Condition Number Estimates for C^0 Interior Penalty Methods

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Summary. In this paper we study the condition number of the system resulting from C^0 interior penalty methods for fourth order elliptic boundary value problems. We show that the condition number can be bounded by Ch^{-4} and that this bound is sharp, where h is the mesh size of the triangulation and C is a positive constant independent of the mesh size.

1 Introduction

C^0 interior penalty methods provide a new approach for the solution of fourth order elliptic problems [10, 4]. These methods combine the ideas of continuous Galerkin methods, discontinuous Galerkin methods and stabilization techniques, which can be illustrated by the following model problem on a bounded polygonal domain Ω in $\mathbb{R}^2$:
Find $u \in H_0^2(\Omega)$ such that

$$\sum_{i,j=1}^{2} \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx = \int_\Omega f v \, dx \qquad \forall\, v \in H_0^2(\Omega), \tag{1}$$

where $f \in L_2(\Omega)$.

Let $\mathcal{T}_h$ be a simplicial or convex quadrilateral triangulation of Ω. In C^0 interior penalty methods, we choose the discrete space $V_h \subset H_0^1(\Omega)$ to be either a $\mathcal{P}_\ell$ ($\ell \geq 2$) triangular Lagrange finite element space or a $\mathcal{Q}_\ell$ ($\ell \geq 2$) tensor product finite element space associated with $\mathcal{T}_h$. By an integration by parts argument [4], it can be shown that the solution u of (1), which belongs to $H^{2+\alpha}(\Omega)$ for some $\alpha > 1/2$ by elliptic regularity [11, 9, 13, 2], satisfies

$$\mathcal{A}_h(u, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h, \tag{2}$$

where

$$\mathcal{A}_h(w,v) = \sum_{D\in\mathcal{T}_h}\sum_{i,j=1}^{2}\int_D \frac{\partial^2 w}{\partial x_i\partial x_j}\frac{\partial^2 v}{\partial x_i\partial x_j}dx + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|}\int_e \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] ds$$
$$+ \sum_{e\in\mathcal{E}_h}\int_e\left(\left\{\!\!\left\{\frac{\partial^2 w}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] + \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\right) ds. \tag{3}$$

In (3), $\mathcal{E}_h$ is the set of all the edges of $\mathcal{T}_h$, and η is a penalty parameter. The jumps $[\![\cdot]\!]$ and averages $\{\!\{\cdot\}\!\}$ are defined as follows.

Let e be an interior edge of $\mathcal{T}_h$ shared by two elements D_- and D_+ and n_e be the unit normal vector of e pointing from D_- to D_+. We define on e, for any function v that is piecewise H^s with respect to the triangulation $\mathcal{T}_h$ for some $s > \frac{5}{2}$,

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] = \frac{\partial v_+}{\partial n_e} - \frac{\partial v_-}{\partial n_e} \quad \texttt{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{1}{2}\left[\frac{\partial^2 v_+}{\partial n_e^2} + \frac{\partial^2 v_-}{\partial n_e^2}\right], \tag{4}$$

where $v_\pm = v\big|_{D_\pm}$. For an edge e that is a subset of $\partial\Omega$, we take n_e to be the outward pointing unit normal vector and define

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] = -\frac{\partial v}{\partial n_e} \quad \texttt{and} \quad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{\partial^2 v}{\partial n_e^2}. \tag{5}$$

Note that $[\![\partial v/\partial n]\!]$ and $\{\!\{\partial^2 v/\partial n^2\}\!\}$ are independent of the choice of n_e.

The discrete problem for (1) is then given by:
Find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h. \tag{6}$$

In view of (2), the C^0 interior penalty method defined by (6) is consistent and for a sufficiently large η, it is also stable. Therefore the discretization error $u - u_h$ is quasi-optimal with respect to appropriate norms [10, 4].

In this paper, we show that the condition number of the system of (6) is of order h^{-4}, where h is the mesh size of the triangulation. This result implies that the system of the discrete problem resulting from C^0 interior penalty methods is very ill-conditioned for small h, in which case the convergence rates of classical iterative methods are very slow. Therefore it is necessary to use modern fast solvers such as multigrid methods [5] and domain decomposition methods [6] to improve the efficiency.

The rest of the paper is organized as follows. We introduce the finite element space and some preliminaries in section 2. In section 3, we derive the upper bound for the condition number of the system. We obtain the lower bound for the condition number in the last section.

2 Preliminaries

In this section, we define the finite element space and derive some preliminary estimates that can help us to obtain the estimates for the condition number.

For simplicity we will focus on the case that $\mathcal{T}_h$ is a quasi-uniform rectangular mesh in this paper. The results we will show are still true for general convex quadrilateral meshes and triangular elements.

To avoid the proliferation of constants, we henceforth use the notation $A \lesssim B$ to represent the statement $A \leq C \times B$, where C is a constant which depends only on the aspect ratios of $\mathcal{T}_h$. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

Let $V_h \subset H_0^1(\Omega)$ be the Q_2 finite element space associated with $\mathcal{T}_h$. For η sufficiently large (which is assumed to be the case), the following relation [4] holds:

$$\mathcal{A}_h(v,v) \approx |v|^2_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, v \in V_h, \tag{7}$$

where

$$|v|^2_{H^2(\Omega,\mathcal{T}_h)} = \sum_{D\in\mathcal{T}_h} |v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \|[\![\partial v/\partial n]\!]\|^2_{L_2(e)}. \tag{8}$$

Here and throughout this paper we follow the standard notation for L_2-based Sobolev spaces [1, 3, 8].

Let

$$\mathbf{A}_h = (\mathcal{A}_h(\varphi_1,\varphi_2))_{1\leq i,j\leq n} \tag{9}$$

be the stiffness matrix, where n is the dimension of V_h and $\varphi_1,\cdots,\varphi_n$ are the nodal basis functions for V_h. We want to estimate the condition number of $\mathbf{A}_h$ given by

$$\kappa(\mathbf{A}_h) = \frac{\lambda_{\max}(\mathbf{A}_h)}{\lambda_{\min}(\mathbf{A}_h)}. \tag{10}$$

Note that

$$\lambda_{\max}(\mathbf{A}_h) = \max_{\substack{x\in\mathbb{R}^n\\ x\neq 0}} \frac{x^T\mathbf{A}_h x}{x^T x} \approx \max_{\substack{v\in V_h\\ v\neq 0}} \frac{\mathcal{A}_h(v,v)}{h^{-2}\|v\|^2_{L_2(\Omega)}}, \tag{11}$$

$$\lambda_{\min}(\mathbf{A}_h) = \min_{\substack{x\in\mathbb{R}^n\\ x\neq 0}} \frac{x^T\mathbf{A}_h x}{x^T x} \approx \min_{\substack{v\in V_h\\ v\neq 0}} \frac{\mathcal{A}_h(v,v)}{h^{-2}\|v\|^2_{L_2(\Omega)}}. \tag{12}$$

3 Upper bound for the condition number

In this section, we obtain an upper bound for the condition number of $\mathbf{A}_h$. From (11) and (12), it is sufficient to find an upper bound for the maximum eigenvalue of $\mathbf{A}_h$ and a lower bound for the minimum eigenvalue of $\mathbf{A}_h$.

Lemma 1. *For all $v \in V_h$, it holds that*

$$\lambda_{\max}(\mathbf{A}_h) \lesssim h^{-2}. \tag{13}$$

Proof. Let $v \in V_h$ be arbitrary, using (7), (8), inverse estimates [3], (4) and the trace theorem (with scaling), we obtain that

$$
\begin{aligned}
\mathcal{A}_h(v,v) &\approx |v|^2_{H^2(\Omega,\mathcal{T}_h)} \\
&= \sum_{D\in\mathcal{T}_h} |v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \| \,[\![\partial v/\partial n]\!]\, \|^2_{L_2(e)} \\
&\lesssim \sum_{D\in\mathcal{T}_h} (\operatorname{diam} D)^{-4} \|v\|^2_{L_2(D)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \sum_{D\in\mathcal{T}_e} \|\partial v_{_D}/\partial n\|^2_{L_2(e)} \qquad (14)\\
&\lesssim \sum_{D\in\mathcal{T}_h} (\operatorname{diam} D)^{-4} \|v\|^2_{L_2(D)} \\
&\quad + \sum_{e\in\mathcal{E}_h} \sum_{D\in\mathcal{T}_e} \Big[(\operatorname{diam} D)^{-2} |v|^2_{H^1(D)} + |v|^2_{H^2(D)} \Big] \\
&\lesssim \sum_{D\in\mathcal{T}_h} (\operatorname{diam} D)^{-4} \|v\|^2_{L_2(D)} + \sum_{e\in\mathcal{E}_h} \sum_{D\in\mathcal{T}_e} (\operatorname{diam} D)^{-4} \|v\|^2_{L_2(D)} \\
&\lesssim \sum_{D\in\mathcal{T}_h} (\operatorname{diam} D)^{-4} \|v\|^2_{L_2(D)} \\
&\lesssim h^{-4} \|v\|^2_{L_2(\Omega)}.
\end{aligned}
$$

where $\mathcal{T}_e$ is the set of all rectangles sharing e as a common edge.

Here we have used the fact that

$$h \approx \operatorname{diam} D \qquad \forall\, D \in \mathcal{T}_h.$$

Therefore, the estimate (13) follows from (11) and (14).

Next we derive a lower bound for the minimum eigenvalue of $\mathbf{A}_h$.

Lemma 2. *It holds that*

$$\lambda_{\min}(\mathbf{A}_h) \gtrsim h^2 \qquad \forall\, v \in V_h. \qquad (15)$$

Proof. For general piecewise H^2 functions v, we have the following Poincaré-Friedrichs inequality [7]:

$$
\begin{aligned}
\|v\|^2_{L_2(\Omega)} + |v|^2_{H^1(\Omega,\mathcal{T}_h)} \lesssim \Big[&|v|^2_{H^2(\Omega,\mathcal{T}_h)} + [\Phi(v)]^2 \\
&+ \sum_{e\in\mathcal{E}_h} \Big(\frac{1}{|e|^3} \|\pi_{e,1} [\![v]\!]_e\|^2_{L_2(e)} + \frac{1}{|e|} \|\pi_{e,0} [\![\partial v/\partial n]\!]_e\|^2_{L_2(e)} \Big) \Big], \qquad (16)
\end{aligned}
$$

where $\Phi : H^2(\Omega,\mathcal{T}_h) \longrightarrow \mathbb{R}$ is a seminorm that satisfies certain properties (cf. (I.2), (I.3), (II.15) and (III.3) of [7]) and the operator $\pi_{e,0}$ (resp. $\pi_{e,1}$) is the orthogonal projection operator from $L_2(e)$ onto $\mathcal{P}_0(e)$ (resp. $\mathcal{P}_1(e)$).

In (16), taking $\Phi(v) = \|\pi_{\partial\Omega,1}\, v\|_{L_2(\Omega)}$ and applying it to $v \in V_h$, we have

$$\|v\|^2_{L_2(\Omega)} + |v|^2_{H^1(\Omega,\mathcal{T}_h)} \lesssim \sum_{D\in\mathcal{T}_h} |v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \| [\![\partial v/\partial n]\!] \|^2_{L_2(e)} = |v|^2_{H^2(\Omega,\mathcal{T}_h)},$$

which implies for all $v \in V_h$

$$\|v\|^2_{L_2(\Omega)} \lesssim |v|^2_{H^2(\Omega,\mathcal{T}_h)}. \tag{17}$$

Therefore, by (12), (7) and (17), we obtain

$$\lambda_{\min}(\mathbf{A}_h) \approx \min_{\substack{v\in V_h\\ v\neq 0}} \frac{\mathcal{A}_h(v,v)}{h^{-2}\|v\|^2_{L_2(\Omega)}} \gtrsim h^2.$$

From Lemma 1 and Lemma 2 we have the following condition number estimate.

Theorem 1. *The condition number of $\mathbf{A}_h$ satisfies the estimate*

$$\kappa(\mathbf{A}_h) = \frac{\lambda_{\max}(\mathbf{A}_h)}{\lambda_{\min}(\mathbf{A}_h)} \lesssim h^{-4}. \tag{18}$$

4 Lower bound for the condition number

In this section we will show that the bound for the condition number obtained in the last section is sharp. We begin with an easy lower bound for $\lambda_{\max}(\mathbf{A}_h)$.

Lemma 3. *It holds that*

$$\lambda_{\max}(\mathbf{A}_h) \gtrsim h^{-2}. \tag{19}$$

Proof. In view of (12) and (7), it suffices to construct a function $v_* \in V_h$ such that

$$|v_*|^2_{H^2(\Omega,\mathcal{T}_h)} \gtrsim h^{-4}\|v_*\|^2_{L_2(\Omega)}. \tag{20}$$

Let D_* be an arbitrary element in $\mathcal{T}_h$. Take $v_* \in V_h$ to be a nodal basis function which is defined by

$$v_*(p) = \begin{cases} 1, & \text{if } p \text{ is the central node of } D_*,\\ 0, & \text{otherwise.}\end{cases} \tag{21}$$

Then it is not difficult to obtain that

$$v_*(x_1,x_2) = (\operatorname{diam} D_*)^{-4}\big((\operatorname{diam} D_*)^2 16x_1x_2 - (\operatorname{diam} D_*)16x_1^2x_2 - (\operatorname{diam} D_*)16x_1x_2^2 + 16x_1^2x_2^2\big). \tag{22}$$

So (21) and (22) imply that

$$\|v_*\|^2_{L_2(\Omega)} = \|v_*\|^2_{L_2(D_*)} = \frac{64}{225}(\operatorname{diam} D_*)^2, \tag{23}$$

and

$$|v_*|^2_{H^2(\Omega,\mathcal{T}_h)} = |v_*|^2_{H^2(D_*)} + \sum_{\substack{e\in\mathcal{E}_h \\ e\subset \bar{D}_*}} \frac{1}{|e|}\|\partial v_*/\partial n\|^2_{L_2(e)} = \frac{5312}{45}(\operatorname{diam} D_*)^{-2}. \quad (24)$$

Therefore, combining (23) and (24), we obtain

$$|v_*|^2_{H^2(\Omega,\mathcal{T}_h)} \geq h^{-4}\|v_*\|^2_{L_2(\Omega)}.$$

We now derive an upper bound for the minimum eigenvalue of $\mathbf{A}_h$.

Lemma 4. *The following estimate for the minimum eigenvalue of* $\mathbf{A}_h$ *holds:*

$$\lambda_{\min}(\mathbf{A}_h) \lesssim h^2. \quad (25)$$

Proof. From the theory of partial differential equations [12], there exist $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $u_1, u_2, \cdots \in H_0^2(\Omega)$ such that

$$\triangle^2 u_i = \lambda_i u_i \quad \text{and} \quad \int_\Omega u_i u_j \, dx = \delta_{ij}.$$

We now consider the following system:

$$\begin{cases} \triangle^2 u_1 = \lambda_1 u_1 & \text{in } \Omega, \\ u_1|_{\partial\Omega} = 0 \quad \text{and} & \frac{\partial u_1}{\partial n}\big|_{\partial\Omega} = 0. \end{cases} \quad (26)$$

Let $\hat{u}_1$ be the Q_2 interpolant of u_1. Then standard interpolation error estimates [3] imply

$$\|u_1 - \hat{u}_1\|^2_{L_2(\Omega)} \lesssim h^4|u_1|^2_{H^2(\Omega)} \lesssim h^4\|u_1\|^2_{L_2(\Omega)}, \quad (27)$$

$$|u_1 - \hat{u}_1|^2_{H^1(\Omega)} \lesssim h^2|u_1|^2_{H^2(\Omega)}, \quad (28)$$

$$\sum_{D\in\mathcal{T}_h} |u_1 - \hat{u}_1|^2_{H^2(D)} \lesssim |u_1|^2_{H^2(\Omega)}. \quad (29)$$

For h small enough, it follows from (27) that

$$\begin{aligned} \|\hat{u}_1\|^2_{L_2(\Omega)} &\gtrsim \|u_1\|^2_{L_2(\Omega)} - \|u_1 - \hat{u}_1\|^2_{L_2(\Omega)} \\ &\gtrsim \|u_1\|^2_{L_2(\Omega)} - h^4\|u_1\|^2_{L_2(\Omega)} \\ &\gtrsim \|u_1\|^2_{L_2(\Omega)}. \end{aligned} \quad (30)$$

On the other hand, since $u_1 \in H_0^2(\Omega)$, by (8), the triangle inequality, (29), the trace theorem with scaling and (28), we obtain that

$$
\begin{aligned}
&|\hat{u}_1|^2_{H^2(\Omega,\mathcal{T}_h)} \\
&= \sum_{D\in\mathcal{T}_h} |\hat{u}_1|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \| [\![\partial\hat{u}_1/\partial n]\!] \|^2_{L_2(e)} \qquad (31)\\
&\lesssim \sum_{D\in\mathcal{T}_h} |\hat{u}_1 - u_1|^2_{H^2(D)} + \sum_{D\in\mathcal{T}_h} |u_1|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \| [\![\partial(\hat{u}_1 - u_1)/\partial n]\!] \|^2_{L_2(e)} \\
&\lesssim \sum_{D\in\mathcal{T}_h} |u_1|^2_{H^2(D)} + \sum_{D\in\mathcal{T}_h} \Big[(\operatorname{diam} D)^{-2} |\hat{u}_1 - u_1|^2_{H^1(D)} + |\hat{u}_1 - u_1|^2_{H^2(D)} \Big] \\
&\lesssim \sum_{D\in\mathcal{T}_h} |u_1|^2_{H^2(D)} \\
&= |u_1|^2_{H^2(\Omega)}.
\end{aligned}
$$

Therefore, the estimate (25) follows from (12), (7), (30) and (31):

$$
\begin{aligned}
\lambda_{\min}(\mathbf{A}_h) &\approx \min_{\substack{v\in V_h\\ v\neq 0}} \frac{\mathcal{A}_h(v,v)}{h^{-2}\|v\|^2_{L_2(\Omega)}} \\
&\lesssim \frac{\mathcal{A}_h(\hat{u}_1,\hat{u}_1)}{h^{-2}\|\hat{u}_1\|^2_{L_2(\Omega)}} \qquad (32)\\
&\approx \frac{|\hat{u}_1|^2_{H^2(\Omega,\mathcal{T}_h)}}{h^{-2}\|\hat{u}_1\|^2_{L_2(\Omega)}} \\
&\lesssim \frac{|u_1|^2_{H^2(\Omega)}}{h^{-2}\|u_1\|^2_{L_2(\Omega)}} \\
&\lesssim h^2.
\end{aligned}
$$

Combining Lemmas 3 and 4, we have the following theorem.

Theorem 2. *The following estimate holds for our model problem:*

$$
\kappa(\mathbf{A}_h) = \frac{\lambda_{\max}(\mathbf{A}_h)}{\lambda_{\min}(\mathbf{A}_h)} \gtrsim h^{-4}. \qquad (33)
$$

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