2,\infty}(\Omega)$ ($u_0 \in W^{2,\infty}(\Omega) \cap H^3(\Omega)$). Then there exists a constant c, such that*

$$|u_\epsilon - u_h|_{1,h} \le c(h + \epsilon)\|u_0\|_{2,\infty}$$
$$(\|u_\epsilon - u_h\|_0 \le c(h^2 + \epsilon^{\frac{3}{2}} + \epsilon h)(|u_0|_{2,\infty} + \|u_0\|_3))$$

6 Numerical Experiments

In this section, we present some numerical results for solving our model problem with

$$a(x) = \left(\frac{2 + P\sin(2\pi x_1/\epsilon)}{2 + P\cos(2\pi x_2/\epsilon)} + \frac{2 + \sin(2\pi x_2/\epsilon)}{2 + P\sin(2\pi x_1/\epsilon)} \right) I_{2\times 2}$$

$$f(x) = -1 \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

We compare the solution obtained by our method with the solution obtained by a second order accurate finite element method in a fine mesh of size h_f, which we call u_ϵ^*. Tables 1 provide absolute errors estimates for $u_\epsilon^* - u_\epsilon^{h,\hat{h},p}$. We have used $p = 2$, $\hat{h} = 1/128$, $h_f = 1/2048$, and a triangular mesh with continuous piecewise linear functions to approximate $\chi_{\hat{h}}^j$ and $v_k^{\hat{h},p}$. From Table

Table 1. $u_\epsilon^* - u_\epsilon^{h,\hat{h},p}$ error

$\epsilon \downarrow \quad h \rightarrow$	1/8	1/16	1/32	1/64
		$\|\cdot\|_0$ error		
1/16	2.3863e-04	1.5793e-04		
1/32	2.3241e-04	8.0169e-05	1.7773e-05	
1/64	2.3540e-04	5.4314e-05	5.1865e-05	5.9606e-05
		$\vert\cdot\vert_{1,h}$ error		
1/16	0.0097	0.0067		
1/32	0.0086	0.0051	0.0036	
1/64	0.0086	0.0044	0.0025	0.0018

Table 2.

$\epsilon = 1/64$, $h = 1/32$, $h_f = 1/1024$

	$\|\cdot\|_0$	$\vert\cdot\vert_{1,h}$
$u_\epsilon^* - u_0^{h,\hat{h}}$	0.0287	0.0215
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}}$	0.0213	0.0026
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon\bar{\phi}^{h,\hat{h},p}$	5.0450e-05	0.0026
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon(\bar{\phi}^{h,\hat{h},p} + \tilde{\phi}_\epsilon^{h,\hat{h},p})$	5.1865e-05	0.0025

1, we see that for $\epsilon << h$ we have errors of order $O(h^2)$ and $O(h)$ for the L^2 norm and semi norm H^1 respectively. We observe that when we fix h and decrease ϵ the errors almost do not change. This is an evidence that in this case the dominant error term is $O(h)$. Also looking the diagonal values in these tables we see clearly that the numerical error agrees with the theoretical rates from Theorem 2.

Table 2 shows the improvement obtained in the final approximation by considering the numerical approximation for the boundary corrector. We observe a better improvement on the $\|\cdot\|_0$ norm rather then on $|\cdot|_{1,h}$ semi norm. The reason for this is that $\bar{\phi}$ is obtained through the homogenized equation associated to Problem (7), therefore it is a good approximation for $\bar{\theta}_\epsilon$ on $L^2(\Omega)$ norm but not on $|\cdot|_1$ semi norm. The term $\tilde{\phi}_\epsilon$ is defined in a thin boundary layer that mostly force the approximation to satisfies the zero Dirichlet boundary condition.

7 Conclusions

We propose a new method for approximating numerically the solution of Equation (1). This method is strongly based on periodicity of the coefficients a_{ij}, and for this reason it has relative low computational cost with optimal error convergence rate.

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